A Generalization of Rumer's Form of Maxwell's Equation in Riemannian Space.

By

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1. Maxwell's electromagnetic equation in vacuum is given by

$$(1.1)\quad \begin{cases} \operatorname{div} \mathfrak{H}=0\,, \qquad \operatorname{div} \mathfrak{E}=0 \\ \operatorname{rot} \mathfrak{E}+\dfrac{1}{c}\dfrac{\partial \mathfrak{H}}{\partial t}=0\,, \qquad \operatorname{rot} \mathfrak{H}-\dfrac{1}{c}\dfrac{\partial \mathfrak{E}}{\partial t}=0\,. \end{cases}$$

In the theory of Relativity, Maxwell's equation in the form (1.1) is generalized in the form of the following tensor equations(1):

$$(1.2)\quad \begin{cases} F_{ij}=\dfrac{\partial \varphi_i}{\partial x^j}-\dfrac{\partial \varphi_j}{\partial x^i} \\ \nabla_j F^{ij}=0 \qquad (i,j=1,2,3,4)\,, \end{cases}$$

where ∇_j denotes the Riemannian covariant derivative, and $\varphi^i=g^{ij}\varphi_j$ is the contravariant vector of a potential whose first three components and fourth component coincide with the vector and the scalar potentials, respectively, in the Galilean coordinate system.

On the other hand, Maxwell's equation (1.1) can be written in a complex form as follows

$$(1.3)\quad \begin{cases} \operatorname{div}(\mathfrak{H}+i\mathfrak{E})=0 \\ \operatorname{rot}(\mathfrak{H}+i\mathfrak{E})-\dfrac{\partial}{ic\partial t}(\mathfrak{H}+i\mathfrak{E})=0 \end{cases}$$

and G. Rumer(2) has rewritten this equation in the following matrix form:

$$\begin{matrix}(1.4)\\(1.4\text{a})\end{matrix}\quad \begin{cases} \mathring{D}\mathfrak{F}=0 \\ F_4=0 \end{cases}$$

$$(1.5)\qquad \mathring{D}=\mathring{\sigma}^i\frac{\partial}{\partial x^i}\,, \qquad [x^1\equiv x,\ x^2\equiv y,\ x^3\equiv z,\ x^4\equiv -ict]$$

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(1) A. S. Eddington, The Mathematical Theory of Relativity, Cambridge. 2nd ed. (1930), 173.

(2) G. Rumer, Zs. f. Physik, **65** (1930), 244.

where $\overset{\circ}{\sigma}{}^i$ are constant matrices[1] and

$$
(1.6)\qquad \mathfrak{F}=\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} \quad \text{where} \quad \begin{cases} F_1=H_1+iE_1 \\ F_2=H_2+iE_2 \\ F_3=H_3+iE_3 \\ F_4=\dfrac{\partial}{\partial x^i}\varphi_i \end{cases}
$$

The purpose of this paper is to obtain Maxwell's equation in Riemannian space by generalizing the matrix form (1.4).

2. In a Riemannian space, if we introduce the constant 4–4 Dirac matrices $\overset{\circ}{\gamma}_i$, the fundamental tensor g_{ij} can be expressed as follows:

$$
(2.1)\qquad \begin{cases} g_{ij}I=\gamma_{(i}\gamma_{j)}\,, \qquad g^{ij}I=\gamma^{(i}\gamma^{j)} \\ \gamma_i=\overset{j}{h}_i\overset{\circ}{\gamma}_j\,, \qquad \gamma^i=\underset{j}{h}{}^i\overset{\circ}{\gamma}{}^j\,, \qquad (\overset{\circ}{\gamma}{}^i\equiv\overset{\circ}{\gamma}_i) \end{cases}
$$

where $\overset{j}{h}_i$ and $\underset{j}{h}{}^i$ are covariant and contravariant vectors respectively such that $\overset{j}{h}_i\underset{k}{h}{}^i=\delta^j_k=\underset{i}{h}{}^j\overset{i}{h}_k$ and $g_{ij}=\sum_a \overset{a}{h}_i\overset{a}{h}_j$. In a Galilean system $g_{ij}=\delta_{ij}$ we can take $\overset{i}{h}_j$ and $\underset{i}{h}{}^j$ such that

$$
(2.2)\qquad \overset{j}{h}_i=\underset{j}{h}{}^i=\delta^i_j\,.
$$

Now we shall restrict ourselves to such g_{ij} as becomes δ_{ij} when all the components of the curvature tensor made of g_{ij} are set to vanish by adjusting certain terms (e. g. parameters) in g_{ij}, leaving the coordinates as they are. For such g_{ij} we can always take $\overset{j}{h}_i$ so that it comes to δ^j_i when g_{ij} becomes δ_{ij} by adjustment of terms as above mentioned.

For such $\overset{j}{h}_i$, if we put[2]

$$
(2.3)\qquad \underset{j}{h}{}^i\overset{\circ}{\sigma}{}^jH^{-1}\equiv\sigma^i_{\text{cov}}\,, \qquad H^{-1}\equiv\begin{pmatrix} \overset{1}{h}_1 & \cdot & \cdot & \overset{4}{h}_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \overset{1}{h}_4 & \cdot & \cdot & \overset{4}{h}_4 \end{pmatrix}^{-1}=\begin{pmatrix} \underset{1}{h}{}^1 & \cdot & \cdot & \underset{1}{h}{}^4 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \underset{4}{h}{}^1 & \cdot & \cdot & \underset{4}{h}{}^4 \end{pmatrix}
$$

and define a differential operator D_{cov} by

$$
(2.4)\qquad D_{\text{cov}}\equiv\sigma^i_{\text{cov}}\nabla_i
$$

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(1) $\overset{\circ}{\sigma}{}^1=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$, $\overset{\circ}{\sigma}{}^2=\begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}$, $\overset{\circ}{\sigma}{}^3=\begin{pmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{pmatrix}$, $\overset{\circ}{\sigma}{}^4=\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$.

(2) $H\widetilde{H}=(g_{ij})\,, \qquad \widetilde{H}^{-1}H^{-1}=(g^{ij})=(g_{ij})^{-1}$

where $\widetilde{H}$ denotes the transposed matrix of H.

which operates on a 1-4 matrix (F_i) where F_i is a covariant vector, then D_{cov} is an invariant operator[1] and becomes $\mathring{D}$ when g_{ij} is set to δ_{ij}. Hence we know that the equation

$$D_{\mathrm{cov}}(F_p)=0 \tag{2.5}$$

is invariant under coordinate transformations and also that (2.5) becomes coincident with (1.4) in the Galilean system. Hence we may take (2.5) as the formal generalization of Maxwell's equation in the g_{ij} field.

It is to be noted that since the potentials φ_i are considered as the components of a vector in the generalization of the type of (1.2), F_i in (1.6) cannot be considered as a vector. On the contrary, F_i in (1.6) is taken to be a vector in the generalization (2.5) and therefore φ_i, which is obtained from the relation (1.6) for given F_i, cannot be a vector.

Before we proceed to rewrite (2.5) in ordinary tensor equations we shall consider some other expressions equivalent to (2.5). In general, for a vector v^i we can consider three kinds of components, i. e. contravariant, covariant, and invariant components, expressed respectively by

$$v^i\,, \qquad v_i=g_{ij}v^j\,, \qquad v_{(i)}=\underset{i}{h^j}v_j=\overset{i}{h_j}v^j\,. \tag{2.6}$$

D_{cov} is the generalization of $\mathring{D}$ in which the operator $\mathring{D}$ is considered as operating on the covariant components of F_i. By putting

$$D_{\mathrm{con}}\equiv\sigma^i_{\mathrm{con}}\nabla_i\equiv\underset{j}{h^i}\mathring{\sigma}^j\dot{\widetilde{H}}\nabla_i \tag{2.7}$$

and

$$D_{\mathrm{inv}}\equiv\underset{j}{h^i}\mathring{\sigma}^j\{\nabla_i-(\nabla_i H^{-1})H\}\equiv\underset{j}{h^i}\mathring{\sigma}^j\{\nabla_i-(\nabla_i\widetilde{H})\widetilde{H}^{-1}\}\,, \tag{2.8}$$

from the relations

$$D_{\mathrm{cov}}(F_p)=\underset{j}{h^i}\mathring{\sigma}^j(\underset{p}{h^l}\nabla_i F_l)=\underset{j}{h^i}\mathring{\sigma}^j(\overset{p}{h_l}\nabla_i F^l)$$

and

$$\nabla_i F_{(m)}=\nabla_i(\underset{m}{h^l}F_l)=\underset{m}{h^l}\nabla_i F_l+F_l\nabla_i\underset{m}{h^l}\,,$$

we have

$$D_{\mathrm{cov}}(F_p)\equiv D_{\mathrm{inv}}(F_{(p)})\equiv D_{\mathrm{con}}(F^p)\text{[2]} \tag{2.9}$$

respectively. Therefore, the generalized Maxwell's equation (2.5) can be

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(1) $'\sigma^i=\dfrac{\partial' x^i}{\partial x^l}\underset{j}{h^l}\mathring{\sigma}^j H^{-1}\left(\widetilde{\dfrac{\partial' x^s}{\partial x^t}}\right)=\dfrac{\partial' x^i}{\partial x^l}\sigma^l\left(\widetilde{\dfrac{\partial' x^s}{\partial x^t}}\right)$, $\;'\nabla_i('F_p)=\dfrac{\partial x^m}{\partial' x^i}\left(\widetilde{\dfrac{\partial x^s}{\partial' x^t}}\right)\nabla_m(F_p)$.

Therefore we have $'\sigma^{i\prime}\nabla_i('F_p)=\sigma^l\nabla_l(F_p)$.

(2) D's can also be written in the following manner

$$D_{\mathrm{cov}}=\mathring{\sigma}^i H^{-1}\nabla_{(i)}\,, \qquad D_{\mathrm{inv}}=\mathring{\sigma}^i\{\nabla_{(i)}-(\nabla_{(i)}H^{-1})H\}\,, \qquad D_{\mathrm{con}}=\mathring{\sigma}^i\widetilde{H}\nabla_{(i)}$$

where $\nabla_{(i)}$ denotes the invariant differential operator i. e. $\nabla_{(i)}=\overset{i}{h_j}\nabla^j=\underset{i}{h^j}\nabla_j$.

written in three ways corresponding to (2.9);

$$(2.10) \qquad D_{\text{cov}}(F_p)=0\,, \qquad D_{\text{inv}}(F_{(p)})=0\,, \qquad D_{\text{con}}(F^p)=0\,.$$

Hence, if we provide that $F_a=H_a+iE_a$, $(a=1, 2, 3)$ and $F_4=\frac{\partial}{\partial x^i}\varphi_i$, we can consider (2.10) as the formal generalization of Rumer's form of Maxwell's equation in Riemannian space.

3. Next we shall express the equation (2.5) in the ordinary tensor form. Substituting the actual values of $\overset{\circ}{\sigma}{}^i$ into (2.5), we have

$$\begin{pmatrix} \underset{1}{h^i} & \underset{2}{h^i} & \underset{3}{h^i} & \underset{4}{h^i} \\ -\underset{2}{h^i} & \underset{1}{h^i} & \underset{4}{h^i} & -\underset{3}{h^i} \\ \underset{3}{h^i} & \underset{4}{h^i} & -\underset{1}{h^i} & -\underset{2}{h^i} \\ \underset{4}{h^i} & -\underset{3}{h^i} & \underset{2}{h^i} & -\underset{1}{h^i} \end{pmatrix} \begin{pmatrix} \underset{1}{h^1} & \underset{1}{h^2} & \underset{1}{h^3} & \underset{1}{h^4} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \underset{4}{h^1} & \underset{4}{h^2} & \underset{4}{h^3} & \underset{4}{h^4} \end{pmatrix} \begin{pmatrix} \nabla_i F_1 \\ \nabla_i F_2 \\ \nabla_i F_3 \\ \nabla_i F_4 \end{pmatrix} = 0$$

Calculating this, we have

$$(3.1) \qquad g^{ij}\nabla_j F_i=0\,,$$

$$(3.2) \qquad \underset{1}{h^i}\underset{2}{h^j}\nabla_{[i}F_{j]}=\underset{3}{h^i}\underset{4}{h^j}\nabla_{[i}F_{j]}\,, \qquad -\underset{1}{h^i}\underset{3}{h^j}\nabla_{[i}F_{j]}=\underset{2}{h^i}\underset{4}{h^j}\nabla_{[i}F_{j]}\,,$$

$$\underset{1}{h^i}\underset{4}{h^j}\nabla_{[i}F_{j]}=\underset{2}{h^i}\underset{3}{h^j}\nabla_{[i}F_{j]}\,.$$

From (3.2), we have

$$(3.3) \qquad \frac{\sqrt{g}}{2}\epsilon_{ijlm}F^{lm}=F_{ij} \qquad (\sqrt{g}=\text{det. of } H)$$

where
$$F_{ij}=\nabla_{[i}F_{j]}=\partial_{[i}F_{j]}\,.$$

Thus (3.1) and (3.3) together are the tensor equation for the generalized Maxwell's equation in Riemannian space. Here it is interesting to observe that equation (3.3) thus obtained is quite similar to Morinaga's equation obtained in Wave Geometry as the condition for integrability of the fundamental equation for ψ.[1]

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(1) K. Morinaga, this Journal, **5** (1935), 170.