Homogeneous Basis for Continuous Geometry.

By

Fumitomo MAEDA.

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J. v. Neumann,[1] in his continuous geometry L, has defined the homogeneous basis, as a system of independent elements (a_i; $i=1, 2, \ldots, m$) which are pairwise perspective and

$$a_1 \cup a_2 \cup \cdots \cup a_m = 1 . \tag{1}$$

When L satisfies the chain-condition, we have a homogeneous basis in which all the elements a_i are minimal. But when L does not satisfy the chain-condition, we cannot have such a homogeneous basis with *minimal* elements.

Thus we meet with a similar situation to that of ring-decomposition in algebra. A ring $\mathfrak{R}$, without radical, with minimum-condition for right ideals, is a direct sum of simple right ideals, i. e.

$$\mathfrak{R} = \mathfrak{a}_1 + \mathfrak{a}_2 + \cdots + \mathfrak{a}_n . \tag{2}$$

But when the ring does not satisfy the minimum-condition, we cannot decompose $\mathfrak{R}$ in a direct sum of *simple* right ideals as (2). To investigate the latter case, in a previous paper[2] I introduced a decomposition system of right ideals $\left\{\mathfrak{a}_U;\ U \in \{U\}\right\}$, where $\{U\}$ is a Boolean algebra. $\mathfrak{a}_U$ satisfies the following conditions:

(α) $$\mathfrak{a}_{U_1} \cap \mathfrak{a}_{U_2} = \mathfrak{a}_{U_1 \frown U_2} ;$$

(β) $$\mathfrak{a}_U = \mathfrak{a}_{U_1} + \mathfrak{a}_{U_2} + \cdots + \mathfrak{a}_{U_n} ,$$

when (U_1, $U_2, \ldots, U_n$) are independent and $U = U_1 \smile U_2 \smile \cdots \smile U_n$;

(γ) $$\mathfrak{a}_V = \mathfrak{R} ,$$

(1) J. v. Neumann [3], 30. The numbers in square brackets refer to the list given at the end of this paper.

(2) F. Maeda [1].

V being the unit element in the Boolean algebra.

Since (1) is merely the direct sum decomposition of the unit element, we can apply the method given above to the homogeneous basis; furthermore, the homogeneous basis requires pairwise perspectivity. In the continuous geometry L the dimension-function $D(a)$ $(a \in L)$, when it does not satisfy the chain-condition, takes all values in $[0, 1]$; and pairwise perspectivity means equidimensionality. Hence, for the homogeneous basis, it is natural to define a decomposition system $\{\mathfrak{a}_U;\ U \in \{U\}\}$, where $\{U\}$ is a system of all measurable sets in $[0, 1]$, and it satisfies, besides conditions similar to (α), (β), and (γ), the following equidimensional condition

$$D(a_U) = m(U),$$

$m(U)$ being the Lebesgue measure of U. If we decompose the inteval $[0, 1]$ in a sum of disjoint sets with the same measure

$$[0, 1] = U_1 + U_2 + \cdots\cdots + U_m,$$

then $(a_{U_i};\ i = 1, 2, \ldots\ldots, m)$ is a homogeneous basis of L. Hence we have different homogeneous bases, according as we decompose $[0, 1]$ in different ways.

In the present paper I show the method of constructing such a decomposition system $\{a_U;\ U \in \{U\}\}$. Following the properties of the Lebesgue measure, I define a_U for U, first for intervals, next for open sets, and lastly for measurable sets.

J. v. Neumann(1) has proved that when $\mathfrak{R}$ is an irreducible, regular, complete rank-ring, the system of all principal right ideals is a continuous geometry. Hence when we apply the result of the present paper to this case, we obtain a decomposition system of right ideals $\{\mathfrak{a}_U;\ U \in \{U\}\}$ and a decomposition system of idempotents $\{e_U;\ U \in \{U\}\}$ such that $\mathfrak{a}_U = (e_U)_r$ and $R(e_U) = m(U)$.(2) In a previous paper(3) I investigated only the relation between the decomposition system of right ideals and the decomposition system of idempotents. From the result of the present paper I can give the concrete construction of such systems, and obtain,

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(1) J. v. Neumann [3], 166.
(2) $R(e_U)$ means the rank of e_U.
(3) F. Maeda [1], 158–166.

moreover, the homogeneous decomposition systems.

1. Let L be a *continuous geometry*,(1) that is, L is a class of elements $a, b, \ldots.$ which satisfies the following axioms:

I. $a<b$ is a *partial ordering* of L, that is:

I_1. Never is $a<a$.

I_2. $a<b$, $b<c$ imply $a<c$.

II. L is a *continuous set*, that is:

II_1. For every subset $S \leqq L$, there is an element $\sum(S)$ in L, which is a *least upper bound* of S, i.e.,

(a) $\sum(S) \geqq a$ for every a in S,

(b) $x \geqq a$ for every a in S implies $x \geqq \sum(S)$.

II_2. For every subset $S \leqq L$, there is an element $\Pi(S)$ in L, which is a *greatest lower bound* of S, i.e.,

(a) $\Pi(S) \leqq a$ for every a in S,

(b) $x \leqq a$ for every a in S implies $x \leqq \Pi(S)$.(2)

III. Let Ω be an infinite aleph.

III_1. If $\alpha<\beta<\Omega$ implies $a_\alpha \geqq a_\beta$, then $\Pi(b \cup a_\alpha;\ \alpha<\Omega) = b \cup \Pi(a_\alpha;\ \alpha<\Omega)$.

III_2. If $\alpha<\beta<\Omega$ implies $a_\alpha \leqq a_\beta$, then $\sum(b \cap a_\alpha;\ \alpha<\Omega) = b \cap \sum(a_\alpha;\ \alpha<\Omega)$.

IV. L fulfils the *modular* axiom, that is:

$$a \leqq c \quad \text{implies} \quad (a \cup b) \cap c = a \cup (b \cap c).$$

V. L is *complemented*, that is:
For every $a \in L$ there exists an inverse x of a, which satisfies

$$a \cup x = 1, \qquad a \cap x = 0.\text{(3)}$$

VI. L is *irreducible*, that is:
If a has a unique inverse, then a is either 0 or 1.

When two elements a, b have a common inverse, then a is said to be *perspective* to b, and we write $a \sim b$. J. v. Neumann(4) has used this releation $\sim$ as equidimensionality, and has obtained a *dimension*

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(1) Cf. J. v. Neumann [1], 94-96; [2], 1-3.

(2) When S has only two elements a, b, we write $\sum(S) = a \cup b$, $\Pi(S) = a \cap b$.

(3) 0, 1 are respectively the zero and the unit elements in L, that is $0 = \Pi(L)$, $1 = \sum(L)$.

(4) J. v. Neumann [1], [2].

function $D(a)$ defined for all $a \in L$. $D(a)$ has the following properties:

(1°) $0 \leqq D(a) \leqq 1$, $D(0)=0$, $D(1)=1$.

(2°) $D(a \cup b)+D(a \cap b)=D(a)+D(b)$.

(3°) $D(a)=D(b)$ when, and only when, $a \sim b$.

(4°) $a>b$ implies $D(a)>D(b)$.

(5°) If $a_1 \leqq a_2 \leqq \ldots . \leqq a_n \leqq \ldots .$, then $D(\sum_n a_n)=\lim_{n\to\infty} D(a_n)$, and if $a_1 \geqq a_2 \geqq \ldots . \geqq a_n \geqq \ldots .$, then $D(\Pi_n a_n)=\lim_{n\to\infty} D(a_n)$.

And the range Δ of $D(a)$ is one of the following two cases, according as L satisfies the chain condition or not:

Case $N=1, 2, \ldots .$: The set Δ_N consists of all real numbers

$$\frac{n}{N}, \qquad n=0, 1, \ldots ., N.$$

Case ∞: The set Δ_∞ consists of all real numbers x, $0 \leqq x \leqq 1$.

2. J. v. Neumann has defined the *homogeneous basis* of L as follows:(1) It is a system $(a_i;\ i=1, 2, \ldots ., m)$ of elements of L such that

$$(a_i;\ i=1, 2, \ldots ., m) \perp,^{(2)} \qquad a_1 \cup a_2 \cup \ldots . \cup a_m = 1,$$

and

$$a_i \sim a_j \qquad (i, j=1, 2, \ldots ., m).$$

The number m is called the *order* of the basis.

When the range of $D(a)$ is Δ_N, we have a homogeneous basis of greatest order. That is

$$(b_\nu;\ \nu=1, 2, \ldots ., N)$$

and

$$D(b_\nu)=\frac{1}{N} \qquad (\nu=1, 2, \ldots ., N).$$

Denote the set of N integers $(1, 2, \ldots ., N)$ by V, the subset of V by U, and the system of all subsets of V by $\{U\}$. When the elements of U are $(n_1, n_2, \ldots ., n_p)$, put

$$a_U = b_{n_1} \cup b_{n_2} \cup \ldots . \cup b_{n_p}.$$

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(1) J. v. Neumann [3], 30.

(2) This means the independence of the system $(a_i;\ i=1, 2, \ldots ., m)$, that is $\sum(a_\sigma;\ \sigma \in J) \cap \sum(a_\sigma;\ \sigma \in K)=0$ for every pair of non-intersecting subsets J, K of the set of m integers $(1, 2, \ldots ., m)$.

Then we have a decomposition system $\{a_U;\ U\in\{U\}\}$ which satisfies the following conditions:

(α) $a_{U_1}\frown a_{U_2}=a_{U_1U_2}$;

(β) $a_{U_1}\smile a_{U_2}=a_{U_1\dot{+}U_2}$;

(γ) $a_V=1$.

If we decompose V in a sum of disjoint sets with the same number of elements

$$V=U_1+U_2+\cdots\cdot+U_m,$$

then $(a_{U_i};\ i=1,2,\ldots.,m)$ is a homogeneous basis of L. In this way, from the decomposition system $\{a_U;\ U\in\{U\}\}$, we can obtain homogeneous bases of all kinds of different orders.

But when the range of $D(a)$ is Δ_∞, we cannot have the homogeneous basis of minimal elements. Hence the decomposition system $\{a_U;\ U\in\{U\}\}$ cannot be obtained easily as in the case of Δ_N.

In what follows, I construct a decomposition system $\{a_U;\ U\in\{U\}\}$, where $\{U\}$ is the σ-field of all measurable sets in $[0,1]$, and

$$D(a_U)=m(U)\qquad\text{for all } U.$$

For this purpose we give a lemma:

LEMMA 2·1. *Let a be any element in L; then there exists an element b such that $b\leqq a$, and $D(b)=\frac{1}{2}D(a)$. If b' be an inverse of b in a, then $D(b')=\frac{1}{2}D(a)$.*

PROOF. Since the range Δ_∞ consists of all numbers x, $0\leqq x\leqq 1$, there exists an element c such that $D(c)=\frac{1}{2}D(a)$. By the property of perspectivity,(1) between a and c, the following two cases occur:

(i) there exists b such that $c\sim b\leqq a$, or

(ii) there exists d such that $a\sim d\leqq c$.

But case (ii) is absurd, since $D(d)=D(a)=2D(c)$. Hence the first part of this ~~the~~ thorem is proved.

Next let b' be an inverse of b in a, that is,

$$b\smile b'=a,\qquad b\frown b'=0.$$

(1) J. v. Neumann [2], 49, Theorem 5·15.

Then, by properties (1°) and (2°) of $D(a)$,

$$D(a)=D(b)+D(b').$$

Hence
$$D(b')=\frac{1}{2}D(a).$$

3. Let $a_{(0,\frac{1}{2}]}$[1] be an element of L such that $D(a_{(0,\frac{1}{2}]})=\frac{1}{2}$. And let $a_{(\frac{1}{2},1]}$ be an inverse of $a_{(0,\frac{1}{2}]}$. Then, since

$$D(a_{(0,\frac{1}{2}]})+D(a_{(\frac{1}{2},1]})=1,$$

we have
$$D(a_{(\frac{1}{2},1]})=\frac{1}{2}.$$

Applying Lemma 2·1 to $a_{(0,\frac{1}{2}]}$, we have two elements $a_{(0,\frac{1}{4}]}$, $a_{(\frac{1}{4},\frac{1}{2}]}$, which are mutually inverses in $a_{(0,\frac{1}{2}]}$, and $D(a_{(0,\frac{1}{4}]})=D(a_{(\frac{1}{4},\frac{1}{2}]})=\frac{1}{4}$. Similarly we have $a_{(\frac{1}{2},\frac{3}{4}]}$, $a_{(\frac{3}{4},1]}$, which are mutually inverse in $a_{(\frac{1}{2},1]}$, and $D(a_{(\frac{1}{2},\frac{3}{4}]})=D(a_{(\frac{3}{4},1]})=\frac{1}{4}$.

Thus we have a homogeneous basis of the 4th order

$$a_{(0,\frac{1}{4}]},\quad a_{(\frac{1}{4},\frac{1}{2}]},\quad a_{(\frac{1}{2},\frac{3}{4}]},\quad a_{(\frac{3}{4},1]}.$$[2]

Proceeding in this way we have homogeneous bases of the 2^pth order

$$a_{(0,\frac{1}{2^p}]},\quad a_{(\frac{1}{2^p},\frac{2}{2^p}]},\ldots,a_{(\frac{2^p-1}{2^p},1]}$$

for $p=1,2,\ldots.$

When $\lambda=\frac{m}{2^p}$, $\nu=\frac{n}{2^p}(m<n)$, we define

$$a_{(\lambda,\nu]}=a_{(\frac{m}{2^p},\frac{m+1}{2^p}]}\cup\ldots\cup a_{(\frac{n-1}{2^p},\frac{n}{2^p}]}.$$

Then, for any dyadic numbers $\lambda,\mu,\nu,\eta(\lambda\leqq\mu\leqq\nu\leqq\eta)$ in $(0,1]$, we can easily prove that

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(1) $(\alpha,\beta]$ means a semi-closed interval $\alpha<x\leqq\beta$.

(2) Independence follows from the properties of the dimension function.

$$a_{(\lambda,\nu]} \cup a_{(\mu,\eta]} = a_{(\lambda,\eta]}, \qquad a_{(\lambda,\nu]} \cap a_{(\mu,\eta]} = a_{(\mu,\nu]}.^{(3)}$$

Next, let $[\lambda, \nu]$ be any closed interval contained in $(0, 1]$. Then there exist two monotone sequences

$$\lambda_1 \leqq \lambda_2 \leqq \cdots\cdots \leqq \lambda_p \leqq \cdots\cdots \quad \text{and} \quad \nu_1 \geqq \nu_2 \geqq \cdots\cdots \geqq \nu_p \geqq \cdots\cdots$$

such that λ_p and ν_p are dyadic numbers of the form $\dfrac{i}{2^p}$ for all p,

and
$$\lim_{p\to\infty} \lambda_p = \lambda, \qquad \lim_{p\to\infty} \nu_p = \nu.$$

Since
$$a_{(\lambda_1,\nu_1]} \geqq a_{(\lambda_2,\nu_2]} \geqq \cdots\cdots \geqq a_{(\lambda_p,\nu_p]} \geqq \cdots\cdots,$$

if we put
$$a_{[\lambda,\nu]} = \Pi(a_{(\lambda_p,\nu_p]};\ p = 1, 2, \ldots\ldots),$$

then, by property (5°) of the dimension function $D(a)$,

$$D(a_{[\lambda,\mu]}) = \lim_{p\to\infty} D(a_{(\lambda_p,\nu_p]}) = \lim_{p\to\infty} (\nu_p - \lambda_p) = \nu - \lambda.$$

From this definition it is evident that $[\lambda, \nu] \subseteqq [\lambda', \nu']$ implies $a_{[\lambda,\nu]} \leqq a_{[\lambda',\nu']}$.

Next let (λ, ν) be any open interval contained in $(0, 1]$; there exist two monotone sequences

$$\lambda_1 \geqq \lambda_2 \geqq \cdots\cdots \geqq \lambda_i \geqq \cdots\cdots \quad \text{and} \quad \nu_1 \leqq \nu_2 \leqq \cdots\cdots \leqq \nu_i \leqq \cdots\cdots$$

where λ_i, ν_i are real numbers such that $\lambda_i < \nu_i$ and $\lim_{i\to\infty} \lambda_i = \lambda$, $\lim_{i\to\infty} \nu_i = \nu$.

Then
$$a_{[\lambda_1,\nu_1]} \leqq \cdots\cdots \leqq a_{[\lambda_i,\nu_i]} \leqq \cdots\cdots \leqq a_{[\lambda,\nu]}.$$

Since $(\lambda, \nu) = \operatorname*{Lim}_{i\to\infty} [\lambda_i, \nu_i]$, it is natural to define

$$a_{(\lambda,\nu)} = \Sigma(a_{[\lambda_i,\nu_i]};\ i = 1, 2, \ldots\ldots).$$

But
$$a_{(\lambda,\nu)} \leqq a_{[\lambda,\nu]},$$

and
$$D(a_{(\lambda,\nu)}) = \lim_{i\to\infty} D(a_{[\lambda_i,\nu_i]}) = \lim_{i\to\infty} (\nu_i - \lambda_i) = \nu - \lambda = D(a_{[\lambda,\nu]}).$$

Hence
$$a_{(\lambda,\nu)} = a_{[\lambda,\nu]}.$$

Therefore we define

$$a_{(\lambda,\nu)} = a_{(\lambda,\nu]} = a_{[\lambda,\nu)} = a_{[\lambda,\nu]}$$

for all
$$0 \leqq \lambda < \nu \leqq 1.$$

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(3) $a_{(\mu,\mu]}$ means 0.

Then, for any real numbers $\lambda, \mu, \nu, \eta (\lambda \leqq \mu \leqq \nu \leqq \eta)$ in $[0, 1]$, we can easily prove that

$$a_{(\lambda, \nu)} \cup a_{(\mu, \eta)} = a_{(\lambda, \eta)}, \qquad a_{(\lambda, \nu)} \cap a_{(\mu, \eta)} = a_{(\mu, \nu)}.$$

4. Next, let O be any open set in $[0, 1]$.(1) Then O is a sum of finite or denumerably infinite, non-overlapping open intervals

$$O = I_1 + I_2 + \cdots\cdot + I_i + \cdots\cdot.$$

Define
$$a_O = \sum(a_{I_i};\ i = 1, 2, \ldots\ldots).$$

Since $(a_{I_i};\ i = 1, 2, \ldots\ldots)\perp$, we have $D(a_O) = \sum_i D(a_{I_i}) = \sum_i m(I_i) = m(O)$.

Of course, $O_1 \geqq O_2$ implies $a_{O_1} \geqq a_{O_2}$.

Lemma 4·1. *When O_1, O_2 are any open sets in $[0, 1]$, then*

$$a_{O_1} \cup a_{O_2} = a_{O_1 \dotplus O_2}, \tag{1}$$

$$a_{O_1} \cap a_{O_2} = a_{O_1 O_2}. \tag{2}$$

Proof. O_1, O_2 are sums of finite or denumerably infinite, non-overlapping open intervals

$$O_1 = I_1^{(1)} + I_2^{(1)} + \cdots\cdot + I_i^{(1)} + \cdots\cdot,$$

$$O_2 = I_1^{(2)} + I_2^{(2)} + \cdots\cdot + I_i^{(2)} + \cdots\cdot.$$

Put $J_i^{(1)} = I_1^{(1)} + I_2^{(1)} + \cdots\cdot + I_i^{(1)}, \qquad J_j^{(2)} = I_1^{(2)} + I_2^{(2)} + \cdots\cdot + I_j^{(2)},$

then $\sum(a_{J_i^{(1)}};\ i = 1, 2, \ldots\ldots) = a_{O_1}, \qquad \sum(a_{J_j^{(2)}};\ j = 1, 2, \ldots\ldots) = a_{O_2}.$

And
$$\begin{aligned} a_{O_1} \cap a_{J_j^{(2)}} &= \sum(a_{J_i^{(1)}};\ i = 1, 2, \ldots\ldots) \cap a_{J_j^{(2)}} \\ &= \sum(a_{J_i^{(1)}} \cap a_{J_j^{(2)}};\ i = 1, 2, \ldots\ldots)^{(2)} \\ &= \sum(a_{J_i^{(1)} J_j^{(2)}};\ i = 1, 2, \ldots\ldots) = a_{O_1 J_j^{(2)}}.^{(3)} \end{aligned}$$

$$a_{O_1} \cap a_{O_2} = a_{O_1} \cap \sum(a_{J_j^{(2)}};\ j = 1, 2, \ldots\ldots) = \sum(a_{O_1} \cap a_{J_j^{(2)}};\ j = 1, 2, \ldots\ldots)$$

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(1) In what follows, "open" means "open relative to $[0, 1]$".

(2) By Axiom III_2.

(3) Since $a_{J_i^{(1)} J_j^{(2)}} \leqq a_{O_1 J_j^{(2)}}$ and $\lim_{i \to \infty} D(a_{J_i^{(1)} J_j^{(2)}}) = D(a_{O_1 J_j^{(2)}})$.

$$=\sum(a_{O_1J_j^{(2)}}\,;\, j=1,2,\ldots\ldots)=a_{O_1O_2}\,.$$

Hence we have (2).

Next we shall prove (1). It is evident that

$$a_{O_1}\cup a_{O_2}\leqq a_{O_1\dotplus O_2}\,.$$

But
$$D(a_{O_1}\cup a_{O_2})=D(a_{O_1})+D(a_{O_2})-D(a_{O_1}\cap a_{O_2})$$
$$=m(O_1)+m(O_2)-m(O_1O_2) \qquad \text{by (2)}$$
$$=m(O_1\dotplus O_2)=D(a_{O_1\dotplus O_2})\,.$$

Hence we have
$$a_{O_1}\cup a_{O_2}=a_{O_1\dotplus O_2}\,.$$

5. Next, let U be any measurable set in $[0,1]$. Then there exists a sequence of open sets

$$O_1\geqq\cdots\cdots\geqq O_n\geqq\cdots\cdots\geqq U \tag{1}$$

such that
$$\lim_{n\to\infty} m(O_n)=m(U)\,.$$

Then
$$a_{O_1}\geqq\cdots\cdots\geqq a_{O_n}\geqq\cdots\cdots.$$

Hence, if we define a_U by

$$a_U=\Pi(a_{O_n}\,;\ n=1,2,\ldots\ldots)\,,$$

then
$$D(a_U)=\lim_{n\to\infty} D(a_{O_n})=\lim_{n\to\infty} m(O_n)=m(U)\,.$$

a_U is uniquely determined independently of sequence (1). For let

$$O_1'\geqq\cdots\cdots\geqq O_n'\geqq\cdots\cdots\geqq U$$

be another sequence such that

$$\lim_{n\to\infty} m(O_n')=m(U)\,,$$

and define $a_U'=\Pi(a_{O_n'}\,;\ n=1,2,\cdots\cdots)$; then $D(a_U')=m(U)$.

Now,
$$O_1O_1'\geqq\cdots\cdots\geqq O_nO_n'\geqq\cdots\cdots\geqq U\,,$$

and
$$\lim_{n\to\infty} m(O_nO_n')=m(U)\,.$$

Put $a''_U = \Pi(a_{O_n O'_n};\ n=1, 2, \ldots.)$; then $D(a''_U) = m(U)$.

Since $$a_{O_n} \geqq a_{O_n O'_n} \geqq a''_U,$$

we have $$a_U = \Pi(a_{O_n};\ n=1, 2, \ldots.) \geqq a''_U.$$

But $$D(a_U) = D(a''_U) = m(U),$$

so we have $$a_U = a''_U.$$

Similarly $$a'_U = a''_U.$$

Consequently $$a_U = a'_U.$$

Let $\{U\}$ be the σ-field of measurable sets in $[0, 1]$. Then we have a system $\left\{a_U;\ U \in \{U\}\right\}$. To investigate the properties of this system, we first prove the following lemmas.

LEMMA 5·1. *When* $U_1 \geqq \cdots \geqq U_i \geqq \cdots$ $\left(\textit{or } U_1 \leqq \cdots \leqq U_i \leqq \cdots\right)$ *is a sequence of measurable sets in* $[0, 1]$, *then*

$$\Pi(a_{U_i};\ i=1, 2, \cdots) = a_U \quad \left(\textit{or } \sum(a_{U_i};\ i=1, 2, \ldots.) = a_U\right)$$

where $U = \operatorname*{Lim}_{i \to \infty} U_i$.

PROOF. Since $$\Pi(a_{U_i};\ i=1, 2, \ldots.) \geqq a_U,$$

and $$m\left(\Pi(a_{U_i};\ i=1, 2, \ldots.)\right) = \lim_{i \to \infty} D(a_{U_i}) = \lim_{i \to \infty} m(U_i) = m(U) = D(a_U),$$

we have $$\Pi(a_{U_i};\ i=1, 2, \ldots.) = a_U.$$

LEMMA 5·2. *When* U_1, U_2 *are any measurable sets in* $[0, 1]$, *then*

$$a_{U_1} \cup a_{U_1} = a_{U_1 \dot{+} U_2}, \qquad a_{U_1} \cap a_{U_2} = a_{U_1 U_2}.$$

PROOF. There exist two sequences of open sets

$$O_1^{(1)} \geqq \cdots \geqq O_i^{(1)} \geqq \cdots \geqq U_1,$$

$$O_1^{(2)} \geqq \cdots \geqq O_i^{(2)} \geqq \cdots \geqq U_2,$$

such that $\lim_{i \to \infty} m(O_i^{(1)}) = m(U_1)$, $\lim_{i \to \infty} m(O_i^{(2)}) = m(U_2)$,

and $a_{U_1} = \Pi(a_{O_i^{(1)}};\ i=1, 2, \ldots.)$, $a_{U_2} = \Pi(a_{O_i^{(2)}};\ i=1, 2, \ldots.)$.

By Axiom III_1, we have

$$\Pi\left(\Pi(a_{O_i^{(1)}} \cup a_{O_j^{(2)}}\,;\ i=1,2,\ldots\ldots)\,;\ j=1,2,\ldots\ldots\right)$$
$$=\Pi(a_{U_1} \cup a_{O_j^{(2)}}\,;\ j=1,2,\ldots\ldots)=a_{U_1} \cup a_{U_2}.$$

On the other hand,

$$\Pi\left(\Pi(a_{O_i^{(1)}} \cup a_{O_j^{(2)}}\,;\ i=1,2,\ldots\ldots)\,;\ j=1,2,\ldots\ldots\right)$$
$$=\Pi\left(\Pi(a_{O_i^{(1)}\dot{+}O_j^{(2)}}\,;\ i=1,2,\ldots\ldots)\,;\ j=1,2,\ldots\ldots\right)$$
$$=\Pi(a_{U_1\dot{+}O_j^{(2)}}\,;\ j=1,2,\ldots\ldots)=a_{U_1\dot{+}U_2}, \qquad \text{as Lemma 5·1.}$$

Hence we have $$a_{U_1} \cup a_{U_2}=a_{U_1\dot{+}U_2}.$$

Similarly we can prove that

$$a_{U_1} \cap a_{U_2}=a_{U_1U_2}.$$

THEOREM. *The system* $\left\{a_U;\ U \in \{U\}\right\}$ *has the follwing properties:*

(i) $D(a_U)=m(U)$ *for all* U.

(ii) $(a_{U_i};\ i=1,2,\ldots\ldots)$ *is independent, when* $(U_i;\ i=1,2,\ldots\ldots)$ *is a system of mutually disjoint sets.*

(iii) $a_U=a_{U_1} \cup a_{U_2} \cup \cdots\cdots \cup a_{U_i} \cup \cdots\cdots$ *when* $U=U_1\dot{+}U_2\dot{+}\cdots\cdots\dot{+}U_i\dot{+}\cdots\cdots$.

(iv) $a_U=a_{U_1} \cap a_{U_2} \cap \cdots\cdots \cap a_{U_i} \cap \cdots\cdots$ *when* $U=U_1U_2\cdots\cdots U_i\cdots\cdots$.

PROOF. (i) is evident by the definition of a_U. (ii) follows from (iii) and Lemma 5·2. To prove (iii), put

$$U^{(i)}=U_1\dot{+}U_2\dot{+}\cdots\cdots\dot{+}U_i;$$

then, by Lemma 5·2, $a_{U^{(i)}}=a_{U_1} \cup a_{U_2} \cup \cdots\cdots \cup a_{U_i}$.
Hence, by Lemma 5·1, we have (iii). Similarly we can prove (iv).

References.

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HIROSIMA UNIVERSITY.