On Space which has the Homogeneous Property for Observation Systems.

By

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§ 1. Introduction and summary.

In Wave Geometry, considering that the momentum-density vector of a moving particle whose existence is defined by ψ is given by $u^l \equiv \psi^\dagger A \gamma^l \psi$, and assuming that the vector u^l generates a congruence of geodesics, we have established the wave-geometrical cosmology.(1) We possess, however, no clear reason why it is only in the case of cosmology that u^l generates a congruence of geodesics.(2) Over this obscurity we feel some dissatisfaction, so we are tempted to study whether there is a way to rectify this unsatisfactory point of the theory.

Since we have seen, in the previous paper,(3) that our cosmology is characterized by *a homogeneous property for observation systems,* the question arises: Is it not possible to establish a cosmology proceeding from the homogeneity of space-time for observation system without assuming that each constituent particle in the universe describes a geodesic?

To answer this question, in this paper, we shall first make clear the conception expressed by the term "*homogeneity of space-time.*" After doing this we shall find all the space-time continuum having the "*homogeneous*" property for observation systems, and by studying the relations between those observation systems, we shall make some contributions to the wave-geometrical cosmology; and in the next paper we shall remove the obscurity indicated above.

The results obtained in this paper are: *There exist three types of homogeneous and statical space, i.e., (I) the Minkowski, (II) the Einstein, and (III) the de-Sitter type. In (I), each system of coordinates is in a uniform motion relative to the other; in (II), the systems do not change relative positions; in (III), each system is in motion, with the velocity* v

(1) T. Iwatsuki, Y. Mimura and T. Sibata: this Journal, **8** (1938), 187 (W. G. No. 27) and the following papers.

(2) In fact,—e. g., in the theory of spiral nebulae—u^l does not generate a congruence of geodesics; cf. T. Iwatsuki and T. Sibata: Theory of Spiral Nebulae, this Journal, **11** (1941), 47 (W. G. No. 44).

(3) T. Sibata: this journal, **11** (1941), 21 (W. G. No. 43).

given by (5.1), *in the direction joining the origins of the respective coordinate-systems.*

These three types of space-time continuums coincide with the statical space-time obtained in relativistic cosmology. In relativistic cosmology the three types of universe—(I) Minkowski, (II) Einstein, and (III) de-Sitter space—are obtained from the field equation

$$K_{jk}-\frac{1}{2}Kg_{jk}+\Lambda g_{jk}=-8\pi T_{jk},$$

together with the three additional assumptions:

(i) the interval ds^2 is spherically symmetric and static, i. e.,

$$ds^2=-e^{\lambda}dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+e^{\nu}dt^2,$$

λ and ν being any functions of r only.

(ii) the proper density of matter is everywhere the same, i. e.,

$$\rho_{00}=\text{constant}.$$

(iii) the proper hydrostatic pressure is constant.

These three additional assumptions are usually regarded as defining homogeneous static space.[(1)] But in this paper, as will be seen, it is on the basis of observation systems, without using the notions of field equation, density, and pressure, that we shall define homogeneous static space (§ 2).

§ 2. Definition for homogeneity of space-time.

In this section we shall consider what is meant when we say vaguely that "space-time is homogeneous for observation systems." On expanding the meaning of this expression from various sides, it seems natural to conclude that at least the following two conditions must be satisfied.

(I) Space-time is spherically symmetric. Mathematically, space-time admits the rotation group in three-dimensional space (x, y, z); in other words, for the coordinate-systems connected by any rotation in three-dimensional space, the interval of the space-time $ds^2=g_{ij}dx^idx^j$ must have the same form.

(II) Condition (I) expresses the homogeneity of space-time for coordinate-systems having a common origin. However, since there can be coordinate-systems which do not have their origin in common, for such coordinate-systems the form of $ds^2=g_{ij}dx^idx^j$ must also be the same.

From what has been said above, with respect to the interval ds^2, we see that by the statement that "space-time is homogeneous for coordinate-systems" it is meant mathematically that the interval ds^2 has the same form for any two systems belonging to a set of coordinate-systems S. Precisely speaking, if x^i and x'^i are the respective coordinates of any world

(1) Cf., e. g., Tolmann: Relativity, Thermodynamics, and Cosmology.

point with respect to K and K' belonging to S, g_{ij} and $g'_{ij}=\frac{\partial x^l}{\partial x'^i}\frac{\partial x'^m}{\partial x'^j}g_{lm}$ are the same functions of x and x':

$$g'_{ij}=g_{ij}(x') . \tag{2.1}$$

Now, if we regard all the relations of x and x' which satisfy (2.1) as transformations from x to x', the relations are to be considered as expressing the transformations of coordinates from K to K'. Then from the "homogeneity of space-time" it follows that in the set of these transformations of coordinates, the rotation group in three-dimensional space and certain transformations which bring the origin to any other point must be contained.

From these considerations, for the "homogeneity of space-time" we can put the following **definition**: *We call a space-time homogeneous when it admits the rotation group in three-dimensional space* (x, y, z) *and other transformations which transform the origin* $r=0$ *to any point.*

In the following sections we shall investigate all the forms of intervals of homogeneous space thus defined, and the relations between any two systems of coordinates.

§ 3. Spherically symmetric space.

When space-time is spherically symmetric, i. e. g_{ij} is invariant for the rotation group in r, θ, φ-space, g_{ij} must have the forms[(1)]:

$$g_{11}(r,t)\,,\quad g_{22}(r,t)\,,\quad g_{44}(r,t)\,,\quad (x^1, x^2, x^3, x^4=r, \theta, \varphi, t)$$
$$g_{14}(r,t)\,,\quad g_{33}=\sin^2\theta g_{22}\,,\quad g_{41}(r,t)\,,\quad \text{the other } g_{ij}=0\,.$$

Now, if we choose the t-axis such that $g_{4a}=0$ $(a=1, 2, 3)$, and assume that the space is static, we have

$$g_{11}(r)\,.\qquad g_{22}(r)\,,\qquad g_{44}(r)\,,$$
$$g_{33}=\sin^2\theta g_{22}\,,\qquad g_{ij}=0 \ \text{for}\ i\neq j\,.$$

When $g_{22}(r)$ involves r, taking $g_{22}(r)$ as $-R^2$ and writing R newly as r, we have

$$g_{11}(r)\,,\quad g_{22}(r)=-r^2\,,\quad g_{33}=-r^2\sin^2\theta\,,\quad g_{44}(r)\,,$$

or

$$ds^2=-e^{\lambda}dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+e^{\nu}dt^2\,, \tag{3.1}$$

λ and ν being any functions of r.

When $g_{22}=\text{constant}=C$, we have

$$ds^2=g_{11}(r)dr^2+C(d\theta^2+\sin^2\theta d\varphi^2)+g_{44}(r)dt^2 \tag{3.2}$$

(1) The form of g_{ij} is obtained from the condition that g_{ij} is invariant for the operators: $U_1\equiv-\cos\varphi\frac{\partial}{\partial\theta}+\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}$, $U_2\equiv\sin\varphi\frac{\partial}{\partial\theta}+\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}$, $U_3\equiv\frac{\partial}{\partial\varphi}$. Cf. this Journal, **11** (1941), 37.

Therefore, for static and spherically symmetric space-time we have two types of intervals as seen in (3.1) and (3.2). But (3.2)[1] does not become Minkowski space whatever g_{11}, g_{44}, and C may be. So that, for static and spherically symmetric space-time, we adopt (3.1) as the usual expression.

§ 4. Homogeneous space-time and the relations of coordinate-systems.

We shall investigate the transformations which transform the origin ($r=0$), making the form of (3.1) invariant. Now, let us confine ourselves to the case when the transformations are generated from the infinitesimal transformations:

$$x'^i = x^i + \xi^i(x)\delta\tau \qquad (i=1,\ldots\ldots,4) \tag{4.1}$$

$$(\xi^1 \neq 0) \qquad (x^1, x^2, x^3, x^4 = r, \theta, \varphi, t)\,.$$

The condition that the form of (3.1) shall be invariant for (4.1), i. e. $g'_{ij} = g_{ij}(x')$, is given by[2]

$$\xi^l \frac{\partial g_{ij}}{\partial x^l} + g_{il}\frac{\partial \xi^l}{\partial x^j} + g_{lj}\frac{\partial \xi^l}{\partial x^i} = 0\,, \tag{4.2}$$

or

$$\nabla_{(i}\xi_{j)} = 0 \qquad \text{(Killing's equation).}$$

Solving (4.2) for g_{ij} of the form (3.1), we have the following three types of intervals[3]

(I) $ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + dt^2$ (Minkowski space),

(II) $ds^2 = -\dfrac{dr^2}{1-r^2/R^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + dt^2$ (Einstein type space),

(III) $ds^2 = -\dfrac{dr^2}{1-k^2r^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + (1-k^2r^2)dt^2$

(de-Sitter type space),

the corresponding transformations which transform the origin ($r=0$) and make g_{ij} invariant being obtained as follows[4]:

$(\mathrm{I})_T$: Lorentz transformations.

$$(\mathrm{II})_T \quad \begin{cases} x' = x\,, \quad y' = y\,, \quad t' = t \\ z' = \sqrt{R^2 - r^2}\sin\dfrac{\tau}{R} + z\cos\dfrac{\tau}{R} \end{cases} \qquad (\tau \text{ is a parameter})$$

and the transformations obtained by cyclic interchange of x, y, z.

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(1) This case will be treated in Note IV.

(2) Cf. this Journal, **11** (1941), 27.

(3) Note **I.**

(4) Note **II.**

$$(\mathrm{III})_T \begin{cases} x'=x\,, \quad y'=y\,, \quad (\tau,\tau' \text{ are parameters}) \\ z'=z+\sqrt{1-k^2r^2}\,[e^{kt}(1-k^2\tau\tau')\tau+e^{-kt}\tau']-2k^2\tau\tau'z \\ e^{kt'}=e^{kt}\left[\dfrac{(1-k^2\tau\tau')^2-k^2e^{-2kt}\tau'^2-2k^2ze^{-kt}(1-k^2\tau\tau')\tau'/\sqrt{1-k^2r^2}}{1-k^2e^{2kt}\tau^2-2k^2ze^{kt}\tau/\sqrt{1-k^2r^2}}\right]^{\frac{1}{2}} \end{cases}$$

and the transformations obtained by cyclic interchange of x, y, z.

§ 5. Velocity of motion of a coordinate-system relative to another system.

For Lorentz transformation, we know that a coordinate-system is in motion with uniform velocity to another system. For the transformations $(\mathrm{II})_T$ in § 4, we see that between any two systems of coordinates there occurs no alternation of their origins.(1) And for the transformations of $(\mathrm{III})_T$, we see that each system of coordinates is in motion relative to the other with the velocity v given by(2)

$$v=\frac{-\tau'(1-k^2\tau\tau')e^{-kt}+\tau e^{kt}}{\tau(1-k^2\tau\tau')e^{-kt}+\tau e^{kt}}kr(1-k^2r^2)\,. \tag{5.1}$$

So we conclude that: *In homogeneous static space-time, each observation system is in motion with a uniform velocity or with the velocity given by* (5.1)*; otherwise they do not alter their relative positions. In the former case the space-time is Minkowski or de-Sitter type, and in the latter case the space-time is Einstein type.*

Note I.

When g_{ij} is the form of (3.1) or (3.2), equation (4.2) is written as

$$\left.\begin{array}{lll}
\xi^1\dfrac{dg_{11}}{dr}+2g_{11}\dfrac{\partial\xi^1}{\partial r}=0\,, & (\text{for } i,j=1,1) & (1.1) \\
\xi^1\dfrac{dg_{22}}{dr}+2g_{22}\dfrac{\partial\xi^2}{\partial\theta}=0\,, & (\quad ,, \quad 2,2) & (2.2) \\
g_{11}\dfrac{\partial\xi^1}{\partial\theta}+g_{22}\dfrac{\partial\xi^2}{\partial r}=0\,, & (\quad ,, \quad 1,2) & (1.2) \\
g_{11}\dfrac{\partial\xi^1}{\partial\varphi}+g_{33}\dfrac{\partial\xi^3}{\partial r}=0\,, & (\quad ,, \quad 1,3) & (1.3) \\
\xi^1\dfrac{\partial g_{33}}{\partial r}+2\xi^2\sin\theta\cos\theta g_{22}+2g_{33}\dfrac{\partial\xi^3}{\partial\varphi}=0\,, & (\quad ,, \quad 3,3) & (3.3) \\
g_{22}\dfrac{\partial\xi^2}{\partial\varphi}+g_{33}\dfrac{\partial\xi^3}{\partial\theta}=0\,, & (\quad ,, \quad 2,3) & (2.3)
\end{array}\right\} \quad (\mathrm{N.\,1})$$

(1) Note **III.**

(2) T. Sibata: this Journal, **11** (1941), 25.

$$g_{11}\frac{\partial\xi^1}{\partial t}+g_{44}\frac{\partial\xi^4}{\partial r}=0\,, \qquad (\;\;,, \quad 1,4) \quad (1.4)$$

$$g_{22}\frac{\partial\xi^2}{\partial t}+g_{44}\frac{\partial\xi^4}{\partial \theta}=0\,, \qquad (\;\;,, \quad 2,4) \quad (2.4)$$

$$g_{33}\frac{\partial\xi^3}{\partial t}+g_{44}\frac{\partial\xi^4}{\partial \varphi}=0\,, \qquad (\;\;,, \quad 3,4) \quad (3.4)$$

$$\xi^1\frac{dg_{44}}{dr}+2g_{44}\frac{\partial\xi^4}{\partial t}=0\,. \qquad (\;\;,, \quad 4,4) \quad (4.4)$$

Putting $g_{11}=-e^{\lambda}$, $g_{44}=e^{\nu}$, we shall solve the equations above for the line element of the form (3.1). For (1.1), (2.2), and (1.2) of (N. 1), we have (since $\xi^1 \neq 0$)

$$\left.\begin{aligned} &e^{-\lambda}=l^2-cr^2 \qquad (l, c \text{ are constants})\\ &\xi^1=-e^{-\frac{\lambda}{2}}l[A(\varphi,t)\cos l\theta-B(\varphi,t)\sin l\theta]\,,\\ &\xi^2=\frac{1}{r}e^{-\frac{\lambda}{2}}[A(\varphi,t)\sin l\theta+B(\varphi,t)\cos l\theta]+E(\varphi,t)\,, \end{aligned}\right\} \qquad (\text{N. }2)$$

where, A, B, E do not contain r and θ. Using the equations above, (1.3), (2.3), and (3.3) of (N. 1) are written as

$$\left.\begin{aligned} &-e^{\frac{\lambda}{2}}l\left[\frac{\partial A}{\partial\varphi}\cos l\theta-\frac{\partial B}{\partial\varphi}\sin l\theta\right]+r^2\sin^2\theta\frac{\partial\xi^3}{\partial r}=0\,,\\ &\frac{1}{r}e^{-\frac{\lambda}{2}}\left[\frac{\partial A}{\partial\varphi}\sin l\theta+\frac{\partial B}{\partial\varphi}\cos l\theta\right]+\frac{\partial E}{\partial\varphi}+\sin^2\theta\frac{\partial\xi^3}{\partial\theta}=0\,,\\ &-\frac{1}{r}e^{-\frac{\lambda}{2}}l[A\cos l\theta-B\sin l\theta]\\ &\qquad+\frac{1}{r}e^{-\frac{\lambda}{2}}[A\sin l\theta+B\cos l\theta]\cot\theta+E\cot\theta+\frac{\partial\xi^3}{\partial\varphi}=0\,. \end{aligned}\right\} \qquad (\text{N. }3)$$

From the first two of (N. 3), in order that ξ^3 shall be integrable, it is necessary that either

$$\frac{\partial A}{\partial\varphi}=\frac{\partial B}{\partial\varphi}=0\,, \qquad \frac{\partial\xi^3}{\partial r}=\frac{\partial\xi^3}{\partial\theta}=0\,,$$

or

$$l^2=1\,, \qquad \frac{\partial A}{\partial\varphi}=0\,.$$

But in the first case, since $\dfrac{\partial\xi^3}{\partial\theta}=0$, from the last equation of (N. 3), it must follow that $l^2=1$. Therefore in either case, from (N. 3), we have

$$l^2=1\,, \quad \frac{\partial A}{\partial\varphi}=0\,, \quad B=T_1(t)\sin\varphi+T_2(t)\cos\varphi\,, \quad E=S_1(t)\sin\varphi+S_2(t)\cos\varphi\,,$$

$$\xi^3=\frac{1}{r}e^{-\frac{\lambda}{2}}\frac{\partial B}{\partial\varphi}\frac{1}{\sin\theta}+\frac{\partial E}{\partial\varphi}\cot\theta+S_3(t)\,.$$

So that, putting together the results obtained above, we have

$$\left.\begin{aligned}
&e^{-\lambda}=1-cr^2\,,\\
&\xi^1=-e^{-\frac{\lambda}{2}}[T(t)\cos\theta-B\sin\theta]\,,\\
&\xi^2=\frac{1}{r}e^{-\frac{\lambda}{2}}[T(t)\sin\theta+B\cos\theta]+E\,,\\
&\xi^3=\frac{1}{r\sin\theta}e^{-\frac{\lambda}{2}}\frac{\partial B}{\partial\varphi}+\frac{\partial E}{\partial\varphi}\cot\theta+S_3(t)\,,
\end{aligned}\right\}\qquad\text{(N. 4)}$$

where $\quad B=T_1(t)\sin\varphi+T_2(t)\cos\varphi\,,\qquad E=S_1(t)\sin\varphi+S_2(t)\cos\varphi\,,$

T, T_1, T_2, S_1, S_2, and S_3 being any functions of t only. Then the remaining equations (1.4), (2.4), (3.4), and (4.4) of (N. 1) become

$$\left.\begin{aligned}
&e^{\frac{\lambda}{2}}\left[\frac{dT}{dt}\cos\theta-\frac{\partial B}{\partial t}\sin\theta\right]+e^{\nu}\frac{\partial\xi^4}{\partial r}=0\,,\\
&-re^{-\frac{\lambda}{2}}\left[\frac{dT}{dt}\sin\theta+\frac{\partial B}{\partial t}\cos\theta\right]-r^2\frac{\partial E}{\partial t}+e^{\nu}\frac{\partial\xi^4}{\partial\theta}=0\,,\\
&-r\sin\theta e^{-\frac{\lambda}{2}}\frac{\partial^2 B}{\partial t\partial\varphi}-r^2\sin\theta\cos\theta\frac{\partial^2 E}{\partial t\partial\varphi}-r^2\sin^2\theta\frac{dS_3}{dt}+e^{\nu}\frac{\partial\xi^4}{\partial\varphi}=0\,,\\
&-\frac{d\nu}{dr}e^{-\frac{\lambda}{2}}[T\cos\theta-B\sin\theta]+2\frac{\partial\xi^4}{\partial t}=0\,.
\end{aligned}\right\}\qquad\text{(N. 5)}$$

From the second and third of (N. 5), in order that ξ^4 shall be integrable, it is necessary that $\dfrac{\partial^2 E}{\partial t\partial\varphi}=0$ and $\dfrac{dS_3}{dt}=0$, i. e., S_1, S_2 and S_3 are constants. Similarly, from the second and the fourth of (N. 5), we have, unless $\xi^1=0$,

$$-\frac{1}{2}e^{\nu}\frac{d\nu}{dr}=h^2r\,,\qquad\text{(N. 6)}$$

h being constant, and

$$\frac{d^2T}{dt^2}-h^2T=0\,,\qquad\frac{\partial B}{\partial t^2}-h^2B=0\,.\qquad\text{(N. 7)}$$

Further, from the first and the fourth of (N. 5), we have

$$h^2[ce^{\nu}-h^2(1-cr^2)]=0\,.\qquad\text{(N. 8)}$$

(I) When $h^2=0$, from (N. 6), we have $e^{\nu}=\text{constant}$; therefore the interval becomes

$$ds^2=-\frac{dr^2}{1-cr^2}-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+dt^2\quad\text{(Einstein type)},\qquad\text{(N. 9)}$$

(II) When $h^2 \neq 0$, from (N. 6), it follows that

$$e^{\nu} = m^2 - h^2 r^2,$$

m^2 being constant. But from (N. 8) it must be true that

$$cm^2 = h^2, \quad \text{or} \quad c = \frac{h^2}{m^2}.$$

Hence, taking mt newly as t, the interval becomes

$$ds^2 = -\frac{dr^2}{1 - cr^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + (1 - cr^2) dt^2 \quad \text{(de-Sitter type).} \qquad \text{(N. 10)}$$

As the special case when $c=0$ we have Minkowski space. So we have the result: *In order that there may exist infinitesimal transformations:* $x'^i = x^i + \xi^i \delta\tau$ $(i=1, \ldots, 4)$ *which make invariant the interval of the form:*

$$ds^2 = -e^{\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + e^{\nu(r)} dt^2,$$

so far as $\xi^r \neq 0$, *the interval must be of Minkowski, Einstein, or de-Sitter type.*

For de-Sitter type space all the transformations making g_{ij} invariant are already obtained.(1) So that now we shall solve (N. 5) for Einstein type space $(h^2=0)$. In this case, from the first and the second of (N. 5), in order that ξ^4 shall be integrable, it is necessary that

$$\frac{dT}{dt} = 0, \qquad \frac{\partial B}{\partial t} = 0;$$

therefore ξ^4 must be constant. So that, combining the equations above with (N. 4), we have

$$\xi^1 = -e^{-\frac{\lambda}{2}} (p \cos\theta - B \sin\theta),$$

$$\xi^2 = \frac{1}{r} e^{\frac{\lambda}{2}} (p \sin\theta + B \cos\theta) + E,$$

$$\xi^3 = e^{-\frac{\lambda}{2}} \frac{1}{r \sin\theta} \frac{\partial B}{\partial \varphi} + \frac{\partial E}{\partial \varphi} \cot\theta + q.$$

$$\xi^4 = q_3$$

where $\qquad B = p_1 \sin\varphi + p_2 \cos\varphi, \qquad E = q_1 \sin\varphi + q_2 \cos\varphi$

p, p_1, p_2, q, q_1, q_2, and q_3 being constants. In these equations, the term $\xi^4 = q_3$ shows that the space is static, and the terms involving q, q_1, and q_2 generate rotations in r, θ, φ-space. Hence the transformations which transform the origin $(r=0)$ are generated from the following three operators:

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(1) T. Sibata: this Journal, **11** (1941), 24.

$$\left.\begin{array}{ll}(\text{coefficient of } p) & e^{-\frac{\lambda}{2}}\left[-\cos\theta\dfrac{\partial}{\partial r}+\dfrac{\sin\theta}{r}\dfrac{\partial}{\partial\theta}\right] \\ (\quad ,, \quad p_1) & e^{-\frac{\lambda}{2}}\left[\sin\varphi\sin\theta\dfrac{\partial}{\partial r}+\dfrac{\sin\varphi\cos\theta}{r}\dfrac{\partial}{\partial\theta}+\dfrac{\cos\varphi}{r\sin\theta}\dfrac{\partial}{\partial\varphi}\right] \\ (\quad ,, \quad p_2) & e^{-\frac{\lambda}{2}}\left[\cos\varphi\sin\theta\dfrac{\partial}{\partial r}+\dfrac{\cos\varphi\cos\theta}{r}\dfrac{\partial}{\partial\theta}-\dfrac{\sin\varphi}{r\sin\theta}\dfrac{\partial}{\partial\varphi}\right].\end{array}\right\}\text{(N. 11)}$$

So we have the result: *The transformations which transform the origin* ($r=0$) *making* g_{ij} *of Einstein type invariant are generated from the three operators given by* (N. 11).

Note II. Finite transformations which transform the origin making g_{ij} of Einstein type invariant.

In order to obtain the finite forms of the transformations which are generated from the operators (N. 11), we shall rewrite (N. 11) in x, y, z-coordinates. In x, y, z-coordinates ($x=r\sin\theta\cos\varphi$, $y=r\sin\theta\sin\varphi$, $z=r\cos\theta$), (N. 11) is written as

$$-e^{-\frac{\lambda}{2}}\frac{\partial}{\partial z},\quad e^{-\frac{\lambda}{2}}\frac{\partial}{\partial y},\quad e^{-\frac{\lambda}{2}}\frac{\partial}{\partial x}, \qquad \text{(N. 12)}$$

$$(e^{-\lambda}=1-cr^2)$$

The finite forms of the transformations generated from $e^{-\frac{\lambda}{2}}\frac{\partial}{\partial z}$ are obtained by solving the equations:

$$\frac{dz'}{d\tau}=\sqrt{1-cr'^2},\quad \frac{dx'}{d\tau}=0,\quad \frac{dy'}{d\tau}=0,\quad \frac{dt'}{d\tau}=0,$$

$$(r'^2\equiv x'^2+y'^2+z'^2),$$

where τ is a parameter and chosen such that when $\tau=0$, x', y', z', and t' coincide with x, y, z, and t respectively. From the equations above we have

$$\left.\begin{array}{l}x'=x,\quad y'=y,\quad t'=t,\\ z'=\sqrt{\dfrac{1}{c}-r^2}\sin c\tau+z\cos c\tau \quad (r^2\equiv x^2+y^2+z^2)\end{array}\right\} \qquad \text{(N. 13)}$$

which is the finite form of the transformations generated from $e^{-\frac{\lambda}{2}}\frac{\partial}{\partial z}$. Similarly, the transformations generated from the last two operators of (N. 12) are obtained by cyclic interchange of x, y, z in (N. 13). We write c as $\frac{1}{R^2}$, as usual.

Note III. Velocity of a coordinate-system to another system which is related by (N.13).

The velocity v of K (x, y, z, t)-system to K' $(x' y', z', t')$ which is related by (N. 13) is obtained by putting $x=y=z=0$ and $\left(\frac{dz}{dt}\right)_{x=y=z=0}=0$ in the expression $\frac{dz'}{dt'}$. The result is $v=0$. So we can say that $K\,(x, y, z, t)$- *and* K' (x', y', z', t')*-systems which are related by* (N. 13) *do not move to each other.*

Note IV. The transformations which make g_{ij} of the form (3.2) invariant.

For the line element of the form (3.2), by putting $g_{11}(r)dr^2=-dR^2$ and writing R newly as r, we can reduce (3.2) to

$$ds^2=-dr^2-C(d\theta^2+\sin^2\theta d\varphi^2)+e^{\nu}dt^2\,, \tag{N. 14}$$

where C is a constant and ν is a function of r. Here we shall solve (N. 1) for g_{ij} of the form (N. 14). In this case (N. 1) becomes

$$\left.\begin{aligned}
&\frac{\partial\xi^1}{\partial r}=0\,,\qquad \frac{\partial\xi^2}{\partial\theta}=0\,,\\
&\frac{\partial\xi^1}{\partial\theta}+C\frac{\partial\xi^2}{\partial r}=0\,,\\
&\frac{\partial\xi^1}{\partial\varphi}+C\sin^2\theta\frac{\partial\xi^3}{\partial r}=0\,,\\
&-\frac{\partial\xi^1}{\partial t}+e^{\nu}\frac{\partial\xi^4}{\partial r}=0\,,\\
&\frac{\partial\xi^2}{\partial\varphi}+\sin^2\theta\frac{\partial\xi^3}{\partial\theta}=0\,,\\
&\cos\theta\xi^2+\sin\theta\frac{\partial\xi^3}{\partial\varphi}=0\,,\\
&-C\frac{\partial\xi^2}{\partial t}+e^{\nu}\frac{\partial\xi^4}{\partial\theta}=0\,,\\
&-C\sin^2\theta\frac{\partial\xi^3}{\partial t}+e^{\nu}\frac{\partial\xi^4}{\partial\varphi}=0\,,\\
&\xi^1\frac{d\nu}{dr}+2\frac{\partial\xi^4}{\partial t}=0\,.
\end{aligned}\right\} \tag{N. 15}$$

From the first three equations of (N. 15), we have

$$\xi^1 = A_1(\varphi, t)\theta + B_1(\varphi, t),$$

$$\xi^2 = -A_1(\varphi, t)r + A_2(\varphi, t),$$

and from the next three equations of (N. 15), it follows that

$$\xi^3 = \cot\theta \frac{\partial \xi^2}{\partial \varphi} + A_3(t),$$

and

$$A_2 = S_1 \sin\varphi + S_2 \cos\varphi,$$

$$A_1 = 0, \qquad \frac{\partial B_1}{\partial \varphi} = 0,$$

where A_3, S_1, and S_2 are functions of t only. That is to say:

$$\xi^1 = B_1(t),$$

$$\xi^2 = S_1 \sin\varphi + S_2 \cos\varphi,$$

$$\xi^3 = \cot\theta \frac{\partial \xi^2}{\partial \varphi} + A_3(t).$$

Further, from the last four equations of (N. 15) it must follow that

S_1, S_2 and S_3 are constants,

and, so far as $\xi^1 \neq 0$.

$$k^2 e^{-\nu} + \frac{1}{2}\frac{d^2\nu}{dr^2} = 0 \qquad \frac{d^2 B_1}{dt^2} = k^2 B_1,$$

or

$$\left(\frac{d\nu}{dr}\right)^2 = 4k^2 e^{-\nu} + a^2, \tag{N. 16}$$

a and k being constants.

(I) When $k=0$, we have

$$\nu = ar + b, \qquad B_1 = pt + q,$$

$$\xi^4 = -\frac{1}{4}(pt^2 + 2qt)a + p\int e^{-\nu} dr + C,$$

$a, b, p, q,$ and c being constants.

(II) When $k \neq 0$, we have

$$B_1 = pe^{kt} + qe^{-kt},$$

$$\xi^4 = -\frac{1}{2k}(pe^{kt} - qe^{-kt})\frac{d\nu}{dr} + c,$$

and either $\quad ae^{\frac{\nu}{2}} = e^{\frac{1}{2}(ar+b)} - k^2 e^{-\frac{1}{2}(ar+b)} \qquad [a \neq 0 \text{ in (N. 16)}]$

or $\quad e^{\nu} = (h \pm kr)^2 \qquad [a = 0 \text{ in (N. 16)}]$

$a, b, h, k, c, p,$ and q being constants.

Therefore, excluding rotation in r, θ, φ-space: $\sin\varphi\frac{\partial}{\partial\theta}+\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}$, $\cos\varphi\frac{\partial}{\partial\theta}-\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}$, $\frac{\partial}{\partial\varphi}$ and translation in t-axis: $\frac{\partial}{\partial t}$, we have the following table:

(I. A) $k=a=0\,,\quad e^{\nu}=e^{b}\,,$

$$\xi^1=pt+q\,,$$
$$\xi^4=pe^{-b}r\,,$$

(I. B) $k=0\,,\quad a\neq 0\,,\quad e^{\nu}=e^{ar+b}\,,$

$$\xi^1=pt+q\,,$$
$$\xi^4=-\frac{a}{4}(pt^2+2qt)-\frac{p}{a}(e^{-ar}e^{-b})\,,$$

(II. A) $k\neq 0\,,\quad a=0\,,\quad e^{\nu}=(h\pm kr)^2\,,$

$$\xi^1=pe^{kt}+qe^{-kt}$$
$$\xi^4=\mp(pe^{kt}-qe^{-kt})\frac{1}{(h\pm kr)}\,,$$

(II. B) $k\neq 0\,,\quad a\neq 0\,,\quad e^{\nu}=\frac{1}{a^2}\left[\frac{1}{4}e^{ar+b}+4k^2e^{-(ar+b)}-2k^2\right]$

$$\xi^1=pe^{kt}+qe^{-kt}\,,$$
$$\xi^4=-\frac{a}{2k}(pe^{kt}-qe^{-kt})\frac{e^{ar+b}-16k^2e^{-(ar+b)}}{e^{ar+b}+16k^2e^{-(ar+b)}-8k^2}\,,$$

and in all cases $\xi^2=\xi^3=0$.

In other words, we have the following four kinds of intervals and the corresponding transformations:

(I. A) $ds^2=-dr^2-C(d\theta^2+\sin^2\theta d\varphi^2)+dt^2\,,$

$$t\frac{\partial}{\partial r}+r\frac{\partial}{\partial t}\,,\qquad \frac{\partial}{\partial r}\,,$$

(I. B) $ds^2=-dr^2-C(d\theta^2+\sin^2\theta d\varphi^2)+e^{ar}dt^2\,,$

$$\begin{cases} t\dfrac{\partial}{\partial r}-\left(\dfrac{a}{4}t^2+\dfrac{1}{a}e^{-ar}\right)\dfrac{\partial}{\partial t}\,, \\ \dfrac{\partial}{\partial r}-\dfrac{a}{2}t\dfrac{\partial}{\partial t}\,, \end{cases}$$

(II. A) $ds^2=-dr^2-C(d\theta^2+\sin^2\theta d\varphi^2)+(h+kr)^2dt^2\,,$

$$\begin{cases} e^{kt}\left[\dfrac{\partial}{\partial r}-\dfrac{1}{h+kr}\dfrac{\partial}{\partial t}\right] \\ e^{-kt}\left[\dfrac{\partial}{\partial r}+\dfrac{1}{h+kr}\dfrac{\partial}{\partial t}\right] \end{cases}$$

(II. B) $$ds^2 = -dr^2 - C(d\theta^2 + \sin^2\theta d\varphi^2) + \left[\frac{1}{4}e^{r+b} + 4k^2 e^{-(r+b)} - 2k^2\right]\frac{1}{a^2}dt^2,$$

$$e^{kt}\left[\frac{\partial}{\partial r} - \frac{a}{2k}\frac{e^{ar+b} - 16k^2 e^{-(ar+b)}}{e^{ar+b} + 16k^2 e^{-(ar+b)} - 8k^2}\frac{\partial}{\partial t}\right],$$

$$e^{-kt}\left[\frac{\partial}{\partial r} + \frac{a}{2k}\frac{e^{ar+b} - 16k^2 e^{-(ar+b)}}{e^{ar+b} + 16k^2 e^{-(ar+b)} - 8k^2}\frac{\partial}{\partial t}\right].$$

In the first and second case we have put $e^{\frac{b}{2}}dt$ newly as dt.

So we have the result: *There exist four kinds of intervals of the form* (N. 14) *which are invariant under the transformations which transform the origin* $r=0$. *These intervals and the corresponding operators which generate the transformations are given by* (I. A), (I. B), (II. A), and (II. B).

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