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Cosmology in Terms of Wave Geometry (IX) Theory of Spiral Nebulae.

By

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§ 1. General outline and summary.

General outline: Though several authors[(1)] have attempted to explain the structure of spiral nebulae, no theory yet, so far as we know, is generally admitted to be satisfactory.

The purpose of this paper is, on a basis of wave geometry, to try to establish a theory concerning the structure of spiral nebulae in a quite natural way. The general outline of the research is as follows:

1. From the fact[(2)] that cosmology in terms of wave geometry is an invariant theory characterised by a group of transformations which are classified in two subgroups; the one (say G_4) is composed of transformations which make a spatial point (it may be any point) invariant, the other (say G_6) of transformations by which the new origin of coordinates is in a motion relative to the old system of coordinates such that the motion exactly satisfies Habble's velocity-distance relation in terms of wave geometrical cosmology, if we obtain a wave geometry which is invariant for the subgroup G_4, it may be regarded as representing a certain physical phenomenon with local irregularity around the fixed point, representing a general stellar existence with a centre.

2. In the expression of the stellar existence obtained as above, we put some restrictions which seem appropriate in characterising spiral nebulae.

3. From the fundamental equation for ψ, as above obtained, solving the actual value of ψ, constructing the particle momentum-density vector $u^i = \psi^\dagger A \gamma^i \psi$, and regarding this u^i as giving the flux and distribution of matter constituting a spiral nebula, we show that our theory successfully explains the actual construction of spiral nebulae.

Summary: In finding the fundamental differential equation for ψ which

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(1) J. H. Jeans, Monthly Notices, R. A. S., **84** (1923), 60.
E. W. Brown, Astrophys. J., **61** (1925), 97.
E. A. Milne, Proc. Roy. Soc., A **156** (1936), 62 and later papers.
H. Vogt, Astr. Nachr., **241** (1931), 217.
H. Jehle, Z. S. f. Astrophys., **15** (1938), 182 and later papers, etc.

(2) T. Sibata, this Journal, **11** (1941), 21.

is invariant for the subgroup G_4, we see that there are two differential equations for ψ which are not transformable by spin-transformation ($\psi'=S\psi$):

$$\frac{\partial\psi}{\partial x^i}=(\Gamma_i+A_i^j\gamma_j+A_i^{j5}\gamma_j\gamma_5)\psi$$

$$\frac{\partial\psi}{\partial x^i}=(\Gamma_i+A_i^j\gamma_j-A_i^{j5}\gamma_j\gamma_5)\psi\,.$$

Solving these equations, and evaluating the respective values of particle momentum-density vectors, we have two flux-systems $\underset{1}{u}^i$ and $\underset{2}{u}^i$; then, superposing these two flux-systems, and normalizing the factor entered in this procedure so that the new flux-system thus obtained satisfies the condition of continuity, we have a particle momentum density vector u^i which may be regarded as representing a general local flux of stellar matter with a centre. Next, to obtain the expression for the construction of spiral nebulae, in restricting this u^i by the condition that u^i describes a stationary plane motion with an axis of axial symmetry, we show that each spatial curve generated by u^i is a logarithmic spiral with the centre at the origin; and from the expression of the invariant density $\sqrt{g_{ij}u^iu^j}$ we conclude that there are several logarithmic spirals with maximum density-distribution of the constituting matter among a system of logarithmic spirals showing the existence of several arms in a spiral nebula. Further, finding the velocity of the constituting matter, we show the relation existing between the velocity and the distance from the centre.

§ 2. Introductory consideration.

In order to establish a theory of spiral nebulae based on the principles of wave geometry, we shall go back again to wave-geometrical cosmology for suggestions.

In wave geometrical cosmology,(1) taking

$$u^i=\psi^\dagger A\gamma^i\psi\,,$$

obtained from the solution of the fundamental equation for ψ

$$\frac{\partial\psi}{\partial x^i}=(\Gamma_i+\Sigma_i)\psi\,, \tag{2.1}$$

as a particle momentum density vector of matter constituting universe, we fixed the form of Σ_i on the presumption that the particle momentum-density vector describes a geodesic of space-time. And, from the condition of integrability of (2.1), we had two kinds of models of universe, the one being of de Sitter type:

(1) T. Iwatsuki, Y. Mimura, T. Sibata, this Journal **8** (1938), 187.
Y. Mimura, T. Iwatsuki, this Journal **8** (1938), 194.

$$ds^2 = -\frac{dr^2}{1-k^2r^2} - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 + (1-k^2r^2)dt^2, \tag{2.2}$$

the other of Einstein type

$$ds^2 = -\frac{dr^2}{1-\dfrac{r^2}{R^2}} - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 + dt^2.$$

Abandoning the latter type of model as unsuitable for explaining Hubble's velocity-distance relation and so on, we adopted the former as the geometry that might well explain the construction of the actual universe. The fundamental equation for ψ was here obtained as follows:

$$\frac{\partial\psi}{\partial x^i} = \left(\Gamma_i + \frac{k}{2}\gamma_i\right)\psi; \tag{2.3}$$

with which solution we could succeed in explaining Hubble's velocity-distance relation and so on.

Here we have a suggestion for a theory to the construction of spiral nebulae, with a spiral nebula regarded as an existence of local irregularity understood by this cosmological theory.

As shown in the previous paper,(1) we recognize that the meaningless coordinates x^1, x^2, x^3, x^4 take the meaning of polar-time coordinates r, θ, φ, t, when, and only when, we identify

$$x^1 = r, \quad x^2 = \theta, \quad x^3 = \varphi, \quad x^4 = t \tag{2.4}$$

in the line element written in the form:

$$-\frac{(dx^1)^2}{1-k^2(x^1)^2} - (x^1)^2(dx^2)^2 - (x^1)^2\sin^2 x^2 (dx^3)^2 + \left(1-k^2(x^1)^2\right)(dx^4)^2. \tag{2.5}$$

That is to say, only when this identification is made can our cosmology represent the phenomena that are observed with polar coordinates in the ordinary meaning and the time indicated by a clock in a laboratory. We have called such a coordinate system, characterised by (2.4) and (2.5), "the r, θ, φ, t observation-system in cosmology."

But, in fact, there are infinitely many transformations which make (2.5) invariant, and they form a 10-parameter group—say G_{10}—, and each coordinate system connected by G_{10} is also an r, θ, φ, t observation-system on account of relations (2.4) and (2.5). So we know that all the r, θ, φ, t observation-systems are connected by transformations of the group G_{10} which make

$$ds^2 = -\frac{dr^2}{1-k^2r^2} - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 + (1-k^2r^2)dt^2$$

invariant.

(1) T. Sibata, this Journal **11** (1941), 22.

On the other hand, it is proved[1] that G_{10} also makes the fundamental equation of our cosmology $\frac{\partial\psi}{\partial x^i}=\left(\Gamma_i+\frac{k}{2}\gamma_i\right)\psi$ invariant. This shows that G_{10} is only one transformation group connecting the r, θ, φ, t observation-systems, by which all the cosmological phenomena are observed as *homogeneous.*

Further, as we have seen,[2] G_{10} is decomposed into two subgroups as follows:

$$G_{10}=G_6+G_4\,,$$

where G_6 is composed of all the transformations, each transformation $(r, \theta, \varphi, t \rightarrow r', \theta', \varphi', t')$ making the new origin of coordinates $r'=0$, with reference to the old system of coordinates, set in a motion which exactly satisfies Hubble's velocity-distance relation in terms of wave geometry, and G_4 is composed of all the transformations which make a spatial point (any point) invariant. So that G_6 is regarded as a group of transformations by which an observation-system resting on a constituent of our universe is transformed to that resting on any other constituent preserving the *homogeneity* of cosmological construction; but, on the contrary, G_4 is regarded as a group by which an observation-system resting on a constituent of universe to any other observation-system on the same constituent preserving the *homogeneity.*

Further, as has been proved,[3] the equation (2.3) is the only equation of the form (2.1) which is invariant for G_6. This shows that all the phenomena which are observed as *homogeneous* by all the observation-systems belonging to G_6 are the phenomena which can be described in terms of wave-geometrical cosmology. Therefore, we can say that all the physical laws obtained in wave geometrical cosmology are deducible from the theory of group basing on G_6.

Contrarily, though G_4 also makes all the cosmological laws invariant, the wave geometry which is invariant for G_4 is not the previous cosmology in terms of wave geometry, but a wider geometry including that cosmology as a part. Now a question arises: What are the phenomena which are observed as *homogeneous* by G_4? They are the phenomena which must be observed as having *homogeneous* (by G_4) constructions surrounding a material point under the control of the law of wave geometrical cosmology. Therefore, such a phenomenon, we venture to say, might be nothing other than that of local irregularity with a *homogeneous* construction surrounding a point in universe, such as a nebula cluster, a spiral nebula, or the solar system.

In the following pages, on a basis of this idea, we shall proceed to establish our theory concerning the construction of spiral nebulae.

(1) T. Sibata, this Journal **11** (1941), 23.
(2) T. Sibata, this Journal **11** (1941), 23.
(3) T. Sibata, this Journal **11** (1941), 23.

§ 3. Theory of the structure of spiral nebulae.

The constituent matter, the stars, of a spiral nebula are in reality distributed not continuously but discretely; mathematically, however, we may regard the constituents of a spiral nebula as a continuous assemblage of geometrical points in a hydrodynamical motion. So that, wave-geometrically, the existence of the constituent matter of a spiral nebula can be regarded as given by a 1–4 matrix ψ, each element being a continuous function of x^1, x^2, x^3, x^4, satisfying a general fundamental equation of the form:

$$\frac{\partial \psi}{\partial x^i}=(\Gamma_i+\Sigma_i)\psi\,, \tag{3.1}$$

where Γ_i is a 4–4 matrix given by the identity:

$$\frac{\partial \gamma_j}{\partial x^i}=\left\{ {k \atop ij} \right\}\gamma_k+\Gamma_i\gamma_j-\gamma_j\Gamma_i \tag{3.2}$$

and Σ_i is, at present, any 4–4 matrix, the elements being functions of x^1, x^2, x^3, x^4.

Here, since we are going to deal only with the invariant theory by the group G_4 of G_{10} based on (2.5), the space-time line element must be of de Sitter form:

$$ds^2=-\frac{dr^2}{1-k^2r^2}-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+(1-k^2r^2)dt^2. \tag{3.3}$$

If we expand Σ_i in sedenion, we have

$$\Sigma_i=A_i^{\cdot pq}\gamma_p\gamma_q+(A_iI+A_i^5\gamma_5)+(A_i^{\cdot j}\gamma_j+A_i^{\cdot j5}\gamma_j\gamma_5)\,. \tag{3.4}$$

The first term of this equation can be transferred into the coefficient of connection of vector parallelism[1] in the form $\left\{ {i \atop jk} \right\}+4A_{kj}^{\cdot\cdot i}$, so that we disregard this term, provided that the coefficient of connection of vector parallelism is Riemannian from the beginning. As for the second term of (3.4), we do not yet know, in general, its definite physical meaning, but geometrically it might be called the term of gauge parallel displacement of ψ corresponding to the gauge transformation[2] of ψ and has geometrical behaviour utterly different[3] from those of the first and third terms. So

(1) The term $A_i^{pq}\gamma_p\gamma_q$ is transferred into the term due to the vector parallelism by taking Γ_i' in place of Γ_i obtained from the equation:

$$\frac{\partial \gamma_j}{\partial x^i}=\Gamma_{ji}^k\gamma_k+\Gamma_i'\gamma_j-\gamma_j\Gamma_i'\,,$$

provided that $\Gamma_{ji}^k=\left\{ {k \atop ji} \right\}+4A_{ij}^{\cdot\cdot k}$.

(2) K. Morinaga, this Journal, **5** (1935), 156.

(3) When A_i and A_i^5 are gradients, the second term can be taken away by a gauge transformation.

that, at the first step of our consideration, we shall disregard this term.

As for the third term of (3.4), judging from all the results[1] hitherto obtained in wave geometry, the antisymmetric parts of A_{ij} and A^{5}_{ij} may be interpreted to express electromagnetic fields of force, and the symmetric parts to express something like fields of disturbance. In considering the structure of spiral nebulae, therefore, we shall preserve this term, considering that there would exist a field of disturbance-like force due to the presence of the constituent matter hydrodynamically distributed.

Thus we realize that the theory concerning the structure of a spiral nebula would be obtained on the following assumptions:

Assumption I. The space-time line element is of de Sitter form, i. e.

$$ds^2=-\frac{dr^2}{1-k^2r^2}-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+(1-k^2r^2)dt^2. \tag{3.5}$$

Assumption II. The fundamental equation for ψ is of the form:

$$\frac{\partial\psi}{\partial x^i}=(\Gamma_i+A^{j}_{i}\gamma_j+A^{j5}_{i}\gamma_j\gamma_5)\psi\,. \tag{3.6}$$

Assumption III. (3.6) is completely integrable.

Assumption IV. (3.6) is invariant in all transformations belonging to G_4.

But, for (3.5), there are infinity of sets of γ_λ for which

$$\gamma_i\gamma_j+\gamma_j\gamma_i=2g_{ij}I\,,\qquad \gamma_i\gamma_5+\gamma_5\gamma_i=0\,,\qquad \gamma_5\gamma_5=-1\,; \tag{3.7}$$

and it is proved[2] that all the sets of γ_λ are classified into two classes; all belonging to a class are equivalent to one another in spin-transformations, and sets belonging to different classes are not equivalent in any spin-transformation. More precisely, if a set of solutions of (3.7) is $\underset{0}{\gamma}_\lambda$, all the sets of the same class are expressed in the form:

$S\underset{0}{\gamma}_iS^{-1}$, $S\underset{0}{\gamma}_5S^{-1}$, S being any 4-4 matrix, and all the sets belonging to the other class are

$$-S\underset{0}{\gamma}_iS^{-1}\,,\qquad -S\underset{0}{\gamma}_5S^{-1}\,.$$

Further, we see that the condition of integrability of (3.6) is the same for all γ_λ satisfying (3.7); for the condition of integrability is[3]

$$\left.\begin{aligned}&K_{lm}^{\cdot\cdot ij}+8(A^{i5}_{[l}A^{j5}_{m]}+A^{i}_{[l}A^{j}_{m]})=0\,,\\&\nabla_{[m}A^{i}_{l]}=0\,,\qquad \nabla_{[m}A^{i5}_{l]}=0\,,\\&A_{[l}^{\cdot i5}A_{m]}^{\cdot j}g_{ij}=0\qquad (i,j,l,m=1,2,3,4)\,,\end{aligned}\right\} \tag{3.8}$$

(1) T. Iwatsuki, Y. Mimura, K. Morinaga, this Journal, **7** (1937), 255. K. Morinaga, this Journal, **7** (1937), 261.

(2) Note I, p. 64.

(3) Note II, p. 65, equations (4)—(7), ($A_i=A_i^{\cdot 5}=0$).

which is independent of choice of γ_λ and its class. So that when g_{ij} and A_i^j, A_i^{j5} are determined from the equations (3.8), there must be two fundamental equations to be considered, which are not transformable by spin-transformations:

$$\frac{\partial \psi}{\partial x^i}=(\Gamma_i+A_i^j\gamma_j+A_i^{j5}\gamma_j\gamma_5)\psi\,, \tag{3.9}$$

$$\frac{\partial \psi}{\partial x^i}=(\Gamma_i+A_i^j\gamma_j-A_i^{j5}\gamma_j\gamma_5)\psi\,. \tag{3.10}$$

Thus, in the theory of spiral nebulae, as the fundamental equation for ψ we have to consider the two equations (3.9) and (3.10) together; otherwise, the theory would be imperfect.[1] So that we come to the conclusion that the existence of matter constituting a spiral nebula is represented by two ψ's—$\underset{1}{\psi}$ and $\underset{2}{\psi}$—defined by the equations

$$\frac{\partial}{\partial x^i}\underset{1}{\psi}=(\Gamma_i+A_i^j\gamma_j+A_i^{j5}\gamma_j\gamma_5)\underset{1}{\psi}\,. \tag{3.11}$$

$$\frac{\partial}{\partial x^i}\underset{2}{\psi}=(\Gamma_i+A_i^j\gamma_j-A_i^{j5}\gamma_j\gamma_5)\underset{2}{\psi}\,. \tag{3.12}$$

Here if we construct the vectors $\underset{1}{u}^l$, $\underset{2}{u}^l$, which are

$$\underset{1}{u}^l=\alpha_1\underset{1}{\psi}^\dagger A\gamma^l\underset{1}{\psi}\,,\qquad \underset{2}{u}^l=\alpha_2\underset{2}{\psi}^\dagger A\gamma^l\underset{2}{\psi}\,,$$

where A is a 4–4 hermitian matrix which makes $A\gamma^l$ also hermitian and α_1, α_2 are any real factors, then each $\underset{1}{u}^l$, $\underset{2}{u}^l$ can be interpreted[2] as a particle momentum-density vector of a certain flux of matter. So it would be quite natural to consider that these two vectors $\underset{1}{u}^l$, $\underset{2}{u}^l$ together make a particle momentum density vector u^l of the flux of matter constituting a spiral nebula. So we make an assumption:

Assumption V. The particle momentum density vector of the matter constituting a spiral nebula is simply made by the two vectors $\underset{1}{u}^l$ and $\underset{2}{u}^l$, without depending on the other physical quantities (tensors or vectors other than $\underset{1}{u}^l$, $\underset{2}{u}^l$).

From this assumption, u^l is obtained[3] as a linear homogeneous function of $\underset{1}{u}^l$ and $\underset{2}{u}^l$, i. e.

(1) In the researches hitherto done in wave geometry, we should have taken into account of the two kinds of equations for ψ, when g_{ij} is solved from the condition of integrability and then γ_λ are obtained from the relation (3.7).

(2) K. Sakuma, this Journal, **11** (1941), 15.

(3) Note V, p. 84.

$$u^l = \underset{1}{a} \underset{1}{u^l} + \underset{2}{a} \underset{2}{u^l}\,, \tag{3.13}$$

where $\underset{1}{a}$ and $\underset{2}{a}$ are, at present, any real scalars.

But all the curves generated by u^l thus obtained do not exactly cover the trajectories of the constituent matters of a spiral nebula, but are very superfluous, because such u^l is obtained only under the restriction that the fundamental equation for ψ is invariant by G_4; therefore u^l is regarded as giving the motion of particle or matter constituting a general stellar existence with a centre at the origin of coordinates.

Accordingly, in order that u^l may be regarded as expressing the motion of particles or matter in a spiral nebula, we have to expect some additional restrictions.

For this purpose, we make the following assumptions:

Assumption $\mathrm{I_N}$. u^l gives a plane motion in each plane (z=const.) perpendicular to an axis (axis of z).

Assumption $\mathrm{II_N}$. A system of trajectories (3 dimensional) generated by u^l is axial symmetric with respect to the axis.

Assumption $\mathrm{III_N}$. The trajectories generated by u^l are stationary.

Assumption $\mathrm{IV_N}$. The distribution of particles or matter is steady.

Assumption $\mathrm{V_N}$. The particle momentum-density vector u^i satisfies the equation of continuity:

$$\nabla_i u^i = 0\,. \tag{3.14}$$

Remarks: (i) In $\mathrm{III_N}$, the term "stationary" means that the curves (3 dimensional) described by u^l are independent of time.

(ii) In $\mathrm{IV_N}$, the term "steady" means that the distribution of matter is independent of time.

(iii) In the actual spiral nebulae, the path-curves and distribuation of flux of stars will change with time, though slightly; but we believe it would be the closest approximation to the truth if we make Assumptions $\mathrm{III_N}$ and $\mathrm{IV_N}$.

(iv) As for $\mathrm{V_N}$, it is quite natural to consider that the matter constituting a spiral nebula satisfies the condition of continuity, since annihilation and creation of the matter in a spiral nebula are considered negligibly small.

§ 4. General form of u^l and distribution of matter in a stellar existence with a centre of spherical symmetricity.

According to the theoretical consideration advanced in § 3, we shall find the actual value of the particle momentum-density vector u^l.

The proup G_4 is obtained in the form of infinitesinal transformation as follows[(1)]:

$$-\cos\varphi\frac{\partial}{\partial\theta}+\cot\theta\sin\varphi\frac{\partial}{\partial\varphi},$$

$$\sin\varphi\frac{\partial}{\partial\theta}+\cot\theta\cos\varphi\frac{\partial}{\partial\varphi},$$

$$\frac{\partial}{\partial\varphi},\qquad\frac{\partial}{\partial t}.$$

Finding the fundamental equations for ψ's of the form (3.11) and (3.12) under the condition that they are invariant for G_4 and are completely integrable, we have[(2)]

$$\left.\begin{aligned}\frac{\partial}{\partial x^i}\underset{1}{\psi}&=(\Gamma_i+A_i^j\gamma_j+A_i^{j5}\gamma_j\gamma_5)\underset{1}{\psi}\\ \frac{\partial}{\partial x^i}\underset{2}{\psi}&=(\Gamma_i+A_i^j\gamma_5-A_i^{j5}\gamma_j\gamma_5)\underset{2}{\psi}\end{aligned}\right\}\tag{4.1}$$

where

$$\left.\begin{aligned}A_i^j&=\frac{k}{2}\cos C\,\delta_i^j-\frac{k}{2}\frac{L}{\sqrt{1-k^2r^2}}\sin C\,\delta_4^j\delta_i^4\\ A_i^{j5}&=\frac{k}{2}\sin C\,\delta_i^j+\frac{k}{2}\frac{L}{\sqrt{1-k^2r^2}}\cos C\,\delta_4^j\delta_i^4\end{aligned}\right\}\tag{4.2}$$

L and C being any constants.

Solving $\underset{1}{\psi}$ from (4.1), we have[(3)]

$$\underset{1}{\psi}_1=\alpha e^{-\frac{i}{2}\varphi}\cos\frac{\theta}{2}+\beta e^{\frac{i}{2}\varphi}\sin\frac{\theta}{2}(\cos\omega+i\sin\omega)$$

$$\underset{1}{\psi}_2=-[i\sqrt{1-k^2r^2}\,\beta+kr\beta']e^{\frac{i}{2}\varphi}\cos\frac{\theta}{2}$$
$$+[i\sqrt{1-k^2r^2}\,\alpha+kr\alpha']e^{-\frac{i}{2}\varphi}\sin\frac{\theta}{2}(\cos\omega+i\sin\omega)$$

$$\underset{1}{\psi}_3=\alpha' e^{-\frac{i}{2}\varphi}\cos\frac{\theta}{2}+\beta' e^{\frac{i}{2}\varphi}\sin\frac{\theta}{2}(\cos\omega-i\sin\omega)$$

$$\underset{1}{\psi}_4=[i\sqrt{1-k^2r^2}\,\beta'+kr\beta]e^{\frac{i}{2}\varphi}\cos\frac{\theta}{2}$$
$$-[i\sqrt{1-k^2r^2}\,\alpha'+kr\alpha]e^{-\frac{i}{2}\varphi}\sin\frac{\theta}{2}(\cos\omega-i\sin\omega),$$

(1) T. Sibata, this Journal, **11** (1940), 22.
(2) Note II, p. 70, equations (A) and (B).
(3) Note III, 75, equations (45)–(52).

where
$$\omega=-\frac{C}{2}$$
and
$$\alpha=a[\sqrt{1-k^2r^2}+ikr]^{\frac{1}{2}}+b\,[\sqrt{1-k^2r^2}+ikr]^{-\frac{1}{2}}$$
$$\alpha'=a[\sqrt{1-k^2r^2}+ikr]^{\frac{1}{2}}-b\,[\sqrt{1-k^2r^2}+ikr]^{-\frac{1}{2}}$$
$$\beta=c\,[\sqrt{1-k^2r^2}+ikr]^{\frac{1}{2}}+d[\sqrt{1-k^2r^2}+ikr]^{-\frac{1}{2}}$$
$$\beta'=c\,[\sqrt{1-k^2r^2}+ikr]^{\frac{1}{2}}-d[\sqrt{1-k^2r^2}+ikr]^{-\frac{1}{2}},$$
where
$$a=pe^{\frac{k}{2}\sqrt{1+L^2}t}+qe^{-\frac{k}{2}\sqrt{1+L^2}t}$$
$$b=-i(\sqrt{1+L^2}+L)pe^{\frac{k}{2}\sqrt{1+L^2}t}+i(\sqrt{1+L^2}-L)qe^{-\frac{k}{2}\sqrt{1+L^2}t}$$
$$c=le^{\frac{k}{2}\sqrt{1+L^2}t}+me^{-\frac{k}{2}\sqrt{1+L^2}t}$$
$$d=-i(\sqrt{1+L^2}+L)le^{\frac{k}{2}\sqrt{1+L^2}t}+i(\sqrt{1+L^2}-L)me^{-\frac{k}{2}\sqrt{1+L^2}t},$$

p, q, l, m being integration constants of ψ.

The solution $\underset{2}{\psi}$ is obtained by putting $-\omega$ and $-L$ instead of ω and L respectively in the solution of $\underset{1}{\psi}$. Using the values of $\underset{1}{\psi}$, $\underset{2}{\psi}$, we have(1), as the r-, θ-, φ-, t- components of $\underset{1}{\psi}^\dagger A\gamma^l\underset{1}{\psi}$ and $\underset{2}{\psi}^\dagger A\gamma^l\underset{2}{\psi}$, $(A=\mathring{\gamma}_4)$,

$$\underset{1}{u}^l\equiv\underset{1}{\psi}^\dagger A\gamma^l\underset{1}{\psi}=\cosh\eta\cdot\underset{1}{v}^l-i\sinh\eta\cdot\underset{1}{v}^l_{\cdot5},$$
$$\underset{2}{u}^l\equiv\underset{2}{\psi}^\dagger A\gamma^l\underset{2}{\psi}=\cosh\eta\cdot\underset{2}{v}^l+i\sinh\eta\cdot\underset{2}{v}^l_{\cdot5},$$
where
$$\underset{1}{v}^r=-kr\sqrt{1-k^2r^2}\,T_0+(1-k^2r^2)\cos\theta\cdot T_1-(1-k^2r^2)\sin\theta\cos\varphi\cdot T_2$$
$$+(1-k^2r^2)\sin\theta\sin\varphi\cdot T_3,$$
$$\underset{1}{v}^\theta=-\frac{1}{r}\cos\theta\cos\varphi\cdot T_2+\frac{1}{r}\cos\theta\sin\varphi\cdot T_3-\frac{1}{r}\sin\theta\cdot T_1,$$
$$\underset{1}{v}^\varphi=\frac{1}{r\sin\theta}(\cos\varphi\cdot T_3+\sin\varphi\cdot T_2),$$
$$\underset{1}{v}^t=\frac{1}{\sqrt{1-k^2r^2}}T_4,$$
$$i\underset{1}{v}^r_{\cdot5}=[\cos\theta\cdot S_0+\sin\theta\cos\varphi\cdot S_1+\sin\theta\sin\varphi\cdot S_2]\sqrt{1-k^2r^2}$$

(1) Note IV, p. 75, equations (53), (55), and (57).

$$\underset{1}{iv}{}^{\theta}_{\cdot 5}=-k\cos\varphi\cdot S_3+k\sin\varphi\cdot S_4 +\frac{\sqrt{1-k^2r^2}}{r}(\cos\theta\cos\varphi\cdot S_1+\cos\theta\sin\varphi\cdot S_2-\sin\theta\cdot S_0)$$

$$\underset{1}{iv}{}^{\varphi}_{\cdot 5}=\frac{\sqrt{1-k^2r^2}}{r\sin\theta}(\cos\varphi\cdot S_2-\sin\varphi\cdot S_1)+k\cot\theta\,(\cos\varphi\cdot S_4+\sin\varphi\cdot S_3)+kS_5$$

$$\underset{1}{iv}{}^{t}_{\cdot 5}=-S_6-\frac{kr}{\sqrt{1-k^2r^2}}(\cos\theta\cdot S_7+\sin\theta\cos\varphi\cdot S_8-\sin\theta\sin\varphi\cdot S_9)\,,$$

$T_0,\ldots,T_4$ and $S_0,\ldots,S_9$ being given in Note IV[1] $\underset{2}{v}{}^l$ and $\underset{2}{v}{}^l_{\cdot 5}$ are given by putting $-L, p', q', l', m'$ instead of L, p, q, l, m in $T_0,\ldots,S_9$.

And, from (3.13), we have

$$u^l=\underset{1}{a}\underset{1}{u}{}^l+\underset{2}{a}\underset{2}{u}{}^l\,. \tag{4.3}$$

But since $\underset{1}{a}$ and $\underset{2}{a}$ may be, at present, any real scalars, without loss of generality, we can normalize a_1, a_2, such that

$$\nabla_i\underset{1}{u}{}^i=0\,,\qquad \nabla_i\underset{2}{u}{}^i=0\,.$$

After this normalization of a_1, a_2, u^l is obtained so as to satisfy the condition (3.14):

$$\nabla_i u^i=0\,,$$

This u^i is the general form of the particle momentum-density vector which is not yet restricted by the assumptions $\mathrm{I_N}$—$\mathrm{IV_N}$.

Next, if we put

$$u^l=D\frac{dx^l}{ds}\,,\qquad\text{where}\qquad D=\sqrt{g_{ij}u^iu^j}\,,$$

and solve the differential equation

$$\frac{dr}{u^1}=\frac{d\theta}{u^2}=\frac{d\varphi}{u^3}=\frac{dt}{u^t}\,,$$

we have the motion of matter constituting general stellar existence with a centre of spherical symmetricity such as nebular cluster, spiral nebulae, the solar system, and so on. And D gives the invariant particle density, by which we can learn the distribution of matter in stellar existence.

§ 5. Motion and distribution of matter in a spiral nebula.

In the previous section we have seen that the particle momentum density vector is given by u^l.

Using the assumptions $\mathrm{I_N}$—$\mathrm{V_N}$ in § 4, we have,[2] in cylindrical coordinates ($\rho=r\sin\theta,\ z=r\cos\theta,\ \varphi,\ t$),

(1) Note IV, p. 82, equations (56) and (58)
(2) Note VI, p. 87, equations (7) and (8).

$$\left.\begin{aligned} &u^z=0\,, \\ &u^\rho=F\cdot\rho^{-2}\cdot k\rho(\bar\sigma+\sigma)\,, \qquad u^\varphi=F\cdot\rho^{-2}\cdot k\mu(1-\sigma\bar\sigma)\,; \\ &u^t=F\cdot\rho^{-2}\{1+\sigma\bar\sigma-\mu i(\bar\sigma-\sigma\}\,, \quad \text{(neglecting } k^2r^2) \end{aligned}\right\} \tag{5.1}$$

where F is an arbitrary function of z and $\rho e^{-\frac{\sigma+\bar\sigma}{\mu(1-\sigma\bar\sigma)}\varphi}$, σ is any complex number, and μ any real constant satisfying $|\mu|\leqq 1$; this u^l is the particle momentum density vector of matter constituting a spiral nebula.

The motion of the matter is then obtained from the equation:

$$\frac{dz}{0}=\frac{d\rho}{\rho}=\frac{(\bar\sigma+\sigma)d\varphi}{\mu(1-\sigma\bar\sigma)}=\frac{(\bar\sigma+\sigma)kdt}{1+\sigma\bar\sigma-\mu i(\bar\sigma-\sigma)}\,, \tag{5.2}$$

from which we have the solution:

$$\text{(i)}\quad z=c_1\,, \qquad \text{(ii)}\quad \rho e^{-\frac{\bar\sigma+\sigma}{\mu(1-\sigma\bar\sigma)}\varphi}=c_2\,, \qquad \text{(iii)}\quad \rho e^{-\frac{(\bar\sigma+\sigma)kt}{1+\sigma\bar\sigma-\mu i(\bar\sigma-\sigma)}}=c_3\,; \tag{5.3}$$

where c_1, c_2, c_3 are integration constants.

The invariant particle density is then obtained[1] as follows (neglecting k^2r^2).

$$D=|\{1+\sigma\bar\sigma-\mu i(\bar\sigma-\sigma)\}\cdot F(c_1,c_2)|\cdot\frac{1}{\rho^2}\,. \tag{5.4}$$

Thus we have the result: *In a spiral nebula, the constituent matter is in a motion, in each plane $z=c_1$, along a curve of a system of logarithmic spirals*

$$\rho e^{-\frac{\sigma+\bar\sigma}{\mu(1-\sigma\bar\sigma)}\varphi}=c_2\,,$$

and the particle density, the distribution of matter, is given by

$$D=|1+\sigma\bar\sigma-\mu i(\bar\sigma-\sigma)|\cdot|F(c_1,c_2)|\cdot\frac{1}{\rho^2}$$

where F is in general any function of the arguments.

§ 6. Physical interpretations of the results obtained in § 5.

From (i), (ii), and (iii) in (5.3), we see that the constituent matter of a spiral nebula moves in each plane $z=c_1$ along each curve of a system of logarithmic spirals having the same centre at $\rho=0$; each particle, starting from the centre at $t=-\infty$, gradually goes further away from the centre, with time, along the logarithmic spiral

$$\rho e^{-\frac{\bar\sigma+\sigma}{\mu(1-\sigma\bar\sigma)}\varphi}=c_2\,,$$

provided that $\dfrac{\sigma+\bar\sigma}{\mu(1-\sigma\bar\sigma)}>0$.

(1) Note VI, p. 88, equation (10).

From (5.4), we see that in each plane $z=c_1$ the particle-density of the constituent matter of the spiral nebula diminishes in proportion to the inverse square of ρ, and also that, since $|F(c_1, c_2)|$, for a fixed c_1, varies as c_2 in general and will take its maximum values for some discrete values of c_2, say $\mathring{c}_2, \mathring{c}_2', \mathring{c}_2'', \ldots$, the constituent matter has denser distribution on the curves

$$\rho e^{-\frac{\bar{\sigma}+\sigma}{\mu(1-\sigma\bar{\sigma})}\varphi}=\mathring{c}_2, \qquad \rho e^{-\frac{\bar{\sigma}+\sigma}{\mu(1-\sigma\bar{\sigma})}\varphi}=\mathring{c}_2', \ldots.$$

than on the others of the system of logarithmic spirals, just suggesting us the discrete existence of arms of a spiral nebula.

The results above concerning the distribution of matter are general statements without any restriction of the function $F(c_1, c_2)$. The number and relative positions of the arms of a spiral nebula, as well as the more precise aspects of the distribution, are not determined until the concrete form of $F(c_1, c_2)$ is given.

As regards the determination of $F(c_1, c_2)$, we have at present no relevant method; we wonder, however, whether we might not have the key to the problem by solving $F(c_1, c_2)$ from a system of equations

$$\nabla_i T^{ij}=0 \qquad \text{and so on,}$$

if we could define the energy-matter tensor of the constituent matter, and so on, as functions of $\underset{1}{\psi}$ and $\underset{2}{\psi}$.

In spite of this, in the following lines we shall consider the possible shapes of spiral nebulae by the forms of the function $F(c_1, c_2)$.

If $|F(c_1, c_2)|$ is taken as a decreasing even function in $c_1\,(=z)$ for all c_2, the distribution D has, with respect to c_1, only one maximum value at $z=0$, and gets rarer as $|z|$ increases at both sides of the plane $z=0$. So that for such a function $F(c_1, c_2)$, our model seems likely to tell the actual construction of spiral nebulae in which the distribution of matter is densest at the central plane and gets rarer with distance at both sides of the plane.

When $F(c_1, c_2)$ is taken as independent of c_2, in each plane $z=c_1$ the distribution of the constituent matter along all the curves of the system of logarithmic spirals is the same; this suggests as model an armless elliptic nebula.

Furthermore, if $F(c_1, c_2)$ is taken such that $F(c_1)=\log\dfrac{1}{|c_1|}+\text{const.}$, the side view of the locus of distribution, namely the locus of $D=\text{const.}$ in (ρ, z) plane, becomes as follows: $z=\pm ke^{-c\rho^2}$; suggesting as model a lens-shaped spiral nebula.

Remark 1. It may be noticed that we have obtained a theoretical result that the density-distribution is proportional to the inverse square of the horizontal distance from the centre. But at present we have no actual data to confirm this theoretical result, leaving confirmation to future observation.

Remark 2. From (5.4), we have $D=\infty$ for $\rho=0$. This is rather contradictory to observational fact, and accordingly our theory seems to have a fatal weak point in it. We can say, however, that the appearance of such an irrational singularity at the origin is an unavoidable defect of mathematical treatments, which always occurs where we estimate the distribution of discretely existing experiential particles with dimension as a continuous aggregate of dimensionless geometrical points. On this account, such an irrational singular point should be avoided in the considerations.

§ 7. Relation between the velocity of constituent matter and the distance from the centre.

From (5.2), we have for the velocity of the constituent matter

$$\frac{d\rho}{dt}=\frac{(\bar{\sigma}+\sigma)\cdot k\rho}{1+\sigma\bar{\sigma}-\mu i(\bar{\sigma}-\sigma)}, \qquad \frac{d\varphi}{dt}=\frac{(1-\sigma\bar{\sigma})\mu k}{1+\sigma\bar{\sigma}-\mu i(\bar{\sigma}-\sigma)},$$

from which we have

$$v=\sqrt{\left(\frac{d\rho}{dt}\right)^2+\rho^2\left(\frac{d\varphi}{dt}\right)^2}=\frac{\sqrt{(\bar{\sigma}+\sigma)^2+(1-\sigma\bar{\sigma})^2\mu^2}}{1+\sigma\bar{\sigma}-\mu i(\bar{\sigma}-\sigma)}\cdot k\rho, \qquad |\mu|\leqq 1$$

Thus we have a theoretical result: *In a spiral nebula the velocity of the constituent matter is proportional to the horizontal distance from the centre.*

But according to J. H. Jeans,[1] from observational data it is concluded that in M 33, N. G. C. 2403, etc., the velocity is, roughly, proportional to $\rho^{0.26}$.

This tells us that in M 33, N. G. C. 2403, etc., at least, our theoretical result deviates somewhat from fact, although our theory, as a whole, well explains the general aspects of the construction of spiral nebulae. At any rate, our theory of spiral nebulae, stated above, having been established as the first explanatory step towards the general views of spiral nebulae, for details some correction of the second approximation are needed.

§ 8. Generalizations of the theory of spiral nebulae and some corrections of the velocity-distance relation.

In the theory of spiral nebulae propounded above, we believe we have succeeded, on the whole, in explaining in a quite natural way the general

(1) J. H. Jeans, Monthly Notice, R. A. S., **84** (1924), 60.

aspects of spiral nebulae; but the theory needs some corrections, at least for the velocity-distance relation.

At the beginning(1) of our theory we dropped, at the first stage of consideration, the terms $A_i I$ and $A_i^5 \gamma_5$ in the sedenion expansion of $\sum_i$. To realise our purpose, here, reviving these terms, we shall find the general form of the fundamental equation for ψ, preserving the same assumptions I—V and $\mathrm{I_N}$—$\mathrm{V_N}$ as were made above.

If we find the equation of the form:

$$\frac{\partial\psi}{\partial x^i}=(\Gamma_i+A_i^j\gamma_j+A_i^{j5}\gamma_j\gamma_5+A_iI+A_i^5\gamma_5)\psi$$

under the restriction that it is invariant by the transformations G_4 and is completely integrable, provided that the metric g_{ij} is of de Sitter type, we have(2)

$$\left.\begin{array}{ll} A_2^{\cdot 2}=A_3^{\cdot 3}=\dfrac{k}{2}\cos\delta\,, & A_2^{\cdot 25}=A_3^{\cdot 35}=\dfrac{k}{2}\sin\delta\,, \\ A_1^{\cdot 1}=\dfrac{k}{2}\cos\delta+\sigma\sin\delta\,, & A_1^{\cdot 15}=\dfrac{k}{2}\sin\delta-\sigma\cos\delta\,, \\ A_4^{\cdot 4}=\dfrac{k}{2}\cos\delta+\tau\sin\delta\,, & A_4^{\cdot 45}=\dfrac{k}{2}\sin\delta-\tau\cos\delta\,, \\ A_{14}=h\sin\delta\,, & A_{14}^{\cdot\cdot 5}=-\mathrm{h}\cos\delta\,, \\ A_{41}=l\sin\delta\,, & A_{41}^{\cdot\cdot 5}=-l\cos\delta\,, \\ \text{other } A_{ij}=0\,, & \text{other } A_{ij}^{\cdot\cdot 5}=0\,, \end{array}\right\} \qquad (8.1)$$

σ, τ, h and l being defined by

$$\sigma=-re^{\frac{\nu}{2}}\frac{d}{dr}(e^{\frac{\nu}{2}}\tau)\,,\qquad h=\frac{\sqrt{1-k^2r^2}}{k^2}\cdot\frac{d}{dr}\left(\frac{\sqrt{1-k^2r^2}}{r}l\right),$$

$$e^\nu\tau^2=P+c\,,\qquad \frac{1}{r^2}e^\nu l^2=k^2(P+d)\,,\qquad (e^\nu\equiv1-k^2r^2)\,,$$

(c, d=arbitrary constants, δ, P=arbitrary functions of r)

$$A_1^{\cdot 5}=-\frac{1}{r}\frac{\sigma}{k}-\frac{1}{2}\frac{d\delta}{dr}\,,\qquad A_2^{\cdot 5}=A_3^{\cdot 5}=0\,,\qquad A_4^{\cdot 5}=\frac{1}{r}\frac{1}{k}e^\nu l\,,$$

$$A_1=\text{arbitrary function of } r,\qquad A_2=A_3=0\,,\qquad A_4=\text{constant};$$

also, from (15) in p. 67,

$$kA_{[ij]}=\frac{1}{2}\sin\delta\left(\frac{\partial A_j^{\cdot 5}}{\partial x^i}-\frac{\partial A_i^{\cdot 5}}{\partial x^j}\right),\qquad kA_{ij}^{\cdot\cdot 5}=-\frac{1}{2}\cos\delta\left(\frac{\partial A_j^{\cdot 5}}{\partial x^i}-\frac{\partial A_i^{\cdot 5}}{\partial x^j}\right).$$

Since the antisymmetric parts of A_{ij} and A_{ij}^5, above, are to be regarded

(1) § 3, p. 52.
(2) Note II, p. 69, equations (13), (18), (19), (20), and (21).

as an electromagnetic field of force, we disregard them on the ground that, in matter constituting a spiral nebula, action due to electromagnetic field of force may be neglected.

Then it is proved[1] that both A_i and A_i^5 become gradient vectors, and the equation (2) in Note VI,[2] again holds good, provided that μ is a function of r only satisfying $|\mu| \leqq 1$.[3]

In other words, neglecting k^2r^2 and disregarding real common factor,

$$\underset{1}{u}^{\rho}=k\rho(\bar{\sigma}+\sigma),$$
$$\underset{1}{u}^{z}=kz(\bar{\sigma}+\sigma)+i(\bar{\sigma}-\sigma)-\mu(1+\sigma\bar{\sigma}),$$
$$\underset{1}{u}^{\varphi}=\mu k(1-\sigma\bar{\sigma}),$$
$$\underset{1}{u}^{t}=1+\sigma\bar{\sigma}-\mu i(\bar{\sigma}-\sigma)-kz\mu(\bar{\sigma}+\sigma);$$

$\underset{2}{u}$ is obtained by putting $L\rightarrow -L$ and $\mu\rightarrow -\mu$, i. e. $\sigma\rightarrow\dfrac{1}{\sigma}$, $\mu\rightarrow-\mu$, in the above.

From the assumption $\mathrm{I_N}$ taking $a_1:a_2=\underset{2}{u}^z:-\underset{1}{u}^z$ in (4.3), we have, from (4.3),

$$\left.\begin{aligned} u^z=0\,,\quad u^{\rho}=\alpha\cdot k\rho(\bar{\sigma}+\sigma)\,,\quad u^{\varphi}=\alpha\cdot k\mu(1-\sigma\bar{\sigma})\,,\\ u^t=\alpha\{1+\sigma\bar{\sigma}-\mu i(\bar{\sigma}-\sigma)\}\,; \end{aligned}\right\} \tag{8.2}$$

where α is, at present, any factor.

Here, if we normalize α such that $\nabla_i u^i=0$, we have, neglecting k^2r^2,

$$\alpha=\frac{1}{\rho^2}\cdot F,$$

where F is a solution of $u^i\dfrac{\partial F}{\partial x^i}=0$, i. e., an arbitrary function of c_1, c_2, c_3;

$$c_1\equiv z,\quad c_2\equiv\int\frac{\mu}{\rho}d\rho-\frac{1-\sigma\bar{\sigma}}{\bar{\sigma}+\sigma}\varphi,\quad c_3\equiv\int\{1+\sigma\bar{\sigma}-\mu i(\bar{\sigma}-\sigma)\}\frac{d\rho}{\rho}-k(\bar{\sigma}+\sigma)t;$$

and we have, from (8.2) neglecting k^2r^2, the formula for the particle-density;

$$D=\sqrt{g_{ij}u^iu^j}=\frac{1}{\rho^2}\,|1+\sigma\bar{\sigma}-\mu i(\bar{\sigma}-\sigma)|\cdot|F(c_1, c_2, c_3)|\,.$$

But from the assumption $\mathrm{IV_N}$, D must not contain c_3, so that

$$D=\frac{1}{\rho^2}\,|1+\sigma\bar{\sigma}-\mu i(\bar{\sigma}-\sigma)|\cdot|F(c_1, c_2)|\,. \tag{8.3}$$

And the motion of the matter is given by

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(1) Note II, p. 65, equations (4) and (15).
(2) Note VI, p. 86.
(3) This is the case when $h=l=\sigma=0$ (i. e. $\alpha=0$) in p. 69.

$$\frac{dz}{0}=\frac{d\rho}{k\rho(\bar{\sigma}+\sigma)}=\frac{d\varphi}{k\mu(1-\sigma\bar{\sigma})}=\frac{dt}{1+\sigma\bar{\sigma}-\mu i(\bar{\sigma}-\sigma)}$$

from which we have

$$\left.\begin{aligned} z=c_1, \qquad \int\frac{\mu}{\rho}d\rho-\frac{1-\sigma\bar{\sigma}}{\bar{\sigma}+\sigma}\varphi=c_2\,, \\ \int\{1+\sigma\bar{\sigma}-\mu i(\bar{\sigma}-\sigma)\}\frac{d\rho}{\rho}-k(\bar{\sigma}+\sigma)t=c_3\,. \end{aligned}\right\} \tag{8.4}$$

To fix the functional form of $\mu(r)$, taking into account that $|\mu|\leqq 1$ and that, when ρ tends to $\pm\infty$, the trajectories would tend to radial straight lines, because of the tendency of motion of matter at a great distance to become radial motion giving Hubble's velocity-distance relation, we can take as the simplest form of $\mu(r)$, neglecting z^2, $\mu=\mu_0 e^{-a\rho^2}$, μ_0, $a(>0)$ being constants.

Then the curves defined by (8.4) are approximately logarithmic spirals near the origin, and the velocity is obtained as follows:

$$v=\left\{\left(\frac{d\rho}{dt}\right)^2+\rho^2\left(\frac{d\varphi}{dt}\right)^2\right\}^{\frac{1}{2}}=\frac{\{(\bar{\sigma}+\sigma)^2+(1-\sigma\bar{\sigma})^2\mu_0^2 e^{-2a\rho^2}\}^{\frac{1}{2}}}{1+\sigma\bar{\sigma}-i(\bar{\sigma}-\sigma)\mu_0 e^{-a\rho^2}}\cdot k\rho\,. \tag{8.5}$$

This is the correction to the velocity-distance relation. If we give suitable numerical values to the constants σ, μ_0, and a, we have the (ρ, v)-graph not so far deviated from that due to Van Maanen's observation (Fig. 1, Fig. 2).

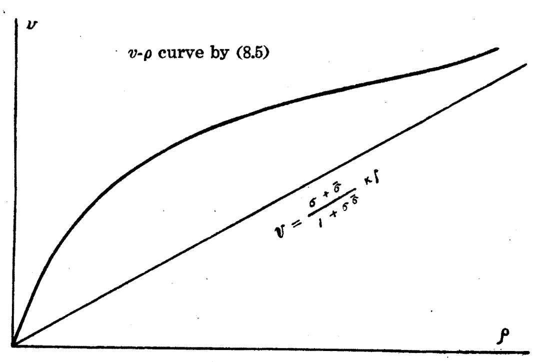

Fig. 1.

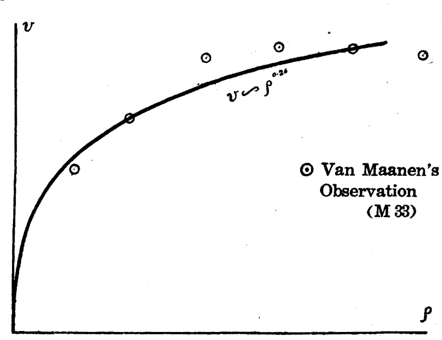

Fig. 2.

As for the distribution of matter, from (8.3) and the relation $|\mu|\leqq 1$, we can say in a keen approximation that in this generalized case there also holds the inverse square law of the density distribution along each path-curve.

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(1) Ap. Journ. **57** (1923), 274; **56** (1922), 207.

NOTES.

Note I.

With respect to Dirac matrices we have the following fundamental theorem: *For the fundamental tenor* $g_{\lambda\mu}$ $(\lambda, \mu=1,\ldots,5)$ *there exists a set of matrices* γ_λ *satisfying the relations*

$$\gamma_\lambda\gamma_\mu+\gamma_\mu\gamma_\lambda=2g_{\lambda\mu}I\,, \qquad (\lambda, \mu=1,\ldots,5) \tag{1}$$

and all the sets of matrices γ'_λ *satisfying the same relations as* (1) *are expressed by*

$$\gamma'_\lambda=S\gamma_\lambda S^{-1} \quad \text{or} \quad \gamma'_\lambda=-S\gamma_\lambda S^{-1}\,,$$

according as

$$\gamma'_{[12345]}=\gamma_{[12345]} \quad \text{or} \quad \gamma'_{[12345]}=-\gamma_{[12345]}\,,$$

S being a certain non-singular matrix (det. $S\neq 0$).

Therefore, all the sets of matrices defined by (1) are classified in two classes such that each set belonging to the same class is transformable by $S\gamma_\lambda S^{-1}$ (called S-transformation) and the sets belonging to different classes are transformable by $-S\gamma_\lambda S^{-1}$.

From the above, by choosing $g_{55}=-1$, $g_{i5}=g_{5i}=0$ $(i=1,\ldots,4)$, we have the theorem: *For the fundamental tensor* g_{ij} $(i, j=1,\ldots,4)$, *all the sets of matrices* $\gamma_1,\ldots,\gamma_5$ *defined by*

$$\gamma_i\gamma_j+\gamma_j\gamma_i=2g_{ij}I\,, \quad \gamma_i\gamma_5+\gamma_5\gamma_i=0\,, \quad \gamma_5\gamma_5=-I\,, \quad (i, j=1,\ldots,5)$$

are classified in two classes represented by

$$\text{(I)} \quad (\gamma_i, \gamma_5) \qquad \text{and} \qquad \text{(II)} \quad (\gamma_i, -\gamma_5)$$

the first being transformable to the second by $-S\gamma_\lambda S^{-1}$ $(S=\gamma_5, \lambda=1,\ldots,5)$ *and each set belonging to the same class* (I) (or (II)) *being transformable by S-transformation to each other.*

Note II.

The fundamental differential equations for ψ which are invariant by G_4.

Let us take the fundamental equation of the form:

$$\left(\frac{\partial}{\partial x^i}-\Gamma_i\right)\psi=(A_i+A_i^{\cdot 5}\gamma_5+A_i^{\cdot j}\gamma_j+A_i^{\cdot j5}\gamma_j\gamma_5)\psi\,. \tag{1}$$

For the above to be invariant in G_4,[(1)] $A_i^{\cdot j}$, $A_i^{\cdot j5}$ and A_i, $A_i^{\cdot 5}$ must be invariant tensors and vectors for the operators: $\cos\varphi\dfrac{\partial}{\partial\theta}-\cot\theta\sin\varphi\dfrac{\partial}{\partial\varphi}$,

(1) T. Sibata, this Journal, **11** (1941), Note, II, 33.

$\sin\varphi\dfrac{\partial}{\partial\theta}+\cot\theta\cos\varphi\dfrac{\partial}{\partial\varphi}$ and $\dfrac{\partial}{\partial\varphi}$ respectively[1]; i. e.

$$\left.\begin{array}{lll} A_{11}(r,t)\,, & A_{2}^{\cdot 2}=A_{3}^{\cdot 3}(r,t)\,, & A_{41}(r,t)\,, \\ A_{14}(r,t)\,, & A_{23}=-A_{32}=\sin\theta\cdot R_{23}(r,t)\,, & A_{44}(r,t)\,, \\ \text{the other} \quad A_{ij}=0\,, & & \end{array}\right\} \tag{2}$$

and

$$A_1(r,t)\,,\qquad A_2=A_3=0\,,\qquad A_4(r,t)\,, \tag{3}$$

($A_i^{\cdot j5}$ and $A_i^{\cdot 5}$ have the same forms as $A_i^{\cdot j}$ and A_i).

But, from the conditions for complete integrability of (1), we have

$$\nabla_{[j}A_{k]}=0\,,\qquad \nabla_{[j}A_{k]}^{\cdot 5}+2A_{[k}^{\cdot p}A_{j]}^{\cdot q5}g_{pq}=0\,, \tag{4}$$

$$\nabla_{[j}A_{k]}^{\cdot p}+2A_{[k}^{\cdot 5}A_{j]}^{\cdot p5}=0\,, \tag{5}$$

$$\nabla_{[j}A_{k]}^{\cdot p5}-2A_{[k}^{\cdot 5}A_{j]}^{\cdot p}=0\,, \tag{6}$$

$$\frac{1}{8}K_{kj}^{\cdot\cdot pq}+A_{[k}^{\cdot p}A_{j]}^{\cdot q}+A_{[k}^{\cdot p5}A_{j]}^{\cdot q5}=0\,. \tag{7}$$

when the fundamental tensor g_{ij} has the form

$$g_{11}=-e^{\lambda(r)},\quad g_{22}=-r^2,\quad g_{33}=-r^2\sin^2\theta,\quad g_{44}=e^{\nu(r)},\quad g_{ij}=0\ (i\neq j)$$

the Christoffel symbols $\{{}^{\,i}_{jk}\}$ are calculated, the actual forms being given in several text books.[2] Then we can solve equations (4)—(7) as follows.

From (5), putting $j, k, p=2, 3, 1$, we have

$$\{{}^{\,1}_{l2}\}A_3^{\cdot l}-\{{}^{\,1}_{l3}\}A_2^{\cdot l}=0\,,$$

or,

$$\{{}^{\,1}_{22}\}A_3^{\cdot 2}-\{{}^{\,1}_{33}\}A_2^{\cdot 3}=0\,,$$

i. e.

$$A_{32}-A_{23}=0\,.$$

Since, from (2), $A_{23}=-A_{32}$, it must be true that

$$A_{23}=A_{32}=0\,. \tag{8}$$

Similarly, from (6), we have

$$A_{23}^{\cdot\cdot 5}=A_{32}^{\cdot\cdot 5}=0\,.$$

Then the surviving equations of (5) become

(1) T. Sibata, this Journal, **11** (1941), , Note IV, 37.
(2) e. g. Eddington: Mathematical Theory of Relativity, 84.

$$\frac{\partial A_2^{;2}}{\partial r}+\frac{1}{r}(A_2^{;2}-A_1^{;1})-2A_1^{;5}A_2^{;25}=0\,, \qquad (\text{for } j, k, p=1, 2, 2) \tag{9.1}$$

$$\left.\begin{aligned}&\frac{\partial A_4^{;1}}{\partial r}-\frac{\partial A_1^{;1}}{\partial t}+\frac{1}{2}\lambda' A_4^{;1}-\frac{1}{2}e^{\nu-\lambda}\nu' A_1^{;4}+2(A_4^{;5}A_1^{;15}-A_1^{;5}A_4^{;15})=0\,,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{for } j, k, p=1, 4, 1)\end{aligned}\right\} \tag{9.2}$$

$$\left.\begin{aligned}&\frac{\partial A_4^{;4}}{\partial t}-\frac{\partial A_1^{;4}}{\partial t}+\frac{1}{2}\nu'(A_4^{;4}-A_1^{;1})+2(A_4^{;5}A_1^{;45}-A_1^{;5}A_4^{;45})=0\,,\\ &\qquad\qquad\qquad\qquad\qquad\qquad (\text{for } j, k, p=1, 4, 4)\end{aligned}\right\} \tag{9.3}$$

$$-\frac{\partial A_2^{;2}}{\partial t}+\frac{1}{r}A_4^{;1}+2A_4^{;5}A_2^{;25}=0\,. \qquad (\text{for } j, k, p=2, 4, 2) \tag{9.4}$$

Similarly, from (6), we have

$$\frac{\partial A_2^{;25}}{\partial r}+\frac{1}{r}(A_2^{;25}-A_1^{;15})+2A_1^{;5}A_2^{;2}=0\,, \tag{10.1}$$

$$\frac{\partial A_4^{;15}}{\partial r}-\frac{\partial A_1^{;15}}{\partial t}+\frac{1}{2}\lambda' A_4^{;15}-\frac{1}{2}e^{\nu-\lambda}\nu' A_1^{;45}-2(A_4^{;5}A_1^{;1}-A_1^{;5}A_4^{;1})=0\,, \tag{10.2}$$

$$\frac{\partial A_4^{;45}}{\partial r}-\frac{\partial A_1^{;45}}{\partial t}+\frac{1}{2}\nu'(A_4^{;45}-A_1^{;15})-2(A_4^{;5}A_1^{;4}-A_1^{;5}A_4^{;4})=0\,, \tag{10.3}$$

$$-\frac{\partial A_2^{;25}}{\partial t}+\frac{1}{r}A_4^{;15}-2A_4^{;5}A_2^{;2}=0\,. \tag{10.4}$$

And (4) becomes

$$\frac{\partial A_4^{;5}}{\partial r}-\frac{\partial A_1^{;5}}{\partial t}+2(A_{41}A_1^{;15}+A_4^{;4}A_{14}^{;5}-A_1^{;1}A_{41}^{;5}-A_{14}A_4^{;45})=0\,. \tag{11}$$

The surviving equations of (7) are

$$\left.\begin{aligned}&\frac{1}{4}K_{12}{}^{12}+A_1^{;1}A_2^{;2}+A_1^{;15}A_2^{;25}=0\,,\\ &\frac{1}{4}K_{23}{}^{23}+A_2^{;2}A_3^{;3}+A^{;25}A_3^{;35}=0\,,\\ &\frac{1}{4}K_{14}{}^{14}+(A_1^{;1}A_4^{;4}-A_4^{;1}A_1^{;4})+(A_1^{;15}A_4^{;45}-A_4^{;15}A_1^{;45})=0\,,\\ &\frac{1}{4}K_{24}{}^{24}+A_2^{;2}A_4^{;4}+A_2^{;25}A_4^{;45}=0\,,\\ &\qquad A_1^{;4}A_2^{;2}+A_1^{;45}A_2^{;25}=0\,,\\ &\qquad A_2^{;2}A_4^{;1}+A_2^{;25}A_4^{;15}=0\,.\end{aligned}\right\} \tag{12}$$

But, for the line element of the form:

$$e^{\nu}=e^{-\lambda}=(1-k^2r^2)\,,$$

$K_{ij}^{\cdot\cdot kl}$ has the form $\qquad K_{ij}^{\cdot\cdot kl}=-2\delta^k_{[i}\delta^l_{j]}k^2$;

hence, from (12), we have (since $A_2^{\cdot 2}=A_3^{\cdot 3}$)

$$\left.\begin{aligned}
&A_2^{\cdot 2}=A_3^{\cdot 3}=\frac{k}{2}\cos\delta\,, && A_2^{\cdot 25}=A_3^{\cdot 35}=\frac{k}{2}\sin\delta\,,\\
&A_1^{\cdot 1}=\frac{k}{2}\cos\delta+\sigma\sin\delta\,, && A_1^{\cdot 15}=\frac{k}{2}\sin\delta-\sigma\cos\delta\,,\\
&A_4^{\cdot 4}=\frac{k}{2}\cos\delta+\tau\sin\delta\,, && A_4^{\cdot 45}=\frac{k}{2}\sin\delta-\tau\cos\delta\,,\\
&A_{14}=h\sin\delta\,, && A_{14}^{\cdot\cdot 5}=-h\cos\delta\,,\\
&A_{41}=l\sin\delta\,, && A_{41}^{\cdot\cdot 5}=-l\cos\delta\,,
\end{aligned}\right\}\tag{13}$$

$$\sigma\tau+hl=0\,,\tag{14}$$

where σ, τ, h, l and δ are certain functions of r and t. Substituting this into (11), we have

$$\frac{\partial A_4^{\cdot 5}}{\partial r}-\frac{\partial A_1^{\cdot 5}}{\partial t}+k(l-h)=0\,.\tag{15}$$

Equations (9) and (10) now become the following four equations:

$$\left.\begin{aligned}
&\frac{k}{2}\frac{\partial\delta}{\partial r}+\frac{1}{r}\sigma+kA_1^{\cdot 5}=0\,, \qquad \text{(from (9.1) and (10.1)),}\\
&\frac{k}{2}\frac{\partial\delta}{\partial t}+\frac{1}{r}g^{11}l+kA_4^{\cdot 5}=0\,, \qquad \text{(from (9.4) and (10.4)),}\\
&\frac{\partial}{\partial r}(g^{11}l)-\frac{1}{r}(g^{11}l)+\frac{1}{2}\lambda'(g^{11}l)-\frac{\partial\sigma}{\partial t}-\frac{1}{2}e^{\nu-\lambda}\nu' g^{44}h=0\,,\\
&\qquad\qquad \text{(from (9.2), (10.2) and (16.1, 2)),}\\
&\frac{\partial\tau}{\partial r}-\frac{\partial}{\partial t}(g^{44}h)+\frac{1}{2}\nu'(\tau-\sigma)+\frac{1}{r}\sigma=0\,,\\
&\qquad\qquad \text{(from (9.3), (10.3) and (16.1, 2)).}
\end{aligned}\right\}\tag{16}$$

((15) is automatically satisfied by (16))

But, since $\frac{\partial}{\partial t}$ is contained in G_4, from invariancy with the operator $\frac{\partial}{\partial t}$, $A_i^{\cdot j}$, $A_i^{\cdot j5}$, A_i and $A_i^{\cdot 5}$ do not contain t. Then σ, τ, h, l, and δ are functions of r only, and accordingly (16) becomes

$$\left.\begin{aligned}
&\frac{k}{2}\frac{d\delta}{dr}+\frac{1}{r}\sigma+kA_1^{;5}=0\,,\\
&\frac{1}{r}g^{11}l+kA_4^{;5}=0\,,\\
&\frac{d}{dr}(g^{11}l)-\frac{1}{r}(g^{11}l)+\frac{1}{2}\lambda'(g^{11}l)-\frac{1}{2}e^{\nu-\lambda}\nu' g^{44}h=0\,,\\
&\frac{d\tau}{dr}+\frac{1}{2}\nu'(\tau-\sigma)+\frac{1}{r}\sigma=0\,.
\end{aligned}\right\}\tag{17}$$

$A_1^{;5}$ and $A_4^{;5}$ are determined from the first two equations of (17). Since $-g^{11}=e^{-\lambda}=e^{\nu}=1-k^2r^2$, the last two equations above become

$$\frac{d}{dr}\left(\frac{l}{r}\right)-k^2r\frac{dl}{dr}-k^2h=0\,,$$

$$\frac{d}{dr}(e^{\frac{\nu}{2}}\tau)+\frac{1}{r}e^{-\frac{\nu}{2}}\sigma=0\,,$$

i. e.

$$\left.\begin{aligned}
&\sigma=-re^{\frac{\nu}{2}}\frac{d}{dr}(e^{\frac{\nu}{2}}\tau)\,,\\
&h=\frac{1}{k^2}\frac{d}{dr}\left(\frac{l}{r}\right)-r\frac{dl}{dr}=\frac{\sqrt{1-k^2r^2}}{k^2}\frac{d}{dr}\left(\frac{\sqrt{1-k^2r^2}}{r}l\right).
\end{aligned}\right\}\tag{18}$$

Substituting (18) into (14), we have

$$-re^{\frac{\nu}{2}}\tau\frac{d}{dr}(e^{\frac{\nu}{2}}\tau)+\left\{\frac{1}{k^2}\frac{d}{dr}\left(\frac{l}{r}\right)-r\frac{dl}{dr}\right\}l=0\,,$$

or

$$-\frac{r}{2}\frac{d}{dr}(e^{\nu}\tau^2)+\frac{1}{k^2}\frac{r}{2}\frac{d}{dr}\left\{\left(\frac{1}{r^2}-k^2\right)l^2\right\}=0\,,$$

i. e.

$$\frac{d}{dr}(e^{\nu}\tau^2)=\frac{1}{k^2}\frac{d}{dr}\left\{\frac{1}{r^2}(1-k^2r^2)l^2\right\}.$$

Therefore, writing the right hand side of the above as $\frac{dP}{dr}$, we have

$$\left.\begin{aligned}
&e^{\nu}\tau^2=P+c\\
&\frac{1}{r^2}e^{\nu}l^2=k^2(P+d)\,,
\end{aligned}\right\}\tag{19}$$

where P is an arbitrary function of r, and c and d are arbitrary constants. So that the solutions of (4)-(7) are given by (13), τ, l, σ, h being given by (19) and (18), and

$$\left.\begin{aligned} A_1^{\cdot 5} &= -\frac{1}{r}\frac{\sigma}{k} - \frac{1}{2}\frac{d\delta}{dr}, \\ A_4^{\cdot 5} &= \frac{1}{r}\frac{1}{k}e^{\nu}l, \qquad (A_2^{\cdot 5} = A_3^{\cdot 5} = 0), \end{aligned}\right\} \tag{20}$$

$$A_1 = \text{arbitrary function of } r, \quad A_2 = A_3 = 0, \quad A_4 = \text{constant}, \tag{21}$$

δ being an arbitrary function of r.

So we have the result: *The fundamental equation* (1) *is invariant in* G_4 *when, and only when,* $A_i^{\cdot j}$, $A_i^{\cdot j5}$, $A_i^{\cdot 5}$, *and* A_i *are given by* (13), (18), (19), (20), *and* (21).

Next, we shall consider the following two cases:

(I) $A_{[ij]} = 0$, $A_{[ij]}^{\cdot\cdot 5} = 0$. From (15), in general, we see that

$$kA_{[14]} = \frac{1}{2}\sin\delta\left(\frac{\partial A_4^{\cdot 5}}{\partial r} - \frac{\partial A_1^{\cdot 5}}{\partial t}\right),$$

$$kA_{[14]}^{\cdot\cdot 5} = -\frac{1}{2}\cos\delta\left(\frac{\partial A_4^{\cdot 5}}{\partial r} - \frac{\partial A_1^{\cdot 5}}{\partial t}\right),$$

i. e. $A_{[ij]}$ and $A_{[ij]}^{\cdot\cdot 5}$ are proportional to $\dfrac{\partial A_i^{\cdot 5}}{\partial x^j} - \dfrac{\partial A_j^{\cdot 5}}{\partial x^i}$. First we shall consider the case when $A_{[ij]} = A_{[ij]}^{\cdot\cdot 5} = 0$. Since $\dfrac{\partial A_1^{\cdot 5}}{\partial t} = 0$, in this case, we have

$$A_4^{\cdot 5} = a \qquad (\text{constant}); \tag{22}$$

accordingly, by (20),

$$h = l = akre^{-\nu}; \tag{23}$$

hence, from (19),

$$a^2k^2e^{-\nu} = k^2(P+d),$$

or

$$\left.\begin{aligned} a^2k^2e^{-\nu} &= k^2(P+d), \\ P &= a^2e^{-\nu} - d \end{aligned}\right\} \tag{24}$$

and

$$\tau = \pm\sqrt{a^2e^{-2\nu} - be^{-\nu}} \qquad (b = d - c); \tag{25}$$

accordingly, from (18),

$$\left.\begin{aligned} \sigma &= \frac{1}{2}\frac{ra^2e^{-\frac{\nu}{2}}\nu'}{\pm\sqrt{a^2e^{-\nu} - b}} \\ &= -\frac{a^2k^2r^2e^{-\frac{3}{2}\nu}}{\pm\sqrt{a^2e^{-\nu} - b}}. \end{aligned}\right\} \tag{26}$$

So we have the result: *When* $A_{[ij]} = A_{[ij]}^{\cdot\cdot 5} = 0$, *equation* (1) *is invariant by* G_4 *when, and only when,* $A_i^{\cdot j}$, $A_i^{\cdot j5}$, $A_i^{\cdot 5}$ *and* A_i *are given by* (13), (20), (21), (22), (23), (25), *and* (26).

(II) When $A_i=A_i^{\cdot 5}=0$. Here, from (20), we have $l=0$, and accordingly, from (19),

$$\tau=ae^{-\frac{\nu}{2}} \qquad (a=\text{constant}),$$

hence, from (18),

$$h=\sigma=0\,,$$

and, from (20),

$$\delta=c \qquad (c=\text{constant}).$$

So we have the result: *For equations of the form:*

$$\left(\frac{\partial}{\partial x^i}-\Gamma_i\right)\psi=(A_i^{\cdot j}\gamma_j+A_i^{\cdot j5}\gamma_j\gamma_5)\psi \tag{A}$$

to be invariant in G_4, it must be true that

$$\left.\begin{aligned} A_i^{\cdot j} &=\frac{k}{2}\cos c\,\delta_i^{\cdot j}+ae^{-\frac{\nu}{2}}\sin c\,\delta_i^4\delta_4^j\,,\\ A_i^{\cdot j5} &=\frac{k}{2}\sin c\,\delta_i^j-ae^{-\frac{\nu}{2}}\cos c\,\delta_i^4\delta_4^j\,,\end{aligned}\right\} \tag{B}$$

a being any constant.

Note III.

The solution ψ of the fundamental equation (1).

If we put

$$\psi=e^A(\cos\omega+\sin\omega\cdot\gamma_5)\phi \qquad \left(\frac{\partial A}{\partial x^i}\equiv A_i\right). \tag{27}$$

Equation (1) becomes

$$\left.\begin{aligned}\left(\frac{\partial}{\partial x^i}-\Gamma_i\right)\phi+\frac{\partial\omega}{\partial x^i}\gamma_5\phi=[&A_i^{\cdot 5}\gamma_5+(\cos 2\omega A_i^{\cdot j}-\sin 2\omega A_i^{\cdot j5})\gamma_j\\ &+(\cos 2\omega A_i^{\cdot j5}+\sin 2\omega A_i^{\cdot j})\gamma_j\gamma_5]\phi\,.\end{aligned}\right\} \tag{28}$$

Hence, if we choose ω such that

$$\cos 2\omega\cdot\cos\delta-\sin 2\omega\sin\delta=1\,,$$

$$\cos 2\omega\sin\delta+\sin 2\omega\cos\delta=0\,,$$

i. e.

$$\omega=-\frac{1}{2}\delta\,, \tag{29}$$

then (28) becomes

$$\left.\begin{aligned}\left(\frac{\partial}{\partial x^i}-\Gamma_i\right)\phi=\Big[&\left\{\delta_i^1\left(-\frac{\sigma}{kr}\right)+\delta_i^4\left(-\frac{1}{r}g^{11}\frac{l}{k}\right)\right\}\gamma_5+\frac{k}{2}\gamma_i\\ &-\delta_i^1(\sigma\gamma_1+h\gamma^4)\gamma_5-\delta_i^4(l\gamma^1+\tau\gamma_4)\gamma_5\Big]\phi\,.\end{aligned}\right\} \tag{30}$$

Therefore, making use of (27) and (29), the solution ψ of (1) is obtained by solving (30) for ϕ.

First we shall take the case when $A_i = A_i^{\cdot 5} = 0$. Here, since $l = h = \sigma = 0$, and $-\tau = \frac{k}{2} L e^{-\frac{\nu}{2}}$ (L=constant), (30) becomes

$$\left(\frac{\partial}{\partial x^i} - \Gamma_i\right)\phi = \frac{k}{2}(\gamma_i + \delta_i^4 \cdot L e^{-\frac{\nu}{2}} \gamma_4 \gamma_5)\phi\,, \tag{31}$$

and the solution ψ of the equation is given by

$$\psi = \left(\cos\frac{c}{2} - \sin\frac{c}{2}\,\gamma_5\right)\phi \tag{32}$$

where c is a constant.

Now we shall solve equation (31), as follows:

For the fundamental tensor g_{ij}:

$$g_{11} = -e^{-\lambda}\,,\quad g_{22} = -r^2\,,\quad g_{33} = -r^2\sin^2\theta\,,\quad g_{44} = e^{\nu}\,,\quad g_{ij} = 0 \quad (i \neq j)\,, \tag{33}$$

we can choose $\gamma_i \equiv \overset{(a)}{h_i}\overset{\circ}{\gamma}_a$ such that

$$\overset{(1)}{h_1} = ie^{\frac{\lambda}{2}}\,,\quad \overset{(2)}{h_2} = ir\,,\quad \overset{(3)}{h_3} = ir\sin\theta\,,\quad \overset{(4)}{h_4} = e^{\frac{\nu}{2}}\,,\quad \overset{(a)}{h_i} = 0 \quad (a \neq i)\,, \tag{34}$$

and $\overset{\circ}{\gamma}_a$ are Dirac matrices of the form

$$\overset{\circ}{\gamma}_1 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix},\quad \overset{\circ}{\gamma}_2 = \begin{pmatrix} & & & -i \\ & & -i & \\ & i & & \\ i & & & \end{pmatrix},\quad \overset{\circ}{\gamma}_3 = \begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix},\quad \overset{\circ}{\gamma}_4 = \begin{pmatrix} & & -i & \\ & & & i \\ i & & & \\ & -i & & \end{pmatrix}, \tag{35}$$

Then Γ_i defined by

$$\frac{\partial \gamma_i}{\partial x^j} - \left\{{k \atop ij}\right\}\gamma_k = \Gamma_j\gamma_i - \gamma_i\Gamma_j\,, \tag{36}$$

is determined in the form

$$\Gamma_j = c_j^{\cdot ab}\overset{\circ}{\gamma}_a\overset{\circ}{\gamma}_b \qquad (c_j^{\cdot ab} = -c_j^{\cdot ba}) \tag{37}$$

where

$$4\sum_{b=1}^{4} c_j^{\cdot ab}\overset{(b)}{h_i} = \frac{\partial \overset{(a)}{h_i}}{\partial x^j} - \left\{{k \atop ij}\right\}\overset{(a)}{h_k}\,, \tag{38}$$

because

$$\Gamma_j\gamma_i - \gamma_i\Gamma_j = c_j^{ab}\overset{(c)}{h_i}(\overset{\circ}{\gamma}_a\overset{\circ}{\gamma}_b\overset{\circ}{\gamma}_c - \overset{\circ}{\gamma}_c\overset{\circ}{\gamma}_a\overset{\circ}{\gamma}_b)$$
$$= 4c_j^{\cdot ac}\overset{(c)}{h_i}\overset{\circ}{\gamma}_a\,.$$

Hence, in our case,

$$c_j^{\cdot ab} = \frac{1}{4}\left(\frac{\partial \overset{(a)}{h_b}}{\partial x^j} - \left\{{a \atop bj}\right\}\overset{(a)}{h_a}\right)\overset{(a)}{h_a}\Big/\overset{(b)}{h_b} \qquad (a, b \text{ not summed})$$

i. e.

$$c_j^{\cdot ab} = -\frac{1}{4}\left\{{a \atop bj}\right\}\frac{\overset{(a)}{h_a}}{\overset{(b)}{h_b}}\,, \qquad (a, b \text{ not summed}) \tag{39}$$

therefore

$$\left.\begin{aligned} &\Gamma_1=0, \quad \Gamma_2=\frac{1}{2}e^{-\frac{\lambda}{2}}\mathring{\gamma}_1\mathring{\gamma}_2, \quad \Gamma_3=\frac{1}{2}\{\sin\theta\, e^{-\frac{\lambda}{2}}\mathring{\gamma}_1\mathring{\gamma}_3+\cos\theta\,\mathring{\gamma}_2\mathring{\gamma}_3\}, \\ &\Gamma_4=-\frac{i}{4}e^{\frac{1}{2}(\nu-\lambda)}\nu'\mathring{\gamma}_1\mathring{\gamma}_4. \end{aligned}\right\} \tag{40}$$

So that $\nabla_i\phi\equiv\left(\frac{\partial}{\partial x^i}-\Gamma_i\right)\phi$ are written down as

$$\left.\begin{aligned} &\nabla_1\phi=\frac{\partial}{\partial r}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}, \qquad \nabla_2\phi=\frac{\partial}{\partial\theta}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}-\frac{i}{2}e^{-\frac{\lambda}{2}}\begin{pmatrix}\phi_2\\ \phi_1\\ -\phi_4\\ -\phi_3\end{pmatrix}, \\ &\nabla_3\phi=\frac{\partial}{\partial\varphi}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}-\frac{1}{2}\left\{\sin\theta\, e^{-\frac{\lambda}{2}}\begin{pmatrix}-\phi_2\\ \phi_1\\ \phi_4\\ -\phi_3\end{pmatrix}+i\cos\theta\begin{pmatrix}-\phi_1\\ \phi_2\\ -\phi_3\\ \phi_4\end{pmatrix}\right\}, \\ &\nabla_4\phi=\frac{\partial}{\partial t}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}-\frac{1}{4}e^{\frac{1}{2}(\nu-\lambda)}\nu'\begin{pmatrix}\phi_1\\ -\phi_2\\ -\phi_3\\ \phi_4\end{pmatrix}. \end{aligned}\right\} \tag{41}$$

Further,

$$\gamma_1\phi=ie^{\frac{\lambda}{2}}\begin{pmatrix}\phi_3\\ \phi_4\\ \phi_1\\ \phi_2\end{pmatrix}, \quad \gamma_2\phi=r\begin{pmatrix}\phi_4\\ \phi_3\\ -\phi_2\\ -\phi_1\end{pmatrix}, \quad \gamma_3\phi=ir\sin\theta\begin{pmatrix}\phi_4\\ -\phi_3\\ -\phi_2\\ \phi_1\end{pmatrix}, \quad \gamma_4\phi=ie^{\frac{\nu}{2}}\begin{pmatrix}-\phi_3\\ \phi_4\\ \phi_1\\ -\phi_2\end{pmatrix} \tag{42}$$

and

$$\gamma_5=i\begin{pmatrix}1&&&\\ &1&&\\ &&-1&\\ &&&-1\end{pmatrix}.$$

Therefore, since $e^{-\lambda}=e^{\nu}=1-k^2r^2$, the actual form of (31) is expressed as

$$\text{for } i=1, \quad \frac{\partial}{\partial r}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}=\frac{k}{2}\frac{i}{\sqrt{1-k^2r^2}}\begin{pmatrix}\phi_3\\ \phi_4\\ \phi_1\\ \phi_2\end{pmatrix}, \tag{43.1}$$

$$\text{for } i=2, \quad \frac{\partial}{\partial\theta}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}-\frac{i}{2}\sqrt{1-k^2r^2}\begin{pmatrix}\phi_2\\ \phi_1\\ -\phi_4\\ -\phi_3\end{pmatrix}=\frac{k}{2}r\begin{pmatrix}\phi_4\\ \phi_3\\ -\phi_2\\ -\phi_1\end{pmatrix}, \tag{43.2}$$

$$\text{for } i=3, \quad \frac{\partial}{\partial\varphi}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}-\frac{1}{2}\sin\theta\sqrt{1-k^2r^2}\begin{pmatrix}-\phi_2\\ \phi_1\\ \phi_4\\ -\phi_3\end{pmatrix}-\frac{i}{2}\cos\theta\begin{pmatrix}-\phi_1\\ \phi_2\\ -\phi_3\\ \phi_4\end{pmatrix}=\frac{k}{2}ir\sin\theta\begin{pmatrix}\phi_4\\ -\phi_3\\ -\phi_2\\ \phi_1\end{pmatrix}, \tag{43.3}$$

$$\text{for } i=4,\quad \frac{\partial}{\partial t}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}+\frac{k^2}{2}r\begin{pmatrix}\phi_1\\ -\phi_2\\ -\phi_3\\ \phi_4\end{pmatrix}=\frac{k}{2}i\sqrt{1-k^2r^2}\begin{pmatrix}-\phi_3\\ \phi_4\\ \phi_1\\ -\phi_2\end{pmatrix}+\frac{k}{2}L\begin{pmatrix}-\phi_3\\ \phi_4\\ -\phi_1\\ \phi_2\end{pmatrix}. \tag{43.4}$$

From (43.2), it must be true that

$$\frac{\partial^2\phi_a}{\partial\theta^2}\doteq-\frac{1}{4}\phi_a \qquad (a=1,2,3,4), \tag{44}$$

i. e.

$$\phi_a=A_a\cos\frac{\theta}{2}+B_a\sin\frac{\theta}{2},$$

where A_a and B_a do not contain θ. Substituting into (43.2), we have the solutions of (43.2) as follows:

$$\left.\begin{aligned}
&\phi_1=A\cos\frac{\theta}{2}+B\sin\frac{\theta}{2},\\
&\phi_2=-\{i\sqrt{1-k^2r^2}B+krB'\}\cos\frac{\theta}{2}+\{i\sqrt{1-k^2r^2}A+krA'\}\sin\frac{\theta}{2},\\
&\phi_3=A'\cos\frac{\theta}{2}+B'\sin\frac{\theta}{2},\\
&\phi_4=\{i\sqrt{1-k^2r^2}B'+krB\}\cos\frac{\theta}{2}-\{i\sqrt{1-k^2r^2}A'+krA\}\sin\frac{\theta}{2},
\end{aligned}\right\} \tag{45}$$

where A, B, A', and B' do not contain θ.

Substituting (45) into (43.3), we get

$$\frac{\partial A}{\partial\varphi}=-\frac{i}{2}A,\qquad \frac{\partial B}{\partial\varphi}=\frac{i}{2}B,$$

$$\frac{\partial A'}{\partial\varphi}=-\frac{i}{2}A',\qquad \frac{\partial B'}{\partial\varphi}=\frac{i}{2}B',$$

hence we have

$$\left.\begin{aligned}
&A=\alpha e^{-\frac{i}{2}\varphi},\qquad B=\beta e^{\frac{i}{2}\varphi},\\
&A'=\alpha' e^{-\frac{i}{2}\varphi},\qquad B'=\beta' e^{\frac{i}{2}\varphi},
\end{aligned}\right\} \tag{46}$$

where α, β, α', and β' do not contain θ and φ.

Substituting (45) and (46) into (43.1), we get

$$\left.\begin{aligned}
&\frac{\partial\alpha}{\partial r}=\frac{i}{2}\frac{k}{\sqrt{1-k^2r^2}}\alpha',\qquad \frac{\partial\beta}{\partial r}=\frac{i}{2}\frac{k}{\sqrt{1-k^2r^2}}\beta',\\
&\frac{\partial\alpha'}{\partial r}=\frac{i}{2}\frac{k}{\sqrt{1-k^2r^2}}\alpha,\qquad \frac{\partial\beta'}{\partial r}=\frac{i}{2}\frac{k}{\sqrt{1-k^2r^2}}\beta.
\end{aligned}\right\} \tag{47}$$

The solutions of the equations above are given by[(1)]

$$
\left.\begin{aligned}
\alpha &= a\{\sqrt{1-k^2r^2}+ikr\}^{\frac{1}{2}}+b\{\sqrt{1-k^2r^2}+ikr\}^{-\frac{1}{2}},\\
\alpha' &= a\{\sqrt{1-k^2r^2}+ikr\}^{\frac{1}{2}}-b\{\sqrt{1-k^2r^2}+ikr\}^{-\frac{1}{2}},\\
\beta &= c\{\sqrt{1-k^2r^2}+ikr\}^{\frac{1}{2}}+d\{\sqrt{1-k^2r^2}+ikr\}^{-\frac{1}{2}},\\
\beta' &= c\{\sqrt{1-k^2r^2}+ikr\}^{\frac{1}{2}}-d\{\sqrt{1-k^2r^2}+ikr\}^{-\frac{1}{2}},
\end{aligned}\right\} \qquad (48)
$$

where $a, b, c,$ and d are functions of t alone.

Lastly, substituting (45) and (46) into (43.4), we have

$$\frac{\partial\alpha}{\partial t}=-\frac{k^2}{2}r\alpha-\frac{k}{2}(i\sqrt{1-k^2r^2}+L)\alpha',$$

$$\frac{\partial\beta}{\partial t}=-\frac{k^2}{2}r\beta-\frac{k}{2}(i\sqrt{1-k^2r^2}+L)\beta',$$

$$\frac{\partial\beta'}{\partial t}=\frac{k^2}{2}r\beta'+\frac{k}{2}(i\sqrt{1-k^2r^2}-L)\beta,$$

$$\frac{\partial\alpha'}{\partial t}=\frac{k^2}{2}r\alpha'+\frac{k}{2}(i\sqrt{1-k^2r^2}-L)\alpha,$$

or, substituting (48) into the equations above, we have

$$\frac{da}{dt}=\frac{k}{2}(ib-La), \qquad \frac{db}{dt}=\frac{k}{2}(-ia+Lb),$$

$$\frac{dc}{dt}=\frac{k}{2}(id-Lc), \qquad \frac{dd}{dt}=\frac{k}{2}(-ic+Ld).$$

The solutions are given, in the case when $L^2+1\neq 0$,[(2)] by

(1) For, putting $\sin^{-1}kr=w$ or $kr=\sin w$, the equations (47) are expressed as

$$\frac{\partial\alpha}{\partial w}=\frac{i}{2}\alpha', \qquad \frac{\partial\beta}{\partial w}=\frac{i}{2}\beta';$$

$$\frac{\partial\alpha'}{\partial w}=\frac{i}{2}\alpha, \qquad \frac{\partial\beta'}{\partial w}=\frac{i}{2}\beta;$$

hence

$$\alpha=ae^{\frac{i}{2}w}+be^{-\frac{i}{2}w},$$

$$\alpha'=ae^{\frac{i}{2}w}-be^{-\frac{i}{2}w},$$

a and b being functions of t alone (β and β' have also similar form). Then, expressing the equations above in terms of r, we have (4.8).

(2) When $L^2+1=0$, we have

$$a=pkt+q+ip, \qquad c=lkt+m+il,$$
$$b=-pkt-q+ip, \qquad d=-lkt-m+il,$$

or

$$a=pkt+q-ip, \qquad c=lkt+m-il,$$
$$b=pkt+q+ip, \qquad d=lkt+m+il,$$

according as $L=i$ or $L=-i$. But here we exclude this special case.

$$\left.\begin{aligned}a&=pe^{\frac{k}{2}\sqrt{1+L^2}\,t}+qe^{-\frac{k}{2}\sqrt{1+L^2}\,t},\\ b&=-i(\sqrt{1+L^2}+L)pe^{\frac{k}{2}\sqrt{1+L^2}\,t}+i(\sqrt{1+L^2}-L)qe^{-\frac{k}{2}\sqrt{1+L^2}\,t},\\ c&=le^{\frac{k}{2}\sqrt{1+L^2}\,t}+me^{-\frac{k}{2}\sqrt{1+L^2}\,t},\\ d&=-i(\sqrt{1+L^2}+L)le^{\frac{k}{2}\sqrt{1+L^2}\,t}+i(\sqrt{1+L^2}-L)me^{-\frac{k}{2}\sqrt{1+L^2}\,t},\end{aligned}\right\}\tag{49}$$

where p, q, l and m are arbitrary constants.

Putting these equations together, we have the result: *The solution* ϕ *of* (31) *is given by* (45), (46), (48), *and* (49).

So that the general solution of

$$\left(\frac{\partial}{\partial x^i}-\Gamma_i\right)\psi=(A_i^{\cdot j}\gamma_j+A_i^{\cdot j5}\gamma_j\gamma_5)\psi\,,\tag{50}$$

where

$$\left.\begin{aligned}A_i^{\cdot j}&=\frac{k}{2}\cos c\,\delta_i^j-\frac{k}{2}Le^{-\frac{\nu}{2}}\sin c\,\delta_i^4\delta_4^j\,,\\ A_i^{\cdot j5}&=\frac{k}{2}\sin c\,\delta_i^j+\frac{k}{2}Le^{-\frac{\nu}{2}}\cos c\,\delta_i^4\delta_4^j\,,\end{aligned}\right\}\tag{51}$$

is given by

$$\psi=(D+E\gamma_5)\phi\,,\qquad\left(D=\cos\frac{c}{2},\ E=-\sin\frac{c}{2}\right),\tag{52}$$

ϕ being given by (45), (46), (48), and (49).

Note IV.

The vector $u^l\equiv\psi^\dagger A\gamma^l\psi$ made from ψ, the solution of equation (50).

By (52), $\psi^\dagger A\gamma^l\psi$ are expressed in terms of ϕ as follows:

$$\begin{aligned}\psi^\dagger A\gamma^l\psi&=\phi^\dagger A(\bar{D}+\bar{E}\gamma_5)\gamma^l(D+E\gamma_5)\phi\\ &=(D\bar{D}+E\bar{E})\phi^\dagger A\gamma^l\phi+(\bar{D}E-D\bar{E})\phi^\dagger A\gamma^l\gamma_5\phi\,,\end{aligned}$$

where $\bar{D}$ etc. denote the conjugate complex of D etc.; hence, if we put

$$c=\xi+i\eta\qquad(\xi,\ \eta\ \text{are real})$$

the equation above becomes

$$\psi^\dagger A\gamma^l\psi=\cos i\eta\cdot\phi^\dagger A\gamma^l\phi-\sin i\eta\cdot\phi^\dagger A\gamma^l\gamma_5\phi\,,$$

i. e.

$$\psi^\dagger A\gamma^l\psi=\cosh\eta\cdot\phi^\dagger A\gamma^l\phi-i\sinh\eta\cdot\phi^\dagger A\gamma^l\gamma_5\phi\,.\tag{53}$$

Therefore, in order to obtain $\psi^\dagger A\gamma^l\psi$, we shall now calculate $v^l\equiv\phi^\dagger A\gamma^l\phi$ and $v_5^l\equiv\phi^\dagger A\gamma^l\gamma_5\phi$. Since

$$v^l = \overset{(a)}{h^l} \phi^\dagger A \mathring{\gamma}_a \phi\,, \qquad v^l_{.5} = \overset{(a)}{h^l} \phi^\dagger A \mathring{\gamma}_a \gamma_5 \phi\,, \tag{54}$$

and(1)

$$\left.\begin{aligned}
\phi^\dagger A \mathring{\gamma}_1 \phi &= i\{-\bar{\phi}_1\phi_1 + \bar{\phi}_3\phi_3 + \bar{\phi}_2\phi_2 - \bar{\phi}_4\phi_4\}\,,\\
\phi^\dagger A \mathring{\gamma}_1 \gamma_5 \phi &= \bar{\phi}_1\phi_1 + \bar{\phi}_3\phi_3 - (\bar{\phi}_2\phi_2 + \bar{\phi}_4\phi_4)\,,\\
\phi^\dagger A \mathring{\gamma}_2 \phi &= \bar{\phi}_1\phi_2 - \bar{\phi}_2\phi_1 + \bar{\phi}_3\phi_4 - \bar{\phi}_4\phi_3\,,\\
\phi^\dagger A \mathring{\gamma}_2 \gamma_5 \phi &= i\{\bar{\phi}_1\phi_2 - \bar{\phi}_2\phi_1 - (\bar{\phi}_3\phi_4 - \bar{\phi}_4\phi_3\}\,,\\
\phi^\dagger A \mathring{\gamma}_3 \phi &= i\{\bar{\phi}_1\phi_2 + \bar{\phi}_2\phi_1 + \bar{\phi}_3\phi_4 + \bar{\phi}_4\phi_3\}\,,\\
\phi^\dagger A \mathring{\gamma}_3 \gamma_5 \phi &= -(\bar{\phi}_1\phi_2 + \bar{\phi}_2\phi_1) + (\bar{\phi}_3\phi_4 + \bar{\phi}_4\phi_3)\,,\\
\phi^\dagger A \mathring{\gamma}_4 \phi &= \bar{\phi}_1\phi_1 + \bar{\phi}_3\phi_3 + \bar{\phi}_2\phi_2 + \bar{\phi}_4\phi_4\,,\\
\phi^\dagger A \mathring{\gamma}_4 \gamma_5 \phi &= i\{\bar{\phi}_1\phi_1 - \bar{\phi}_3\phi_3 + \bar{\phi}_2\phi_2 - \bar{\phi}_4\phi_4\}\,,
\end{aligned}\right\} \tag{55}$$

first, we must calculate $\bar{\phi}_2\phi_1$ etc.

From (45) and (46), we have

$$\bar{\phi}_1\phi_1 = \frac{\alpha\bar{\alpha} + \beta\bar{\beta}}{2} + \frac{\alpha\bar{\alpha} - \beta\bar{\beta}}{2}\cos\theta + \frac{\sin\theta}{2}\{(\bar{\alpha}\beta + \alpha\bar{\beta})\cos\varphi + i(\bar{\alpha}\beta - \alpha\bar{\beta})\sin\varphi\}\,,$$

or, using (48),

$$\begin{aligned}
2\bar{\phi}_1\phi_1 = (a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d}) &+ \sqrt{1-k^2r^2}\,(a\bar{b} + \bar{a}b + c\bar{d} + \bar{c}d)\\
&+ ikr(a\bar{b} - \bar{a}b + c\bar{d} - \bar{c}d)\\
+[a\bar{a} + b\bar{b} - c\bar{c} - d\bar{d} &+ \sqrt{1-k^2r^2}\,(a\bar{b} + \bar{a}b - c\bar{d} - \bar{c}d)\\
&+ ikr(a\bar{b} - \bar{a}b - c\bar{d} + \bar{c}d)]\cos\theta\\
+[\bar{a}c + a\bar{c} + \bar{b}d + b\bar{d} &+ \sqrt{1-k^2r^2}\,(\bar{b}c + b\bar{c} + \bar{a}d + a\bar{d})\\
&+ ikr(\bar{b}c - b\bar{c} - \bar{a}d + a\bar{d})]\sin\theta\cos\varphi\\
+i[\bar{a}c - a\bar{c} + \bar{b}d - b\bar{d} &+ \sqrt{1-k^2r^2}\,(\bar{b}c - b\bar{c} + \bar{a}d - a\bar{d})\\
&+ ikr(\bar{b}c + b\bar{c} - \bar{a}d - a\bar{d})]\sin\theta\sin\varphi\,.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
2\bar{\phi}_3\phi_3 = a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} &- \sqrt{1-k^2r^2}\,(a\bar{b} + \bar{a}b + c\bar{d} + \bar{c}d)\\
&- ikr(a\bar{b} - \bar{a}b + c\bar{d} - \bar{c}d)\\
+[a\bar{a} + b\bar{b} - c\bar{c} - d\bar{d} &- \sqrt{1-k^2r^2}\,(a\bar{b} + \bar{a}b - c\bar{d} - \bar{c}d)\\
&- ikr(a\bar{b} - \bar{a}b - c\bar{d} + \bar{c}d)]\cos\theta\\
+[(\bar{a}c + a\bar{c} + \bar{b}d + b\bar{d}) &- \sqrt{1-k^2r^2}\,(\bar{b}c + b\bar{c} + \bar{a}d + a\bar{d})\\
&- ikr(\bar{b}c - b\bar{c} - \bar{a}d + a\bar{d})]\sin\theta\cos\varphi\\
+i[\bar{a}c - a\bar{c} + \bar{b}d - b\bar{d} &- \sqrt{1-k^2r^2}\,(\bar{b}c - b\bar{c} + \bar{a}d - a\bar{d})\\
&- ikr(\bar{b}c + b\bar{c} - \bar{a}d - a\bar{d})]\sin\theta\sin\varphi\,.
\end{aligned}$$

(1) T. Sibata, this Journal **8** (1938), 173.

Therefore,

$$\bar{\phi}_1\phi_1+\bar{\phi}_3\phi_3=(a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d})+\cos\theta(a\bar{a}+b\bar{b}-c\bar{c}-d\bar{d})$$
$$+\sin\theta\cos\varphi(\bar{a}c+a\bar{c}+\bar{b}d+b\bar{d})+i\sin\theta\sin\varphi(\bar{a}c-a\bar{c}+\bar{b}d-b\bar{d}),$$

$$\bar{\phi}_1\phi_1-\bar{\phi}_3\phi_3=\sqrt{1-k^2r^2}(a\bar{b}+\bar{a}b+c\bar{d}+\bar{c}d)+ikr(a\bar{b}-\bar{a}b+c\bar{d}-\bar{c}d)$$
$$+\cos\theta[\sqrt{1-k^2r^2}(a\bar{b}+\bar{a}b-c\bar{d}-\bar{c}d)+ikr(a\bar{b}-\bar{a}b-c\bar{d}+\bar{c}d)]$$
$$+\sin\theta\cos\varphi[\sqrt{1-k^2r^2}(\bar{b}c+b\bar{c}+\bar{a}d+a\bar{d})+ikr(\bar{b}c-b\bar{c}-\bar{a}d+a\bar{d})]$$
$$+i\sin\theta\sin\varphi[\sqrt{1-k^2r^2}(\bar{b}c-b\bar{c}+\bar{a}d-a\bar{d})+ikr(\bar{b}c+b\bar{c}-\bar{a}d-a\bar{d})].$$

Further,

$$\bar{\phi}_2\phi_2=\cos^2\frac{\theta}{2}[(1-k^2r^2)\alpha\bar{\alpha}+k^2r^2\bar{\beta}'\beta'+i\sqrt{1-k^2r^2}\,kr(\beta\bar{\beta}'-\bar{\beta}\beta')]$$
$$+\sin^2\frac{\theta}{2}[(1-k^2r^2)\alpha\bar{\alpha}+k^2r^2\alpha'\bar{\alpha}'+i\sqrt{1-k^2r^2}\,kr(\alpha\bar{\alpha}'-\bar{\alpha}\alpha')]$$
$$-\sin\frac{\theta}{2}\cos\frac{\theta}{2}\begin{bmatrix}e^{i\varphi}\{(1-k^2r^2)\beta\bar{\alpha}+k^2r^2\beta'\bar{\alpha}'+ikr\sqrt{1-k^2r^2}(\beta\bar{\alpha}'-\beta'\bar{\alpha})\}\\+e^{-i\varphi}\{(1-k^2r^2)\alpha\bar{\beta}+k^2r^2\alpha'\bar{\beta}'+ikr\sqrt{1-k^2r^2}(\alpha\bar{\beta}'-\bar{\beta}\alpha')\}\end{bmatrix}$$
$$=\frac{(\alpha\bar{\alpha}+\beta\bar{\beta})}{2}(1-k^2r^2)+\frac{\alpha'\bar{\alpha}'+\beta'\bar{\beta}'}{2}k^2r^2+ikr\sqrt{1-k^2r^2}\,\frac{\beta\bar{\beta}'-\bar{\beta}\beta'+\alpha\bar{\alpha}'-\alpha'\bar{\alpha}}{2}$$
$$+\cos\theta\Big[\frac{\beta\bar{\beta}-\alpha\bar{\alpha}}{2}(1-k^2r^2)+\frac{\beta'\bar{\beta}'-\alpha'\bar{\alpha}'}{2}k^2r^2$$
$$+ikr\sqrt{1-k^2r^2}\,\frac{\beta\bar{\beta}'-\bar{\beta}\beta'-\alpha\bar{\alpha}'+\bar{\alpha}\alpha'}{2}\Big]$$
$$-\frac{\sin\theta}{2}\begin{bmatrix}\cos\varphi\{(\beta\bar{\alpha}+\alpha\bar{\beta})(1-k^2r^2)+(\beta'\bar{\alpha}'+\bar{\beta}'\alpha')k^2r^2\\ \qquad+ikr\sqrt{1-k^2r^2}(\beta\bar{\alpha}'-\beta'\bar{\alpha}+\alpha\bar{\beta}'-\bar{\beta}\alpha')\}\\+i\sin\varphi\{\beta\bar{\alpha}-\alpha\bar{\beta})(1-k^2r^2)+(\beta'\bar{\alpha}'-\bar{\beta}'\alpha')k^2r^2\\ \qquad+ikr\sqrt{1-k^2r^2}(\beta\bar{\alpha}'-\beta'\bar{\alpha}-\alpha\bar{\beta}'+\bar{\beta}\alpha')\}\end{bmatrix}$$

$$\bar{\phi}_4\phi_4=\cos^2\frac{\theta}{2}[(1-k^2r^2)\beta'\bar{\beta}'+\beta\bar{\beta}k^2r^2+ikr\sqrt{1-k^2r_2}(\beta'\bar{\beta}-\bar{\beta}'\beta)]$$
$$+\sin^2\frac{\theta}{2}[(1-k^2r^2)\alpha'\bar{\alpha}'+\alpha\bar{\alpha}k^2r^2+ikr\sqrt{1-k^2r^2}(\alpha'\bar{\alpha}-\bar{\alpha}'\alpha)]$$
$$-\sin\frac{\theta}{2}\cos\frac{\theta}{2}\begin{bmatrix}e^{i\varphi}\{(1-k^2r^2)\beta'\bar{\alpha}'+\beta\bar{\alpha}k^2r^2+ikr\sqrt{1-k^2r^2}(\beta'\bar{\alpha}-\beta\bar{\alpha}')\}\\+e^{-i\varphi}\{(1-k^2r^2)\alpha'\bar{\beta}'+\alpha\bar{\beta}k^2r^2+ikr\sqrt{1-k^2r^2}(\alpha'\bar{\beta}-\alpha\bar{\beta}')\}\end{bmatrix}$$
$$=(1-k^2r^2)\frac{\beta'\bar{\beta}'+\alpha'\bar{\alpha}'}{2}+k^2r^2\frac{\alpha\bar{\alpha}+\beta\bar{\beta}}{2}+ikr\sqrt{1-k^2r^2}\,\frac{\beta'\bar{\beta}-\bar{\beta}'\beta+\alpha'\bar{\alpha}-\bar{\alpha}'\alpha}{2}$$

$$+\Big[(1-k^2r^2)\frac{\beta'\bar\beta'-\alpha'\bar\alpha'}{2}+k^2r^2\frac{\beta\bar\beta-\alpha\bar\alpha}{2}$$
$$+ikr\sqrt{1-k^2r^2}\,\frac{\beta'\bar\beta-\bar\beta'\beta-\alpha'\bar\alpha+\bar\alpha'\alpha}{2}\Big]\cos\theta$$
$$-\left[\begin{array}{l}\cos\varphi\Big\{(1-k^2r^2)\dfrac{\beta'\bar\alpha'+\alpha'\bar\beta'}{2}+k^2r^2\dfrac{\beta\bar\alpha+\alpha\bar\beta}{2}\\ \qquad\qquad +ikr\sqrt{1-k^2r^2}\,\dfrac{\beta'\bar\alpha-\beta\bar\alpha'+\alpha'\bar\beta-\alpha\bar\beta'}{2}\Big\}\\ +i\sin\varphi\Big\{(1-k^2r^2)\dfrac{\beta'\bar\alpha'-\alpha'\bar\beta'}{2}+k^2r^2\dfrac{\beta\bar\alpha-\alpha\bar\beta}{2}\\ \qquad\qquad +ikr\sqrt{1-k^2r^2}\,\dfrac{\beta'\bar\alpha-\beta\bar\alpha'-\alpha'\bar\beta+\alpha\bar\beta'}{2}\Big\}\end{array}\right]\sin\theta\,,$$

hence,

$$\bar\phi_2\phi_2+\bar\phi_4\phi_4=\frac{\alpha\bar\alpha+\beta\bar\beta+\alpha'\bar\alpha'+\beta\bar\beta'}{2}+\frac{\beta\bar\beta-\alpha\bar\alpha+\beta'\bar\beta'-\alpha'\bar\alpha'}{2}\cos\theta$$
$$-\Big[\cos\varphi\,\frac{\beta\bar\alpha+\alpha\bar\beta+\beta'\bar\alpha'+\alpha'\bar\beta'}{2}+i\sin\varphi\,\frac{\beta\bar\alpha-\alpha\bar\beta+\beta'\bar\alpha'-\bar\beta'\alpha'}{2}\Big]\sin\theta\,,$$

$$\bar\phi_2\phi_2-\bar\phi_4\phi_4=\frac{\alpha\bar\alpha+\beta\bar\beta-\alpha'\bar\alpha'-\beta'\bar\beta'}{2}+k^2r^2(\beta'\bar\beta'+\alpha'\bar\alpha'-\alpha\bar\alpha-\beta\bar\beta)$$
$$+ikr\sqrt{1-k^2r^2}\,(\beta\bar\beta'-\bar\beta\beta'+\alpha\bar\alpha'-\alpha'\bar\alpha)$$
$$+\Big[\frac{\beta\bar\beta-\alpha\bar\alpha-\beta'\bar\beta'+\alpha'\bar\alpha'}{2}+k^2r^2(\beta'\bar\beta'-\alpha'\bar\alpha'-\beta\bar\beta+\alpha\bar\alpha)$$
$$+ikr\sqrt{1-k^2r^2}\,(\beta\bar\beta'-\bar\beta\beta'-\alpha\bar\alpha'+\bar\alpha\alpha')\Big]\cos\theta$$
$$-\Big[\frac{\beta\bar\alpha+\alpha\bar\beta-\beta'\bar\alpha'-\alpha'\bar\beta'}{2}+k^2r^2(\beta'\bar\alpha'+\alpha'\bar\beta'-\beta\bar\alpha-\alpha\bar\beta)$$
$$+ikr\sqrt{1-k^2r^2}\,(\beta\bar\alpha'-\beta'\bar\alpha+\alpha\bar\beta'-\alpha'\bar\beta)\Big]\sin\theta\cos\varphi$$
$$-i\Big[\frac{\beta\bar\alpha-\alpha\bar\beta-\beta'\bar\alpha'+\alpha'\bar\beta'}{2}+k^2r^2(\beta'\bar\alpha'-\bar\beta'\alpha'-\beta\bar\alpha+\alpha\bar\beta)$$
$$+ikr\sqrt{1-k^2r^2}\,(\beta\bar\alpha'-\beta'\bar\alpha-\alpha\bar\beta'+\bar\beta\alpha')\Big]\sin\theta\sin\varphi\,,$$

or, using (48),

$$\bar\phi_2\phi_2+\bar\phi_4\phi_4=(a\bar a+b\bar b+c\bar c+d\bar d)+\cos\theta\,(c\bar c+d\bar d-a\bar a-b\bar b)$$
$$-\sin\theta\cos\varphi\,(a\bar c+\bar ac+b\bar d+\bar bd)+i\sin\theta\sin\varphi\,(a\bar c-\bar ac+b\bar d-\bar bd)\,,$$
$$\bar\phi_2\phi_2-\bar\phi_4\phi_4=\sqrt{1-k^2r^2}\,(a\bar b+\bar ab+c\bar d+\bar cd)+ikr(a\bar b-\bar ab+c\bar d-\bar cd)$$
$$+[\sqrt{1-k^2r^2}\,(-a\bar b-\bar ab+c\bar d+\bar cd)-ikr(-a\bar b+\bar ab+c\bar d-\bar cd)]\cos\theta$$
$$-[\sqrt{1-k^2r^2}\,(a\bar d+\bar ad+b\bar c+\bar bc)+ikr(b\bar c-\bar bc-a\bar d+\bar ad)]\sin\theta\cos\varphi$$
$$+i[\sqrt{1-k^2r^2}\,(a\bar d-\bar ad+b\bar c-\bar bc)+ikr(b\bar c+\bar bc-a\bar d-\bar ad)]\sin\theta\sin\varphi\,.$$

Moreover,

$$\bar{\phi}_1\phi_2=-\cos^2\frac{\theta}{2}e^{i\varphi}(i\sqrt{1-k^2r^2}\,\bar{\alpha}\beta-kr\bar{\alpha}\beta')+\sin^2\frac{\theta}{2}e^{-i\varphi}(i\sqrt{1-k^2r^2}\,\alpha\bar{\beta}+kr\alpha'\bar{\beta})$$

$$+\sin\frac{\theta}{2}\cos\frac{\theta}{2}[i\sqrt{1-k^2r^2}\,(\alpha\bar{\alpha}-\beta\bar{\beta})+kr(\bar{\alpha}\alpha'-\bar{\beta}\beta')]$$

i. e.

$$\begin{aligned}2\bar{\phi}_1\phi_2=&-e^{i\varphi}(i\sqrt{1-k^2r^2}\,\bar{\alpha}\beta+kr\bar{\alpha}\beta')+e^{-i\varphi}(i\sqrt{1-k^2r^2}\,\alpha\bar{\beta}+kr\alpha'\bar{\beta})\\&-[e^{i\varphi}(i\sqrt{1-k^2r^2}\,\bar{\alpha}\beta+kr\bar{\alpha}\beta')+e^{-i\varphi}(i\sqrt{1-k^2r^2}\,\alpha\bar{\beta}+kr\alpha'\bar{\beta})]\cos\theta\\&+[i\sqrt{1-k^2r^2}\,(\alpha\bar{\alpha}-\beta\bar{\beta})+kr(\bar{\alpha}\alpha'-\bar{\beta}\beta')]\sin\theta\\&=\cos\varphi\,[i\sqrt{1-k^2r^2}\,(-\bar{\alpha}\beta+\alpha\bar{\beta})+kr(-\bar{\alpha}\beta'+\alpha'\bar{\beta})]\\&-i\sin\varphi\,[i\sqrt{1-k^2r^2}\,(\bar{\alpha}\beta+\alpha\bar{\beta})+kr(\bar{\alpha}\beta'+\alpha'\bar{\beta})]\\&-\cos\theta\cos\varphi\,[i\sqrt{1-k^2r^2}\,(\bar{\alpha}\beta+\alpha\bar{\beta})+kr(\bar{\alpha}\beta'+\alpha'\bar{\beta})]\\&+i\cos\theta\sin\varphi\,[i\sqrt{1-k^2r^2}\,(-\bar{\alpha}\beta+\alpha\bar{\beta})+kr(-\bar{\alpha}\beta'+\alpha'\bar{\beta})]\\&+\sin\theta\,[i\sqrt{1-k^2r^2}\,(\alpha\bar{\alpha}-\beta\bar{\beta})+kr(\alpha\bar{\alpha}'-\bar{\beta}\beta')]\,,\end{aligned}$$

hence,

$$\begin{aligned}2(\bar{\phi}_1\phi_2+\phi_1\bar{\phi}_2)=&\cos\varphi\,[kr(\alpha'\bar{\beta}+\bar{\alpha}'\beta-\bar{\alpha}\beta'-\alpha\bar{\beta}')+2i\sqrt{1-k^2r^2}\,(\alpha\bar{\beta}-\bar{\alpha}\beta)]\\&+\sin\varphi\,[-ikr(\alpha'\bar{\beta}-\bar{\alpha}'\beta+\bar{\alpha}\beta'-\alpha\bar{\beta}')+2\sqrt{1-k^2r^2}\,(\alpha\bar{\beta}+\bar{\alpha}\beta)]\\&-\cos\theta\cos\varphi\cdot kr[(\alpha'\bar{\beta}+\bar{\alpha}'\beta)+(\bar{\alpha}\beta'+\alpha\bar{\beta}')]\\&+\cos\theta\sin\varphi\cdot ikr[(\alpha'\bar{\beta}-\bar{\alpha}'\beta)-(\bar{\alpha}\beta'-\alpha\bar{\beta}')]\\&+\sin\theta\cdot kr[(\bar{\alpha}\alpha'+\alpha\bar{\alpha}')-(\bar{\beta}\beta'+\beta\bar{\beta}')]\,,\end{aligned}$$

$$\begin{aligned}2(\bar{\phi}_1\phi_2-\phi_1\bar{\phi}_2)=&\cos\varphi\;kr(\alpha'\bar{\beta}-\bar{\alpha}'\beta-\bar{\alpha}\beta'+\alpha\bar{\beta}')-\sin\varphi\, ikr(\alpha'\bar{\beta}+\bar{\alpha}'\beta+\bar{\alpha}\beta'+\alpha\bar{\beta}')\\&+\cos\theta\cos\varphi\,[-kr(\alpha'\bar{\beta}-\bar{\alpha}'\beta+\bar{\alpha}\beta'-\alpha\bar{\beta}')-2i\sqrt{1-k^2r^2}\,(\alpha\bar{\beta}+\bar{\alpha}\beta)]\\&+\cos\theta\sin\varphi\,[ikr(\alpha'\bar{\beta}+\bar{\alpha}'\beta-\bar{\alpha}\beta'-\alpha\bar{\beta}')-2\sqrt{1-k^2r^2}\,(\alpha\bar{\beta}-\bar{\alpha}\beta)]\\&+\sin\theta\,[kr(\bar{\alpha}\alpha'-\alpha\bar{\alpha}'-\bar{\beta}\beta'+\beta\bar{\beta}')+2i\sqrt{1-k^2r^2}\,(\alpha\bar{\alpha}-\beta\bar{\beta})]\,.\end{aligned}$$

Also

$$\bar{\phi}_3\phi_4=\cos^2\frac{\theta}{2}e^{i\varphi}\bar{\alpha}'(i\sqrt{1-k^2r^2}\,\beta'+kr\beta)-\sin^2\frac{\theta}{2}e^{-i\varphi}\bar{\beta}'(i\sqrt{1-k^2r^2}\,\alpha'+kr\alpha)$$

$$+\sin\frac{\theta}{2}\cos\frac{\theta}{2}[-\bar{\alpha}'(i\sqrt{1-k^2r^2}\,\alpha'+kr\alpha)+\bar{\beta}'(i\sqrt{1-k^2r^2}\,\beta'+kr\beta)]$$

i. e.

$$\begin{aligned}2\bar{\phi}_3\phi_4=&e^{i\varphi}\bar{\alpha}'(i\sqrt{1-k^2r^2}\,\beta'+kr\beta)-e^{-i\varphi}\bar{\beta}'(i\sqrt{1-k^2r^2}\,\alpha'+kr\alpha)\\&+[e^{i\varphi}\bar{\alpha}'(i\sqrt{1-k^2r^2}\,\beta'+kr\beta)+e^{-i\varphi}\bar{\beta}'(i\sqrt{1-k^2r^2}\,\alpha'+kr\alpha)]\cos\theta\end{aligned}$$

$$+[i\sqrt{1-k^2r^2}(-\alpha'\bar{\alpha}'+\beta'\bar{\beta}')+kr(-\alpha\bar{\alpha}'+\beta\bar{\beta}')]\sin\theta$$

$$=\cos\varphi\,[kr(\bar{\alpha}'\beta-\alpha\bar{\beta}')+i\sqrt{1-k^2r^2}(\bar{\alpha}'\beta'-\alpha'\bar{\beta}')]$$

$$+\cos\theta\cos\varphi\,[kr(\bar{\alpha}'\beta+\alpha\bar{\beta}')+i\sqrt{1-k^2r^2}(\bar{\alpha}'\beta'+\alpha'\bar{\beta}')]$$

$$+i\sin\varphi\,[kr(\bar{\alpha}'\beta+\alpha\bar{\beta}')+i\sqrt{1-k^2r^2}(\bar{\alpha}'\beta'+\alpha'\bar{\beta}')]$$

$$+i\cos\theta\sin\varphi\,[kr(\bar{\alpha}'\beta-\alpha\bar{\beta}')+i\sqrt{1-k^2r^2}(\bar{\alpha}'\beta'-\alpha'\bar{\beta}')]$$

$$+\sin\theta\,[kr(-\alpha\bar{\alpha}'+\beta\bar{\beta}')+i\sqrt{1-k^2r^2}(\alpha'\bar{\alpha}'+\beta'\bar{\beta}')],$$

hence,

$$\begin{aligned}2(\bar{\phi}_3\phi_4+\phi_3\bar{\phi}_4)=&\cos\varphi\,[kr(\bar{\alpha}'\beta+\alpha'\bar{\beta}-\alpha\bar{\beta}'-\bar{\alpha}\beta')+2i\sqrt{1-k^2r^2}(\bar{\alpha}'\beta'-\alpha'\bar{\beta}')]\\&+\sin\varphi\,[ikr(\bar{\alpha}'\beta-\alpha'\bar{\beta}+\alpha\bar{\beta}'-\bar{\alpha}\beta')-2\sqrt{1-k^2r^2}(\bar{\alpha}'\beta'+\alpha'\bar{\beta}')]\\&+\cos\theta\cos\varphi\cdot kr(\bar{\alpha}'\beta+\alpha'\bar{\beta}+\alpha\bar{\beta}'+\bar{\alpha}\beta')\\&+\sin\theta\cdot kr(-\alpha\bar{\alpha}'-\bar{\alpha}\alpha'+\beta\bar{\beta}'+\bar{\beta}\beta')\\&+i\cos\theta\sin\varphi\cdot kr(\bar{\alpha}'\beta-\alpha'\bar{\beta}-\alpha\bar{\beta}'+\bar{\alpha}\beta')\end{aligned}$$

$$\begin{aligned}2(\bar{\phi}_3\phi_4-\phi_3\bar{\phi}_4)=&\cos\varphi\cdot kr(\bar{\alpha}'\beta-\alpha'\bar{\beta}-\alpha\bar{\beta}'+\bar{\alpha}\beta')+i\sin\varphi kr(\bar{\alpha}'\beta+\alpha'\bar{\beta}+\alpha\bar{\beta}'+\bar{\alpha}\beta')\\&+\cos\theta\cos\varphi\,[kr(\bar{\alpha}'\beta-\alpha'\bar{\beta}+\alpha\bar{\beta}'-\bar{\alpha}\beta')+2i\sqrt{1-k^2r^2}(\bar{\alpha}'\beta'+\alpha'\bar{\beta}')]\\&+\cos\theta\sin\varphi\,[ikr(\bar{\alpha}'\beta+\alpha'\bar{\beta}-\alpha\bar{\beta}'-\bar{\alpha}\beta')-2\sqrt{1-k^2r^2}(\bar{\alpha}'\beta'-\alpha'\bar{\beta}')]\\&+\sin\theta\,[kr(-\alpha\bar{\alpha}'+\bar{\alpha}\alpha'+\beta\bar{\beta}'-\bar{\beta}\beta')+2i\sqrt{1-k^2r^2}(-\alpha'\bar{\alpha}'+\beta'\bar{\beta}')].\end{aligned}$$

Using the equations above, from (55), we have

$$\begin{aligned}\phi^\dagger A\mathring{\gamma}_2\phi=&\cos\theta\cos\varphi[-kr(\alpha'\bar{\beta}-\bar{\alpha}'\beta+\bar{\alpha}\beta'-\alpha\bar{\beta}')\\&\qquad+i\sqrt{1-k^2r^2}(-\alpha\bar{\beta}-\bar{\alpha}\beta+\bar{\alpha}'\beta'+\alpha'\bar{\beta}')]\\&+\cos\theta\sin\varphi[ikr(\alpha'\bar{\beta}+\bar{\alpha}'\beta-\bar{\alpha}\beta'-\alpha\bar{\beta}')\\&\qquad-\sqrt{1-k^2r^2}(\alpha\bar{\beta}-\bar{\alpha}\beta+\bar{\alpha}'\beta'-\alpha'\bar{\beta}')]\\&+\sin\theta[kr(\bar{\alpha}\alpha'-\alpha\bar{\alpha}'-\bar{\beta}\beta'+\beta\bar{\beta}')+i\sqrt{1-k^2r^2}(\alpha\bar{\alpha}-\beta\bar{\beta}+\beta'\bar{\beta}'-\alpha'\bar{\alpha}')]\end{aligned}$$

$$\begin{aligned}-i\phi^\dagger A\mathring{\gamma}_2\gamma_5\phi=&\cos\varphi kr(\alpha'\bar{\beta}-\bar{\alpha}'\beta-\bar{\alpha}\beta'+\alpha\bar{\beta}')\\&\qquad-i\cos\theta\cos\varphi\sqrt{1-k^2r^2}(\alpha\bar{\beta}+\bar{\alpha}\beta+\bar{\alpha}'\beta'+\alpha'\bar{\beta}')\\&-i\sin\varphi kr(\alpha'\bar{\beta}+\bar{\alpha}'\beta+\bar{\alpha}\beta'+\alpha\bar{\beta}')\\&\qquad-\cos\theta\sin\varphi\sqrt{1-k^2r^2}(\alpha\bar{\beta}-\bar{\alpha}\beta-\bar{\alpha}'\beta'+\alpha'\bar{\beta}')\\&+i\sin\theta\sqrt{1-k^2r^2}(\alpha\bar{\alpha}-\beta\bar{\beta}-\beta'\bar{\beta}'+\alpha'\bar{\alpha}').\end{aligned}$$

$$\begin{aligned}-i\phi^\dagger A\mathring{\gamma}_3\phi=&\cos\varphi[kr(\alpha'\bar{\beta}+\bar{\alpha}'\beta-\bar{\alpha}\beta'-\alpha\bar{\beta}')+i\sqrt{1-k^2r^2}(\alpha\bar{\beta}-\bar{\alpha}\beta+\bar{\alpha}'\beta'-\alpha'\bar{\beta}')]\\&+\sin\varphi[-ikr(\alpha'\bar{\beta}-\bar{\alpha}'\beta+\bar{\alpha}\beta'-\alpha\bar{\beta}')+\sqrt{1-k^2r^2}(\alpha\bar{\beta}+\bar{\alpha}\beta-\bar{\alpha}'\beta'-\alpha'\bar{\beta}')]\end{aligned}$$

$$\begin{aligned}\phi^\dagger A\mathring{\gamma}_3\gamma_5\phi = & i\cos\varphi\sqrt{1-k^2r^2}\,(-\alpha\bar\beta+\bar\alpha\beta+\bar\alpha'\beta'-\alpha'\bar\beta') \\ & -\sin\varphi\sqrt{1-k^2r^2}\,(\alpha\bar\beta+\bar\alpha\beta+\bar\alpha'\beta'+\alpha'\bar\beta') \\ & +\cos\theta\cos\varphi kr(\alpha'\bar\beta+\bar\alpha'\beta+\bar\alpha\beta'+\alpha\bar\beta') \\ & -i\cos\theta\sin\varphi kr(\alpha'\bar\beta-\bar\alpha'\beta-\bar\alpha\beta'+\alpha\bar\beta') \\ & +\sin\theta kr(-\alpha\bar\alpha'-\bar\alpha\alpha'+\beta\bar\beta'+\bar\beta\beta').\end{aligned}$$

or, using (48),

$$\begin{aligned}\phi^\dagger A\mathring{\gamma}_2\phi = & -2i(a\bar d+\bar a d+b\bar c+\bar b c)\cos\theta\cos\varphi-2(a\bar d-\bar a d+b\bar c-\bar b c)\cos\theta\sin\varphi \\ & +2i(a\bar b+\bar a b-c\bar d-\bar c d)\sin\theta,\end{aligned}$$

$$\begin{aligned}-i\phi^\dagger A\mathring{\gamma}_2\gamma_5\phi = & -2kr(\bar a c-a\bar c-\bar b d+b\bar d)\cos\varphi \\ & \qquad -2i\sqrt{1-k^2r^2}\,(\bar a c+a\bar c+\bar b d+b\bar d)\cos\theta\cos\varphi \\ & -2ikr(\bar a c+a\bar c-\bar b d-b\bar d)\sin\varphi \\ & \qquad -2\sqrt{1-k^2r^2}\,(a\bar c-\bar a c+b\bar d-\bar b d)\cos\theta\sin\varphi \\ & +2i\sqrt{1-k^2r^2}\,(a\bar a+b\bar b-c\bar c-d\bar d)\sin\theta,\end{aligned}$$

$$-i\phi^\dagger A\mathring{\gamma}_3\phi = 2i(b\bar c-\bar b c+a\bar d-\bar a d)\cos\varphi+2(b\bar c+\bar b c+a\bar d+\bar a d)\sin\varphi,$$

$$\begin{aligned}\phi^\dagger A\mathring{\gamma}_3\gamma_5\phi = & -2i\sqrt{1-k^2r^2}\,(a\bar c-\bar a c+b\bar d-\bar b d)\cos\varphi \\ & \qquad +2kr(\bar a c+a\bar c-\bar b d-b\bar d)\cos\theta\cos\varphi \\ & -2\sqrt{1-k^2r^2}\,(a\bar c+\bar a c+b\bar d+\bar b d)\sin\varphi \\ & \qquad +2ikr(\bar a c-a\bar c-\bar b d+b\bar d)\cos\theta\sin\varphi \\ & +2kr(-a\bar a+b\bar b+c\bar c-d\bar d)\sin\theta,\end{aligned}$$

and

$$\begin{aligned}-i\phi^\dagger A\mathring{\gamma}_1\phi = & 2ikr(-a\bar b+\bar a b-c\bar d+\bar c d)-2\sqrt{1-k^2r^2}\sin\theta\cos\varphi(a\bar d+\bar a d+b\bar c+bc) \\ & +2\sqrt{1-k_2r^2}\,(-a\bar b-\bar a b+c\bar d+\bar c d)\cos\theta \\ & \qquad +2\sqrt{1-k^2r^2}\,i\sin\theta\sin\varphi(a\bar d-\bar a d+b\bar c-\bar b c),\end{aligned}$$

$$\begin{aligned}-i\phi^\dagger A\mathring{\gamma}_4\gamma_5\phi = & 2\sqrt{1-k^2r^2}\,(a\bar b+\bar a b+c\bar d+\bar c d)+2ikr\sin\theta\cos\varphi(\bar b c-b\bar c-\bar a d+a\bar d) \\ & +2ikr\cos\theta(a\bar b-\bar a b-c\bar d+\bar c d)-2kr\sin\theta\sin\varphi(\bar b c+b\bar c-\bar a d-a\bar d),\end{aligned}$$

$$\phi^\dagger A\mathring{\gamma}_4\phi = 2(a\bar a+b\bar b+c\bar c+d\bar d),$$

$$\begin{aligned}\phi^\dagger A\mathring{\gamma}_1\gamma_5\phi = & 2\cos\theta(a\bar a+b\bar b-c\bar c-d\bar d) \\ & +2\sin\theta\cos\varphi(a\bar c+\bar a c+b\bar d+\bar b d) \\ & +2i\sin\theta\sin\varphi(\bar a c-a\bar c+\bar b d-b\bar d).\end{aligned}$$

Then, by (54), we have

$$\left.\begin{aligned} v^r &= -kr\sqrt{1-k^2r^2}\,T_0+(1-k^2r^2)\cos\theta\cdot T_1-(1-k^2r^2)\sin\theta\cos\varphi\cdot T_2 \\ &\quad +(1-k^2r^2)\sin\theta\sin\varphi\cdot T_3\,, \\ v^\theta &= -\frac{1}{r}\cos\theta\cos\varphi\cdot T_2+\frac{1}{r}\cos\theta\sin\varphi\cdot T_3-\frac{1}{r}\sin\theta\cdot T_1\,, \\ v^\varphi &= \frac{1}{r\sin\theta}(\cos\varphi\cdot T_3+\sin\varphi\cdot T_2)\,, \\ v^t &= \frac{1}{\sqrt{1-k^2r^2}}T_4\,, \end{aligned}\right\} \tag{55}$$

where

$$\left.\begin{aligned} &T_0=2i(a\bar{b}-\bar{a}b+c\bar{d}-\bar{c}d)\,, && T_1=2\,(-a\bar{b}-\bar{a}b+c\bar{d}+\bar{c}d)\,, \\ &T_2=2\,(a\bar{d}+\bar{a}d+b\bar{c}+\bar{b}c)\,, && T_3=2i(a\bar{d}-\bar{a}d+b\bar{c}-\bar{b}c)\,, \\ &T_4=2\,(a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d})\,, \end{aligned}\right\} \tag{56}$$

and

$$\left.\begin{aligned} iv^r_{\cdot 5} &= [\cos\theta\cdot S_0+\sin\theta\cos\varphi\cdot S_1+\sin\theta\sin\varphi\cdot S_2]\sqrt{1-k^2r^2}\,, \\ iv^\theta_{\cdot 5} &= -k\cos\varphi\cdot S_3+k\sin\varphi\cdot S_4+\frac{\sqrt{1-k^2r^2}}{r}(\cos\theta\cos\varphi\cdot S_1 \\ &\quad +\cos\theta\sin\varphi\cdot S_2-\sin\theta\cdot S_0)\,, \\ iv^\varphi_{\cdot 5} &= \frac{\sqrt{1-k^2r^2}}{r\sin\theta}(\cos\varphi\cdot S_2-\sin\varphi\cdot S_1)+k\cot\theta\,(\cos\varphi\cdot S_4+\sin\varphi\cdot S_3)+kS_5\,, \\ iv^t_{\cdot 5} &= -S_6-\frac{kr}{\sqrt{1-k^2r^2}}(\cos\theta\cdot S_7+\sin\theta\cos\varphi\cdot S_8-\sin\theta\sin\varphi\cdot S_9)\,, \end{aligned}\right\} \tag{57}$$

where

$$\left.\begin{aligned} &S_0=2\,(a\bar{a}+b\bar{b}-c\bar{c}-d\bar{d})\,. && S_1=2\,(a\bar{c}+\bar{a}c+b\bar{d}+\bar{b}d)\,, \\ &S_2=2i(\bar{a}c-a\bar{c}+\bar{b}d-b\bar{d})\,, && S_3=2i(\bar{a}c-a\bar{c}-\bar{b}d+b\bar{d})\,, \\ &S_4=2\,(\bar{a}c+a\bar{c}-\bar{b}d-b\bar{d})\,, && S_5=2\,(-a\bar{a}+b\bar{b}+c\bar{c}-d\bar{d})\,, \\ &S_6=2\,(a\bar{b}+\bar{a}b+c\bar{d}+\bar{c}d)\,, && S_7=2i(a\bar{b}-\bar{a}b-c\bar{d}+\bar{c}d)\,, \\ &S_8=2i(\bar{b}c-b\bar{c}-\bar{a}d+a\bar{d})\,, && S_9=2\,(\bar{b}c+b\bar{c}-\bar{a}d-a\bar{d})\,. \end{aligned}\right\} \tag{58}$$

In cylindrical coordinates $\rho=r\sin\theta$, $z=r\cos\theta$, φ and t, v^l and $v^l_{\cdot 5}$ are expressed as follows:

$$\left.\begin{aligned} v^\rho &= -k\rho\sqrt{1-k^2r^2}\,T_0-k^2\rho zT_1-(1-k^2\rho^2)\cos\varphi\cdot T_2+(1-k^2\rho^2)\sin\varphi\cdot T_3\,, \\ v^z &= -kz\sqrt{1-k^2r^2}\,T_0+(1-k^2z^2)T_1+k^2\rho z\cos\varphi\cdot T_2-k^2\rho z\sin\varphi\cdot T_3\,, \\ v^\varphi &= \frac{\cos\varphi}{\rho}T_3+\frac{\sin\varphi}{\rho}T_2\,, \\ v^t &= \frac{1}{\sqrt{1-k^2r^2}}T_4\,, \end{aligned}\right\} \tag{59}$$

and

$$\left.\begin{aligned}
&iv^{\rho}_{\cdot 5}=(\cos\varphi\cdot S_1+\sin\varphi\cdot S_2)\sqrt{1-k^2r^2}-kz\cos\varphi\cdot S_3+kz\sin\varphi\cdot S_4\,,\\
&iv^{z}_{\cdot 5}=\sqrt{1-k^2r^2}\,S_0+k\rho\cos\varphi\cdot S_3-k\rho\sin\varphi\cdot S_4\,,\\
&iv^{\varphi}_{\cdot 5}=\frac{\sqrt{1-k^2r^2}}{\rho}(\cos\varphi\cdot S_2-\sin\varphi\cdot S_1)+\frac{kz}{\rho}(\cos\varphi\cdot S_4+\sin\varphi\cdot S_3)+kS_5\,,\\
&iv^{t}_{\cdot 5}=-S_6-\frac{k}{\sqrt{1-k^2r^2}}(zS_7+\rho\cos\varphi\cdot S_8-\rho\sin\varphi\cdot S_9)\,.
\end{aligned}\right\}\tag{60}$$

Using (40), T's and S's are expressed as follows:

$$\left.\begin{aligned}
-\frac{1}{2}T_0=&\ (l\bar{l}+p\bar{p})(\bar{\sigma}+\sigma)e^{\bar{\tau}+\tau}-(m\bar{m}+q\bar{q})(\bar{\epsilon}+\epsilon)e^{-\bar{\tau}-\tau}\\
&+(\bar{l}m+\bar{p}q)(\bar{\sigma}-\epsilon)e^{\bar{\tau}-\tau}-(l\bar{m}+p\bar{q})(\bar{\epsilon}-\sigma)e^{\tau-\bar{\tau}}\,,\\
\frac{1}{2}T_1=&\ i[(l\bar{l}-p\bar{p})(\bar{\sigma}-\sigma)e^{\bar{\tau}+\tau}-(m\bar{m}-q\bar{q})(\bar{\epsilon}-\epsilon)e^{-\bar{\tau}-\tau}\\
&+(\bar{l}m-\bar{p}q)(\bar{\sigma}+\epsilon)e^{\bar{\tau}-\tau}-(l\bar{m}-p\bar{q})(\bar{\epsilon}+\sigma)e^{\tau-\bar{\tau}}]\,,\\
\frac{1}{2}T_2=&\ i[(p\bar{l}+\bar{p}l)(\bar{\sigma}-\sigma)e^{\bar{\tau}+\tau}-(q\bar{m}+\bar{q}m)(\bar{\epsilon}-\epsilon)e^{-\bar{\tau}-\tau}\\
&+(q\bar{l}+\bar{p}m)(\bar{\sigma}+\epsilon)e^{\bar{\tau}-\tau}-(p\bar{m}+\bar{q}l)(\bar{\epsilon}+\sigma)e^{\tau-\bar{\tau}}]\,,\\
\frac{1}{2}T_3=&-(p\bar{l}-\bar{p}l)(\bar{\sigma}-\sigma)e^{\bar{\tau}+\tau}+(q\bar{m}-\bar{q}m)(\bar{\epsilon}-\epsilon)e^{-\bar{\tau}-\tau}\\
&-(q\bar{l}-\bar{p}m)(\bar{\sigma}+\epsilon)e^{\bar{\tau}-\tau}+(p\bar{m}-\bar{q}l)(\bar{\epsilon}+\sigma)e^{\tau-\bar{\tau}}\,,\\
\frac{1}{2}T_4=&\ (l\bar{l}+p\bar{p})(1+\sigma\bar{\sigma})e^{\bar{\tau}+\tau}+(q\bar{q}+m\bar{m})(1+\epsilon\bar{\epsilon})e^{-\bar{\tau}-\tau}\\
&+(\bar{p}q+\bar{l}m)(1-\bar{\sigma}\epsilon)e^{\bar{\tau}-\tau}+(p\bar{q}+l\bar{m})(1-\sigma\bar{\epsilon})e^{\tau-\bar{\tau}}\,,\\
\frac{1}{2}S_0=&\ (p\bar{p}-l\bar{l})(1+\sigma\bar{\sigma})e^{\bar{\tau}+\tau}+(q\bar{q}-m\bar{m})(1+\epsilon\bar{\epsilon})e^{-\bar{\tau}-\tau}\\
&+(\bar{p}q-\bar{l}m)(1-\bar{\sigma}\epsilon)e^{\bar{\tau}-\tau}+(p\bar{q}-l\bar{m})(1-\sigma\bar{\epsilon})e^{\tau-\bar{\tau}}\,,\\
\frac{1}{2}S_1=&\ (\bar{p}l+p\bar{l})(1+\sigma\bar{\sigma})e^{\bar{\tau}+\tau}+(\bar{q}m+q\bar{m})(1+\epsilon\bar{\epsilon})e^{-\bar{\tau}-\tau}\\
&+(\bar{p}m+q\bar{l})(1-\bar{\sigma}\epsilon)e^{\bar{\tau}-\tau}+(\bar{q}l+p\bar{m})(1-\sigma\bar{\epsilon})e^{\tau-\bar{\tau}}\,,\\
\frac{1}{2}S_2=&\ i[(\bar{p}l-p\bar{l})(1+\sigma\bar{\sigma})e^{\bar{\tau}+\tau}+(\bar{q}m-q\bar{m})(1+\epsilon\bar{\epsilon})e^{-\bar{\tau}-\tau}\\
&+(\bar{p}m-q\bar{l})(1-\bar{\sigma}\epsilon)e^{\bar{\tau}-\tau}+(\bar{q}l-p\bar{m})(1-\sigma\bar{\epsilon})e^{\tau-\bar{\tau}}]\,,\\
\frac{1}{2}S_3=&\ i[(\bar{p}l-p\bar{l})(1-\sigma\bar{\sigma})e^{\bar{\tau}+\tau}+(\bar{q}m-q\bar{m})(1-\epsilon\bar{\epsilon})e^{-\bar{\tau}-\tau}\\
&+(\bar{p}m-q\bar{l})(1+\bar{\sigma}\epsilon)e^{\bar{\tau}-\tau}+(\bar{q}l-p\bar{m})(1+\sigma\bar{\epsilon})e^{\tau-\bar{\tau}}]\,,\\
\frac{1}{2}S_4=&\ (\bar{p}l+p\bar{l})(1-\sigma\bar{\sigma})e^{\bar{\tau}+\tau}+(\bar{q}m+q\bar{m})(1-\epsilon\bar{\epsilon})e^{-\bar{\tau}-\tau}\\
&+(\bar{p}m+q\bar{l})(1+\bar{\sigma}\epsilon)e^{\bar{\tau}-\tau}+(\bar{q}l+p\bar{m})(1+\sigma\bar{\epsilon})e^{\tau-\bar{\tau}}\,,
\end{aligned}\right\}\tag{61}$$

$$\left.\begin{aligned}
-\frac{1}{2}S_5 &= \quad (p\bar{p}-l\bar{l})(1-\sigma\bar{\sigma})e^{\bar{\tau}+\tau}+(q\bar{q}-m\bar{m})(1-\epsilon\bar{\epsilon})^{-\bar{\tau}-\tau} \\
&\quad +(\bar{p}q-\bar{l}m)(1+\bar{\sigma}\epsilon)e^{\bar{\tau}-\tau}+(p\bar{q}-l\bar{m})(1+\sigma\bar{\epsilon})e^{\tau-\bar{\tau}}, \\
\frac{1}{2}S_6 &= i[(l\bar{l}+p\bar{p})(\bar{\sigma}-\sigma)e^{\bar{\tau}+\tau}-(m\bar{m}+q\bar{q})(\bar{\epsilon}-\epsilon)e^{-\bar{\tau}-\tau} \\
&\quad +(\bar{l}m+\bar{p}q)(\bar{\sigma}+\epsilon)e^{\bar{\tau}-\tau}-(l\bar{m}+p\bar{q})(\bar{\epsilon}+\sigma)e^{\tau-\bar{\tau}}], \\
\frac{1}{2}S_7 &= \quad (l\bar{l}-p\bar{p})(\bar{\sigma}+\sigma)e^{\bar{\tau}+\tau}-(m\bar{m}-q\bar{q})(\bar{\epsilon}+\epsilon)e^{-\bar{\tau}-\tau} \\
&\quad +(\bar{l}m-\bar{p}q)(\bar{\sigma}-\epsilon)e^{\bar{\tau}-\tau}-(l\bar{m}-p\bar{q})(\bar{\epsilon}-\sigma)e^{\tau-\bar{\tau}}, \\
\frac{1}{2}S_8 &= -(p\bar{l}+\bar{p}l)(\bar{\sigma}+\sigma)e^{\bar{\tau}+\tau}+(q\bar{m}+\bar{q}m)(\bar{\epsilon}+\epsilon)e^{-\bar{\tau}-\tau} \\
&\quad -(q\bar{l}+\bar{p}m)(\bar{\sigma}-\epsilon)e^{\bar{\tau}-\tau}+(p\bar{m}+\bar{q}l)(\bar{\epsilon}-\sigma)e^{\tau-\bar{\tau}}, \\
-\frac{1}{2}S_9 &= i[(p\bar{l}-\bar{p}l)(\bar{\sigma}+\sigma)e^{\bar{\tau}+\tau}-(q\bar{m}-\bar{q}m)(\bar{\epsilon}+\epsilon)^{-\bar{\tau}-\tau} \\
&\quad +(q\bar{l}-\bar{p}m)(\bar{\sigma}-\epsilon)e^{\bar{\tau}-\tau}-(p\bar{m}-\bar{q}l)(\bar{\epsilon}-\sigma)e^{\tau-\bar{\tau}}],
\end{aligned}\right|$$

where $\tau\equiv\frac{1}{2}\sqrt{1+L^2}\,kt$, $\sigma\equiv\sqrt{1+L^2}+L$, $\epsilon\equiv\sqrt{1+L^2}-L$, and l, m, p, q are any constants.

Note V.

Theorem. *The most general form of the vector U^l which is made from two vectors $\underset{1}{u}^l$ and $\underset{2}{u}^l$ is given by*

$$U^l=\underset{1}{a}\underset{1}{u}^l+\underset{2}{a}\underset{2}{u}^l \qquad (l=1,\dots\dots,n)$$

Proof. Let the required form of U^l be

$$U^l=f^l(\underset{1}{u}^1,\dots\dots,\underset{1}{u}^n;\ \underset{2}{u}^1,\dots\dots,\underset{2}{u}^n), \qquad (l=1,\dots\dots,n),$$

or, for brevity,

$$U=f(\underset{1}{u};\ \underset{2}{u}).$$

Since U is a vector, in any linear transformations $x'=Tx$, it must be true that

$$Tf(\underset{1}{u};\ \underset{2}{u})=f(T\underset{1}{u};\ T\underset{2}{u}). \tag{L}$$

Specially, when T has the forms:

$$T=\begin{pmatrix} 1 & 0\cdots 0 \\ 0 & \\ \vdots & 0 \\ 0 & \end{pmatrix}, \qquad T=\begin{pmatrix} 0, & 1, 0\cdots 0 \\ 0 & \\ \vdots & 0 \\ 0 & \end{pmatrix}, \qquad \text{etc.,}$$

respectively, (L) becomes

$$f^1=f^1(\underset{1}{u}^1, 0, \ldots, 0\,;\ \underset{2}{u}^1, 0, \ldots, 0)\,,$$
$$f^2=f^1(\underset{1}{u}^2, 0, \ldots, 0\,;\ \underset{2}{u}^2, 0, \ldots, 0)\,,$$
etc.,

respectively. Therefore $f^1, f^2, \ldots$ are the same functions of $(\underset{1}{u}^1, \underset{2}{u}^1)$, $(\underset{1}{u}^2, \underset{2}{u}^2) \ldots$, i. e.

$$f^l=F(\underset{1}{u}^l, \underset{2}{u}^l) \tag{F}$$

Further, when

$$T=\begin{pmatrix} a, b, 0 \cdots 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \ \end{pmatrix},$$

(L) becomes

$$aF(\underset{1}{u}^1, \underset{2}{u}^1)+bF(\underset{1}{u}^2, \underset{2}{u}^2)=F(a\underset{1}{u}^1+b\underset{1}{u}^2, a\underset{2}{u}^1+b\underset{2}{u}^2)\,.$$

If we choose u^l, at a point, such that

$$\underset{1}{u}^1=1\,, \quad \underset{1}{u}^2=0\,; \quad \underset{2}{u}^1=0\,, \quad \underset{2}{u}^2=1\,,$$

the equation above becomes

$$aF(1, 0)+bF(0, 1)=F(a, b)\,,$$

which shows that $F(a, b)$ is the form of $\underset{1}{a}a+\underset{1}{b}b$. Therefore by (F) we have

$$f^l=\underset{1}{a}\underset{1}{u}^l+\underset{2}{a}\underset{2}{u}^l\,.$$

Note VI.

By the assumptions $\mathrm{II_N}$ and $\mathrm{III_N}$, $u^r : u^\theta : u^\varphi$ must not contain φ and t. Under this condition, from (53), (55), (57) and (61), we have the following four cases:

$$\text{(I)}\quad p=q=m=0\ (l\neq 0)\,, \qquad \text{(II)}\quad l=m=q=0\ (p\neq 0)\,,$$
$$\text{(III)}\quad p=q=l=0\ (m\neq 0)\,, \qquad \text{(IV)}\quad l=m=p=0\ (q\neq 0)\,.$$

Now we take the case (I).(1) In this case we have, from (55) and (57),

$$v^r=2l\bar{l}e^{\bar{\tau}+\tau}[kr\sqrt{1-k^2r^2}(\bar{\sigma}+\sigma)+(1-k^2r^2)\cos\theta i(\bar{\sigma}-\sigma)]\,,$$
$$v^\theta=2l\bar{l}e^{\bar{\tau}+\tau}\cdot\frac{\sin\theta}{r}i(\sigma-\bar{\sigma})\,,$$
$$v^\varphi=0\,,$$
$$v^t=2l\bar{l}e^{\bar{\tau}+\tau}\cdot\frac{1}{\sqrt{1-k^2r^2}}(1+\sigma\bar{\sigma})\,,$$

(1) The other three cases (II), (III), and (IV) are treated by the same way obtaining the same result as in the case (I), so we shall omitt the treatment of them.

and $$iv^r_{.5}=2l\bar{l}e^{\bar{\tau}+\tau}\cdot(-1)\cos\theta\sqrt{1-k^2r^2}(1+\sigma\bar{\sigma}),$$

$$iv^\theta_{.5}=2l\bar{l}e^{\bar{\tau}+\tau}\cdot\frac{\sin\theta}{r}\sqrt{1-k^2r^2}(1+\sigma\bar{\sigma}),$$

$$iv^\varphi_{.5}=2l\bar{l}e^{\bar{\tau}+\tau}\cdot k(1-\sigma\bar{\sigma}),$$

$$iv^t_{.5}=2l\bar{l}e^{\bar{\tau}+\tau}\cdot\left[i(\sigma-\bar{\sigma})-\frac{kr}{\sqrt{1-k^2r^2}}\cos\theta(\bar{\sigma}+\sigma)\right].$$

Then $\underset{1}{u}^l$ is determined by the equation

$$\underset{1}{u}^l=\cosh\eta\cdot v^l+i\sinh\eta\cdot v^l_{.5}.$$

$\underset{1}{u}^l$ is obtained by putting $\eta\to-\eta$, $L\to-L$ $\left(\text{accordingly } \sigma\to\frac{1}{\sigma}\right)$ and $l\to l'$ in the aboves.

Now, we will normalize $\underset{1}{u}^l$ and $\underset{2}{u}^l$ such that

$$\nabla_i(\alpha_1\underset{1}{u}^i)=0,\qquad \nabla_i(\alpha_2\underset{2}{u}^i)=0.$$

Disregarding real common factor, $\underset{1}{u}^l$ is expressed as

$$\left.\begin{aligned}
&\underset{1}{u}^r=kr\sqrt{1-k^2r^2}(\bar{\sigma}+\sigma)+(1-k^2r^2)\cos\theta\cdot i(\bar{\sigma}-\sigma)-\mu\sqrt{1-k^2r^2}\cos\theta(1+\sigma\bar{\sigma}),\\
&\underset{1}{u}^\theta=\frac{\sin\theta}{r}i(\sigma-\bar{\sigma})+\mu\frac{\sin\theta}{r}\sqrt{1-k^2r^2}(1+\sigma\bar{\sigma}),\\
&\underset{1}{u}^\varphi=\mu k(1-\sigma\bar{\sigma}),\\
&\underset{1}{u}^t=\frac{1}{\sqrt{1-k^2r^2}}(1+\sigma\bar{\sigma})-\mu i(\bar{\sigma}-\sigma)-\mu\frac{kr}{\sqrt{1-k^2r^2}}\cos\theta(\bar{\sigma}+\sigma),
\end{aligned}\right\}\quad(1)$$

or, in cylindrical coordinates,

$$\left.\begin{aligned}
&\underset{1}{u}^\rho=k\rho\sqrt{1-k^2r^2}(\bar{\sigma}+\sigma)\\
&\underset{1}{u}^z=kz\sqrt{1-k^2r^2}(\bar{\sigma}+\sigma)+i(1-k^2r^2)(\bar{\sigma}-\sigma)-\mu\sqrt{1-k^2r^2}(1+\sigma\bar{\sigma})\\
&\underset{1}{u}^\varphi=\mu k(1-\sigma\bar{\sigma})\\
&\underset{1}{u}^t=\frac{1}{\sqrt{1-k^2r^2}}(1+\sigma\bar{\sigma})-\mu i(\bar{\sigma}-\sigma)-\mu\frac{kz}{\sqrt{1-k^2r^2}}(\bar{\sigma}+\sigma)
\end{aligned}\right\}\quad(2)$$

where $$\mu\equiv\tanh\eta,\qquad(3)$$

and α_1 is the solution of

$$\alpha_1(\nabla_i\underset{1}{u}^i)+\underset{1}{u}^i\frac{\partial\alpha_1}{\partial x^i}=0.\qquad(4)$$

Since, by actual calculation,

$$\nabla_i\underset{1}{u}^i=\frac{k(3-4k^2r^2)}{\sqrt{1-k^2r^2}}(\bar{\sigma}+\sigma)-kz\left\{4ki(\bar{\sigma}-\sigma)-\mu k\frac{1+\sigma\bar{\sigma}}{\sqrt{1-k^2r^2}}\right\},$$

a particular solution of α_1 is given by, regarding $1-k^2r^2 \doteq 1$,

$$[kz(\bar{\sigma}+\sigma)+i(\bar{\sigma}-\sigma)-\mu(1+\bar{\sigma}\sigma)]^{-1}\rho^{-2}.$$

Hence the general form of α_1 is given by

$$\alpha_1=[kz(\bar{\sigma}+\sigma)+i(\bar{\sigma}-\sigma)-\mu(1+\sigma\bar{\sigma})]^{-1}\rho^{-2}F_1, \tag{5}$$

F_1 being the general solution of

$$u^l\frac{\partial F_1}{\partial x^l}=0. \tag{6}$$

But, since (6) has the three independent solutions:

$$\left.\begin{aligned}&\frac{\rho}{kz(\sigma+\bar{\sigma})+i(\bar{\sigma}-\sigma)-\mu(1+\sigma\bar{\sigma})}, \qquad \rho e^{-\frac{\sigma+\bar{\sigma}}{\mu(1-\sigma\bar{\sigma})}\varphi},\\ &e^{-\frac{(\sigma+\bar{\sigma})k}{(1+\sigma\bar{\sigma})(1-\mu^2)}(\mu z+t)}[k(\sigma+\bar{\sigma})z+i(\bar{\sigma}-\sigma)-\mu(1+\sigma\bar{\sigma})],\end{aligned}\right\} \tag{7}$$

F_1 is an arbitrary function of (7). Similarly, α_2 is obtained by putting $\mu\rightarrow-\mu$, $L\rightarrow-L$ $\left(\text{accordingly } \sigma\rightarrow\frac{1}{\sigma}\right)$ in the above. Namely α_2 is given by

$$\alpha_2=[kz(\sigma+\bar{\sigma})+i(\sigma-\bar{\sigma})+\mu(1+\sigma\bar{\sigma})]^{-1}\cdot\rho^{-2}F_2,$$

F_2 being an arbitrary function of

$$\frac{\rho}{kz(\sigma+\bar{\sigma})+i(\sigma-\bar{\sigma})+\mu(1+\sigma\bar{\sigma})}, \qquad \rho e^{-\frac{\sigma+\bar{\sigma}}{\mu(1-\sigma\bar{\sigma})}\varphi},$$

$$e^{\frac{(\sigma+\bar{\sigma})k}{(1+\sigma\bar{\sigma})(1-\mu^2)}(\mu z-t)}[k(\sigma+\bar{\sigma})z-i(\bar{\sigma}-\sigma)+\mu(1+\sigma\bar{\sigma})].$$

Then taking into account of the assumption $\mathrm{I_N}$, it must be that

$$F_1=-F_2=F$$

i.e. F is an arbitrary function of $\rho e^{-\frac{\sigma+\bar{\sigma}}{\mu(1-\sigma\bar{\sigma})}\varphi}$. So that Neglecting k^2r^2 compared with 1, the actual form of u^l is reduced to the form

$$\left.\begin{aligned}&u^z=0,\\ &u^\rho=\lambda\rho^{-2}\cdot k\rho(\bar{\sigma}+\sigma)[-i(\bar{\sigma}-\sigma)+\mu(1+\sigma\bar{\sigma})],\\ &u^\varphi=\lambda\rho^{-2}\cdot\mu k(1-\sigma\bar{\sigma})[-i(\bar{\sigma}-\sigma)+\mu(1+\sigma\bar{\sigma})],\\ &u^t=\lambda\rho^{-2}\cdot[\{1+\sigma\bar{\sigma}-\mu i(\bar{\sigma}-\sigma)\}\{-i(\bar{\sigma}-\sigma)+\mu(1+\sigma\bar{\sigma})\}-\mu k^2z^2(\bar{\sigma}+\sigma)^2].\end{aligned}\right\} \tag{8}$$

where λ is any real factor satisfying

$$u^l\frac{\partial\lambda}{\partial x^l}=0.$$

Since the solutions of the differential equations

$$\frac{dz}{u^z}=\frac{d\rho}{u^\rho}=\frac{d\varphi}{u^\varphi}=\frac{dt}{u^t}$$

are obtained as

$$\left.\begin{array}{l} z=c_1\,,\quad \rho e^{-\frac{(\sigma+\bar\sigma)}{\mu(1-\sigma\bar\sigma)}\varphi}=c_2\,,\qquad \rho e^{-ct}=c_3 \\ \left(c\equiv\dfrac{k(\bar\sigma+\sigma)[\mu(1+\sigma\bar\sigma)-i(\bar\sigma-\sigma)]}{\mu(1+\sigma\bar\sigma)^2-i(\bar\sigma-\sigma)(1+\sigma\bar\sigma)(1+\mu^2)-\mu(\bar\sigma-\sigma)^2-\mu k^2c_1^2(\bar\sigma+\sigma)^2}\right)\end{array}\right\} \quad (9)$$

c_1, c_2, c_3 being constants of integration, λ is an arbitrary function of (9).

Moreover, from the assumption $\mathrm{IV_N}$, particle density $D=\sqrt{g_{ij}u^iu^j}$ i. e.

$$D=\lambda\rho^{-2}[1+\sigma\bar\sigma-\mu i(\bar\sigma-\sigma)] \quad (10)$$

must not contain t. Hence λ is an arbitrary function of z and $\rho e^{-\frac{(\sigma+\bar\sigma)}{\mu(1-\sigma\bar\sigma)}\varphi}$.

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