Almost Periodic Functions in Groups.

By

Tôzirô OGASAWARA.

(Received June 30, 1941.)

The object of this paper is to study the theory[1] of J. v. Neumann's almost periodic functions (a. p. f.), with the view that the theory is reduced to the study of the space of continuous functions (a. p. f.) on a uniquely determined bicompact topological group. This idea originates from A. Weil[2] and E. R. van Kampen.[3] In Part 1 we study the writings of F. Wecken[4] and S. Bochner[5] on a. p. f. In Part 2 we study the theory[6] of Bochner-Neumann's vector-valued a. p. f. under the same idea.

Here I express my hearty thanks to Prof. F. Maeda for his kind guidance.

Part 1.

§ **1.** Here we are concerned with J. v. Neumann's a. p. f. in an abstract group G, the element of which we denote by a, b, x, y, z. Let T be any set of a. p. f., and let T^* be the smallest group-invariant set of a. p. f. containing T, i. e. the set of all $f(axb)$, $f \in T$. Let H be the set of elements a of G such that $f(a)=f(e)$ for all $f \in T^*$, where e denotes the unit element of G; then H is the invariant subgroup of G. We assume $H=(e)$, for the general case is reduced to this. We make G a topological group by introducing neighbourhoods

$$U_{(f_1, f_2, \dots . f_n;\ \varepsilon)}(a)=\left(x\,;\ |f_i(x)-f_i(a)|<\varepsilon,\ f_i \in T^*,\ i=1, 2, \dots . n\right).$$

Let $\rho_f(a, b)$ be the translation function of a. p. f. $f(x) \in T$; that is,

$$\rho_f(a, b)=\underset{x, y \in G}{\text{l. u. b.}}\, |f(xay)-f(xby)|.$$

Put

$$V_{(f_1, f_2, \dots . f_n;\ \varepsilon)}(a)=\left(x\,;\ \rho_{f_i}(a, x)<\varepsilon,\ f_i \in T,\ i=1, 2, \dots . n\right);$$

then, by means of these neighbourhoods, G becomes another topological space.

---

(1) J. v. Neumann, Trans. Amer. Math. Soc. **36** (1934), 445–492.
S. Bochner and J. v. Neumann, Trans. Amer. Math. Soc. **37** (1937), 21–50.
E. R. van Kampen, Annals of Math. **37** (1935), 78–91.

(2) A. Weil. C. R. Paris, **200** (1935), 38–40.

(3) loc. cit.

(4) F. Wecken, Math. Zeitschrift **45** (1939), 377–404.

(5) S. Bochner, Annals of Math. **40** (1939), 769–799.

(6) loc. cit.

THEOREM 1. The topologizations of G mentioned above are isomorphic as a uniform space.

PROOF. Any $f(x) \in T^*$ is written $g(axb)$, $g \in T$; therefore $\rho_f(a, b) = \rho_g(a, b)$. For any given $f_1, f_2, \ldots. f_n$, $f_i \in T^*$, we take such $g_1 g_2, \ldots. g_n$, $g_i \in T$; then

$$V_{(g_1, g_2, \ldots. g_n; \epsilon)}(a) \leqq U_{(f_1, f_2, \ldots. f_n; \epsilon)}(a).$$

Consider any $f(x) \in T$; then there exist[1] $y_\mu, z_\mu, \mu = 1, 2, \ldots. m$ such that

$$\text{l. u. b.} \left| f(yxz) - f(y_\mu x z_\mu) \right| < \frac{\epsilon}{4} \qquad \text{for some } \mu.$$

Therefore we have

$$\begin{aligned} |f(yxz) - f(yaz)| &\leqq |f(yxz) - f(y_\mu x z_\mu)| + |f(y_\mu x z_\mu) - f(y_\mu a z_\mu)| \\ &+ |f(y_\mu a z_\mu) - f(yaz)| < \frac{\epsilon}{2} + |f(y_\mu x z_\mu) - f(y_\mu a z_\mu)|. \end{aligned}$$

Put $f_\mu(x) = f(y_\mu x z_\mu)$, $\mu = 1, 2, \ldots. m$; then we have

$$U_{(f_1 f_2, \ldots. f_m; \frac{\epsilon}{4})}(a) \leqq V_{(f; \epsilon)}(a).$$

From this we see that the theorem is valid.

We denote by $\bar{G}$ the completion of G considered as a uniform space.

THEOREM 2. $\bar{G}$ is a uniquely determined bicompact group.

PROOF. We see from the second topologization of G that the group operations xy, x^{-1} are uniformly continuous in x, y and x, so it is sufficient to show[2] that G is totally bounded. But this follows from the definition of a. p. f. Thus the theorem is proved.

Now we consider another characterization of $\bar{G}$. Let $\mathfrak{M}$ be smallest modul containing T. Here we say $\mathfrak{M}$ is a modul[3] when the following conditions are satisfied:

(1) If $f(x), g(x) \in \mathfrak{M}$ and α is any constant, then

$$\alpha f(x),\ f(x) + g(x),\ f(x)g(x),\ \overline{f(x)},\ f(ax),\ f(xa) \in \mathfrak{M}.$$

(2) $\mathfrak{M}$ is closed in the uniform topology with the metrization

$$\text{l. u. b.}_x \left| f(x) - g(x) \right|.$$

Any $f(x) \in \mathfrak{M}$ is uniformly continuous in the uniform space G, so $f(x)$ is extensible to a uniquely determined continuous function on $\bar{G}$, which we denote by the same symbol $f(x)$. Consider the set of all continuous functions on $\bar{G}$ corresponding to functions $\in \mathfrak{M}$, which we denote by the same symbol $\mathfrak{M}$. Then $\mathfrak{M}$ is the smallest modul containing T on $\bar{G}$.

THEOREM 3. $\mathfrak{M}$ on $\bar{G}$ is the set of all continuous functions on $\bar{G}$. By

(1) S. Bochner and J. v. Neumann, loc. cit., 23.
(2) A. Weil, Sur les espaces à structure uniforme et sur la topologie générale, (1937), 25.
(3) E. R. van Kampen, loc. cit., 79.

this property $\bar{G}$ is uniquely determined.

PROOF. From the theory of function ring on a bicompact space it is sufficient to show[1] that $1 \in \mathfrak{M}$, for then we see that $\mathfrak{M}$ is the closed ring with these properties:

(1) If $f(x) \in \mathfrak{M}$, then $\overline{f(x)} \in \mathfrak{M}$.

(2) $1 \in \mathfrak{M}$.

(3) For any distinct two elements $a, b \in \bar{G}$, there exists an $f(x) \in \mathfrak{M}$ such that $f(a) \neq f(b)$.

Without use of the existence theorem of the mean due to J. v. Neumann we can prove $1 \in \mathfrak{M}$. For it is easy to see that a real positive $f(x)$ on $\bar{G}$ exists, and l. u. b. $|f(x)|$ can be expressed as the uniform limit of polynomials of $f(x)$ without the constant term, so that $1 \in \mathfrak{M}$.

The second part of this theorem follows from the fact that any bicompact space is uniquely determined by the set of all continuous functions on it.

REMARK. The above theorem is the special case of the following theorem: Let S be any uniform space weakly topologized by a set T of real bounded functions on S with the property (3) in Theorem 3, then the completion $\bar{S}$ of S is bicompact and the set of all uniformly continuous functions on S, or the set of all continuous functions on $\bar{S}$, coincides with the smallest closed ring containing T and 1. Weiertsrass approximation theorem is a special case of this theorem.

$\bar{G}$ may be obtained as follows: Let G_f be the function space of elements $f(xay)$ considered as a function of x, y, which we denote by a_f. The distance of a_f, b_f is written $\rho_f(a, b)$. Then $\bar{G}_f$ is a compact metric group. Consider the direct product $\otimes \bar{G}_f$ with the usual topologization; then $\otimes \bar{G}_f$ is a bicompact gronp. $\bar{G}$ is obtained as the closure of the set of elements $\{a_f\} \in \otimes \bar{G}_f$ corresponding to all $a \in G$.

In connection with the first topologization of G we may have another way of obtaining $\bar{G}$.

Consider G_a, T_a corresponding to G, T with the same properties, where a belongs to any index set.

THEOREM 4.

$$\overline{\otimes G_a} = \otimes \bar{G}_a .$$

PROOF. This follows from the corresponding theorem in a uniform space, the proof of which offers little difficulty.

§ **2.** In the following lines we assume that G is maximal a. p., i. e., that for any distinct two elements a, b there exists an a. p. f. $f(x)$ such that $f(a) \neq f(b)$. Let $\bar{G}$ be the completion of G considered as the uniform space topologized by the set of all a. p. f. on G. Let T be any set of a. p. f. on

(1) M. H. Stone, Trans. Amer. Math. Soc. **41** (1937), 375–481.

G, and let $\mathfrak{M}$ be the smallest modul containing T. Let H be the maximal invariant subgroup as defined in § 1. Then $\overline{G/H}$, the completion of G/H considered as the uniform space topologized by T, is isomorphic to $\overline{G}/\mathfrak{H}$, where $\mathfrak{H}$ is the maximal invariant subgroup of $\overline{G}$ on which any $f(x) \in \mathfrak{M}$ is constant. Thus the moduls of a. p. f. on G and the invariant subgroups of $\overline{G}$ correspond one to one. The lattice-theoretic treatment of this relation will simplify and complete the work(1) of E. R. van Kampen on moduls of a. p. f.

§ 3. The invariant measure can be introduced in two ways. One is the modified Haar- Banach method,(2) in which directed sets play a part instead of sequences. By this invariant measure we can define the mean $M_x f(x)$ of a. p. f. on G, and show that for any given $\varepsilon > 0$ there exist elements a_μ, $\mu = 1, 2, \ldots . N$ such that

$$\left| \frac{1}{N} \sum_{\mu} f(x a_\mu y) - M_x f(x) \right| < \varepsilon \qquad \text{for all } x, y.$$

But this is explained in Part 2 for the case of vector-valued a. p. f.

Another way is the modified J. v. Neumann method,(3) which is closely related to the work of S. Bochner on a. p. f. Here we prefer the second way of obtaining the invariant measure.

Let us allow the existence(4) of the mean $M_x f(x)$ of a. p. f. $f(x)$ on G with properties:

(1) $M_x[\alpha f(x)] = \alpha M_x[f(x)]$, where α is constant.

(2) $M_x[f_1(x) + f_2(x)] = M_x f_1(x) + M_x x_2(x)$.

(3) $M_x[f(x)] \geqq 0$ for $f(x) \geqq 0$.

(4) $M_x 1 = 1$.

(5) $M_x f(xa) = M_x f(x)$

(6) $M_x f(ax) = M_x f(x)$

(7) $M_x f(x^{-1}) = M_x f(x)$

And (5) (7) or (6) (7) follows from the other properties.(5)

By the preceding theorems we may assume that $M_x f(x)$ is defined for any continuous function $f(x)$ on $\overline{G}$ with the same properties. Let S be the set of all continuous functions on $\overline{G}$ such that $0 \leqq f(x) \leqq 1$ on $\overline{G}$, and let $S(E)$, for any subset E, be the set of functions of S such that $f(x) = 0$ for $x \bar{\in} E$. Put for any open set $O \leqq \overline{G}$

$$\mu^*(O) = \underset{f \in S(O)}{\text{l. u. b.}}\, M_x f(x) .$$

and put for any set $E \leqq \overline{G}$

(1) loc. cit.

(2) S. Saks, Théorie de l'intégrale, (1933), 264–272.

(3) J. v. Neumann, Compositio Math. **1** (1934), 106–114.

(4) J. v. Neumann, Trans. Amer. Math. Soc. **36** (1934), 450-452.

(5) J. v. Neumann, ibid. 452.

$$\mu^*(E)=\underset{O \subseteq E}{\text{g. l. b.}}\ \mu^*(O)\,.$$

Then we can show by the same argument due to A. Markoff[(1)] that $\mu^*(E)$ is a group invariant regular outer measure in the sense of C. Caratheodory, and any Borel set is measurable, and the relation

$$M_x f(x)=\int_{\bar{G}} f(x)\mu(dE)\,,$$

holds good, where we write $\mu(E)$ for measurable sets E instead of $\mu^*(E)$.

Following S. Bochner,[2)] we define the Jordan inner and outer volumes $\nu_*(E)$ and $\nu^*(E)$ of sets $E \subset \bar{G}$ as follows:

$$\nu_*(E)=\underset{f \in S(E)}{\text{l. u. b.}}\ M_x f(x)\,, \qquad \nu^*(E)=1-\nu_*(\bar{G}-E)\,,$$

and we say that E is measurable in his sense when $\nu_*(E)=\nu^*(E)$ holds good. Denote by $\underline{E}$ and $\bar{E}$ the inner part and the closure of E. E is measurable in his sense when, and only when, $\mu(\underline{E})=\mu(\bar{E})$ or the boundary of E is a null set; for we easily see that $\nu_*(\underline{E})=\mu(E)$, $\nu^*(E)=\mu(\bar{E})$. S. Bochner's measurable sets form the metrically dense Boolean subalgebra of the field of our measurable sets modulo null sets. S. Bochner has defined the Jordan volumes for only subsets of G. We see that S. Bochner's measurable sets in G and $\bar{G}$ correspond, one to one, by the set $S(E)$. From this our field of measurable sets of $\bar{G}$ modulo null sets is the metrical completion of the field of S. Bochner's measurable sets of G modulo null sets. For the Bohr a. p. f. the same statement holds good between our field of measurable sets and F. Wecken's a. p. sets.[(3)]

§ **4.** Invariant measure $\mu^*(E)$ introduced in the preceding § has the following properties:

(1) $\mu(E)$ is the unique completely additive measure function on Borel sets $E \subset \bar{G}$ with $\mu(\bar{G})=1$ which is group-invariant with respective to G. If $F(E)$ is a right invariant completely additive set function on Borel sets in $\bar{G}$, i. e, $F(Ea)$ is independent of $a \in G$, then there exists a constant α such that $F(E)=\alpha\mu(E)$ for any Borel set E.

(2) $\mu(E)$ is ergodic, i. e. if $\mu(E-Fa)+\mu(Ea-E)=0$ for all $a \in G$ and a fixed Borel set E, then $\mu(E)=0$ or 1.

(3) $\mu(E)$ has the property of "Zerlegungsgleichheit," i. e. if $\mu(E)=\mu(F)$, then E, F can be written $E=\sum E_n$, $F=\sum F_n$, where E_n, F_n, $n=1, 2, \ldots.$ are measurable sets and $F_n=E_n a_n$ modulo a null set for some $a_n \in G$.

---

(1) A. Markoff, Rec. Math. Moscou, N. s. **4** (1938), 168–190.

(2) loc. cit., 773–774.

(3) loc. cit., 392.

(4) If G has an infinite number of elements, and $\mu(E)$ takes a value $\alpha > 0$, then for any positive β, $\beta < \alpha$, there exists a subset $F \leqq E$ such that $\mu(F) = \beta$.

(1) is proved by the same argument of S. Bochner.(1) (2) is a consequence of (1). Our Borel field of measurable sets modulo null sets is a special case of reducible geometries(2) with no infinite element which are irreducible under a group of automorphisms.

§ **5.** Kovanko a. p. f. and Besicovitch a. p. f. are obtained by the completions of the space of Bohr a. p. f. in certain ways.(3) By the completions of the space of J. v. Neumann a. p. f. we have the spaces of generalized Kovanko a. p. f. and Besicovitch a. p. f. We have seen in § 1 that a. p. f. on G is considered as a continuous function on $\bar{G}$, and conversely. Therefore generalized Kovanko a. p. f. are represented by measurable functions on $\bar{G}$, and generalized Besicovitch a. p. f. by functions of L_p, $1 \leqq p < +\infty$ in $\bar{G}$. We note that the elements of matrices of inequivalent irreducible unitary representations form a complete and closed orthogonal system for $1 \leqq p < +\infty$, but a complete system for $p = +\infty$

Next consider the space of set functions, V_p, $1 \leqq p < +\infty$. V_p is the set of all completely additive set functions $F(E)$ on Borel sets $E \leqq \bar{G}$, for which $\sum_{i=1} \frac{|F(E_i)|^p}{\mu(E_i)^{p-1}}$ are bounded for all decompositions of $\bar{G}$ into disjoint Borel sets E_i, $i = 1, 2, \dots, n$. And the norm is defined in the usual way. Put $T(a) = F(Ea^{-1})$, then $T(a)$ is an a. p. f. with values in V_p, $p > 1$. Any $F(E) \in V_p$, $p > 1$, is absolutely continuous, so that it is an indefinite integral. But for the case $p = 1$, $F(E)$ is absolutely continuous if, and only if, $T(a)$ is a. p. We can prove this by the argument due to S. Bochner.(4) From this we see that Besicovitch a. p. f. are special cases of vector-valued a. p. f. and are also defined as functionals. These considerations have some applications in S. Bochner's paper.(5)

Part 2.

§ **6.** Let L be a complex linear topological space, and let $\mathfrak{U}$ be a distinguished neighbourhood system of 0. We assume that L is convex, i. e. $\alpha U + (1-\alpha) U \leqq U$ for $U \in \mathfrak{U}$ and α, $0 \leqq \alpha \leqq 1$. Moreover, we impose on L that $\alpha U \leqq U$ for $|\alpha| \leqq 1$. Then L is a uniform topological space. We say L is complete, if, and only if, any totally bounded closed subset of L is bicompact. We give a second equivalent definition of completeness: L is

(1) loc. cit., 787.
(2) I. Halperin, Annals of Math. **40** (1939), 581–599.
(3) loc. cit.
(4) loc. cit., 788.
(5) loc. cit.

complete if, and only if, any Cauchy family of subsets from any totally bounded closed subset of L converges to an element of L. This follow from the theorem due to A. Weil.(1) We say that a directed set(2) $\{\mathfrak{f}_\alpha\}$ is fundamental if for any $U \varepsilon \mathfrak{U}$ there exists an α such that $\mathfrak{f}_\beta - \mathfrak{f}_\alpha \varepsilon U$ for any $\beta \geqq \alpha$, and a directed set $\{\mathfrak{f}_\alpha\}$ converges to an element $\mathfrak{f} \varepsilon L$ if for any $U \varepsilon \mathfrak{U}$ there exists an α such that $\mathfrak{f}_\beta - \mathfrak{f} \varepsilon U$ for any $\beta \geqq \alpha$. Now we have a third equivalent definition of completeness: L is complete if, and only if, any fundamental directed set from any totally bounded closed subset of L converges to an element of L. This is easily seen from the second definition. We note that Banach space with the usual weak topology is complete if, and only if, it is regular.

Let L_b^D be the set of all bounded functions on the set D to L, and let U' be the set of functions $\mathfrak{f}(x)$ such that $\mathfrak{f}(x) \in U$; and let $\mathfrak{U}'$ be the set of U'. Then L_b^D becomes a convex linear topological space by topologization by a distinguished neighbourhood system $\mathfrak{U}'$ of 0.

THEOREM 5. L is complete if, and only if, L_b^D is complete.

PROOF. By use of the second definition of completeness we can easily see the validity of the theorem.

§ **7.** In the following lines we assume that L is complete, for otherwise this property is attained by the extension of L.(3) We assume also that G is maximal a. p. A function $\mathfrak{f}(x)$ on G to L is said to be an a. p. f. if the set $\{\mathfrak{f}(xa)\}$ for all $a \in G$ is totally bounded in L_b^G. Put $\rho_U(a, b) = \underset{x,y}{\text{l. u. b.}} \|f(xay) - f(xby)\|_U^+$ where $\|\mathfrak{f}\|_U^+$ denotes the pseudo-metric determined by U. $\rho_U(a, b)$ is an a. p. f. of a, b in $G \times G$, and it is a translation function of $f_U(x)$ defined by $f_U(x) = \rho_U(x, e)$.

THEOREM 6. Any vector-valued a. p. f. $\mathfrak{f}(x)$ is uniquely extensible to a continuous function on $\overline{G}$.

PROOF. We see that $\mathfrak{f}(x)$ is uniformly continuous on the uniform space G mentioned in § 1. By combining this fact with the theorem due to A. Weil we have the theorem.

Let H be the maximal invariant subgroup determined by the smallest modul $\mathfrak{M}$ containing $f_U(x)$ for all $U \in \mathfrak{U}$. Then $\mathfrak{f}(x)$ is uniformly continuous in G/H considered as the uniform space with the topologization by $f_U(x)$, $U \in \mathfrak{U}$ as mentioned in § 1. Hence $\mathfrak{f}(x)$ is extensible to a uniquely determined continuous function on $\overline{G/H}$. If L has J. v. Neumann's countability condition,(4) i. e. if $\mathfrak{U}$ is enumerable and $\Pi U = (0)$, then $\overline{G/H}$ is a compact metric group, the theory of any a. p. f. on G is reduced to the theory of continuous functions on a compact metric group. The enumerability of non-

(1) A. Weil, Sur les espaces à structure uniforme et sur la topologie générale, (1937), 26.
(2) G. Birkhoff, Annals of Math. **38** (1937), 40.
(3) G. Birkhoff, loc., cit., 48.
(4) J. v. Neumann, Trans, Amer. Math. Soc. **37** (1935), 4.

trivial Fourier coefficients is closely related to this fact. It is easy to see that $\mathfrak{M}$ is the small modul containing all $\xi(\mathfrak{f}(x))$, where ξ is a linear functional on L. From the following discussion we shall see, also, that $\mathfrak{M}$ is the smallest modul containing the elements of matrices of irreducible unitary representations with non-trivial Fourier coefficients of $\mathfrak{f}(x)$.

§ **8.** S. Bochner and J. v. Neumann[1] have established the existence of the mean of vector-valued a. p. f., the proof of which may be modified to be valid for our case, where the bicompactness of the convex closure of the set $\mathfrak{f}(xa)$ in L_b^G plays a essential part. But in this § we show the existence of the mean by use of integration with respect to the invariant measure introduced in $\bar{G}$.

Let $\varDelta$ be any decomposition of $\bar{G}$ into a finite number of disjoint Borel sets E_i, $i=1, 2, \ldots . n$: and consider the sum

$$S_\varDelta = \sum \mathfrak{f}(x_i)\mu(E_i)$$

where x_i is any element of E_i.

Then we see that the set $S_\varDelta$ for all decompositions $\varDelta$ is totally bounded and forms a fundamental directed set with the usual ordering of $\varDelta$, so that the limit exists, which we denote by

$$\int_{\bar{G}} \mathfrak{f}(x)\mu(dE)\,, \qquad \text{or} \qquad M_x\mathfrak{f}(x)\,.$$

From the definition of the integral we can verify that $M_x\mathfrak{f}(x)=M_x\mathfrak{f}(axb)$. If in this definition we consider $L_b^{\bar{G}}$ instead of L, and $T(x)=\mathfrak{f}(xz)$ instead of $\mathfrak{f}(x)$, where $T(x)$ is a function with the range in $L_b^{\bar{G}}$; then

$$M_xT(x)=M_x(xz)=M_x\mathfrak{f}(x)\,,$$

hence $M_xT(x)$ is a constant function. From this we see for any given $U \in \mathfrak{U}$ we can choose $a_i \in G$, $i=1, 2, \ldots . n$ such that

$$\frac{1}{n}\sum \mathfrak{f}(a_ix) - M_xf(x) \in U \qquad \text{for all } x \in G.$$

If we put $T(x)=\mathfrak{f}(yxz)$ and consider it as a function with the range in $L_b^{\bar{G}\times\bar{G}}$, we have an analogous result; that is, for any $U \in \mathfrak{U}$ we can choose a_i, $i=1, 2, \ldots . n$ such that

$$\frac{1}{n}\sum \mathfrak{f}(xa_iy) - M_x\mathfrak{f}(x) \in U \qquad \text{for all } x, y \in \bar{G}.$$

Let $D_{\rho\sigma}(x)$ be an element of any irreducible unitary representation matrix which does not enter into the smallest modul $\mathfrak{M}$ determined by $f_U(x)$ for all $U \in \mathfrak{U}$, then $M_x\{D_{\rho\sigma}(x)\mathfrak{f}(x)\}=0$. For if we put $\mathfrak{f}_0=M_x\{D_{\rho\sigma}(x)\mathfrak{f}(x)\}$ and suppose $\mathfrak{f}_0 \neq 0$, then there exists a linear functional $\xi(\mathfrak{f})$ such that

(1) loc. cit., 29.

$\xi(\mathfrak{f}_0) \neq 0$. But $\xi(\mathfrak{f}(x)) \in \mathfrak{M}$, so that $\xi\{M_x(D_{\rho\sigma}(x)\mathfrak{f}(x)\} = M_x\{D_{\rho\sigma}(x)\xi(\mathfrak{f}(x))\} = 0$, which contradicts $\xi(\mathfrak{f}_0) \neq 0$. Thus we must have $\mathfrak{f}_0 = 0$.

That any vector-valued a. p. f. all of whose Fourier coefficients vanish is identically zero follows from an argument analogous to that above.

The work(1) of S. Bochner and J. v. Neumann on a. p. f. is generalized with some necessary modifications, but we do not reproduce it here.

The cost of this research has been defrayed from the Scientific Research Expenditure of the Department of Education.

(1) loc. cit.