Wave Geometry No. 41

On Some Characters of Time.

By

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1. Introduction.

In Wave Geometry, we saw the validity(1) of taking $u^i \equiv \psi^\dagger A\gamma^i\psi$ as the momentum density vector of the matter constituting space-time, the existence of that matter being asserted by ψ, the solution of the fundamental equation:

$$\left(\frac{\partial}{\partial x^i} - \Gamma_i\right)\psi = \Sigma_i\psi\,;$$

and by using this u^i we were able to establish a promising theory concerning universe.

The main reasons for taking u^i as momentum density vector of matter were the four following(2):

(1) The equation of motion of matter should be included as a part of the field theory in consideration.

(2) When we choose the coordinates so that

$$ds^2 = -\sum_{a,b=1}^{3} g_{ab}dx^a dx^b + g_{44}(dx^4)^2\,, \qquad g_{ab},\, g_{44} > 0\,, \tag{1}$$

and identify x^4 with the coordinate t, the fourth component of u^i becomes $\psi^\dagger\psi$ except for a real factor $\frac{1}{\sqrt{g_{44}}}$, $\psi\psi^\dagger$ expressing the meaning of density or existence probability of matter represented by ψ.

(3) From the relation(3):

$$g_{ij}u^i u^j \equiv M^2 + N^2 > 0 \qquad (\text{where } M = \psi^\dagger A\psi,\ N = \psi^\dagger A\gamma_5\psi)\,,$$

if we express by $'u^i$ the component of the vector u^i in a Minkowski local coordinate system at any point of the space-time whose metric is gived by (1), then we have the relation:

$$-('u^1)^2 - ('u^2)^2 - ('u^3)^2 + ('u^4)^2 > 0\,,$$

proving that the above-given relation satisfies the condition that u^i can be taken to represent a momentum density vector.

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(1) T. Iwatsuki, Y. Mimura and T. Sibata; This Journal **8** (1938), 187 (W. G. No. 27).
(2) loc. cit., 189, 192.
(3) T. Sibata; This Journal, **8** (1938), 175 (W. G. No. 26).

(4) The curve (4-dimensional) generated by u^i is regarded as most appropriate in representing geodesics or trajectories of particle in both gravitational and non-gravitational field defined from the angle of Wave Geometry.

In this paper, we shall show that the u^i thus taken as representing momentum density vector has physically some remarkable characters in connection with interpretations of time.

2. Unindirectional character of time in physical phenomena.

From the identification of u^i with momentum density vector of matter, we have equations of motion of matter:

$$\rho \frac{dx^i}{ds} = u^i , \tag{2}$$

(ρ being interpreted as the invariant density).
But from the relation:

$$g_{ij}u^i u^j = M^2 + N^2 > 0 ,$$

we have

$$\rho^2 = M^2 + N^2 > 0 ,$$

from which

$$\rho = +\sqrt{M^2+N^2} \qquad \text{(real)};$$

here + sign must be taken from the meaning of ρ—an invariant density.

If we take coordinates as shown in (1) and identify $x^4 = t$, for the fourth component of u^i, we have

$$\frac{dt}{ds} = \frac{1}{\sqrt{g_{44}}\sqrt{M^2+N^2}} (\psi^2\psi) = \frac{1}{\sqrt{g_{44}}\sqrt{M^2+N^2}} (\psi_1\bar{\psi}_1 + \cdots\cdots + \psi_4\bar{\psi}_4) > 0 .$$

So that dt and ds must have the same sign. But since from (2), $ds(=\sqrt{g_{ij}dx^i dx^j})$ must be real and does not become to vanish (for, ρ, dx^i, u^i are all real), ds must retain the same sign $+\sqrt{g_{ij}dx^i dx^j}$ or $-\sqrt{g_{ij}dx^i dx^j}$ while we are dealting with the equation: $\rho \frac{dx^i}{ds} = u^i$, so that the dt in consideration must have a definite sign.

This may be physically interpreted as follows:

The phenomenon described by u^i, a flux of matter so to speak, takes place in a definite direction of time without changing its sense.

In reality, all natural phenomena are irreversible with respect to time. It is a remarkable fact, and how could we mathematically describe about this fact is a problem not yet solved and might be one of the fundamental problems of mathematical physits.

Nevertheless, judging from the result above obtained, it seems likely we have a suggestion that the Wave Geometry might present a key to this problem, the irreversibility of time.

Remark: Here the absolute sign of dt, i.e., $dt>0$, does not so much indicate essential importance as it is mere mathematical convention showing whether the positive direction of the coordinate time is taken in the sense of the lapse of actual time or not.

3. Physical meaning of eigenvalue for the operator $ds=\gamma_i dx^i$ with respect to $ds\psi$.

In Wave Geometry in which we took as the foundation on the metric of the form $ds\psi$ which is composed of the metric-operator $ds(\equiv\gamma_i dx^i)$ and the operand ψ, the eigenvalue μ of the operator ds obtained through the equation:

$$ds\psi=\mu I\psi \tag{2}$$

must have important meanings both physically and geometrically.

Hitherto, in this geometry, the four eigenvalues obtained from equation (2) above,

$$+\sqrt{g_{ij}dx^i dx^j}\,,\quad +\sqrt{g_{ij}dx^i dx^j}\,,\quad -\sqrt{g_{ij}dx^i dx^j}\,,\quad -\sqrt{g_{ij}dx^i dx^j}$$

were interpreted[1] so as to give the values of actual length realized by the metric-operator $ds(=\gamma_i dx^i)$, while the corresponding eigenfunction ψ was observed so as to indicate the existence or existence-probability of value μ of the length measured. This is physically interpreted, as the physical space whose ordinary form of metric is given by the eigenvalue $\mu=\sqrt{g_{ij}dx^i dx^j}$ is conceivable by the fact that ψ, i.e. the matter constituting space, exists. But the physical meaning for the sign of eigenvalue $+\sqrt{g_{ij}dx^i dx^j}$ or $-\sqrt{g_{ij}dx^i dx^j}$ has not yet been unsolved.

In this section, we shall try to interpret some physical meanings for the lengths with signs: $+\sqrt{g_{ij}dx^i dx^j}$, $-\sqrt{g_{ij}dx^i dx^j}$.

Since dx's in the equation: $\quad \gamma_i dx^j\psi=\mu I\psi \qquad (3)$

are required both mathematically and physically to be real numbers, if we solve[2] for real dx's from (3),

we have
$$\frac{dx^1}{u^1}=\frac{dx^2}{u^2}=\frac{dx^3}{u^3}=\frac{dx^4}{x^4}\doteq\frac{\mu}{M}, \qquad N=0, \tag{4}$$

where
$$u^i=\psi^\dagger A\gamma^i\psi\,,\qquad M=\psi^\dagger A\psi\,,\qquad N=\psi^\dagger A\gamma_5\psi\,.$$

From this we come to a conclusion:

For $ds(\equiv\gamma_i dx^i)$ to be an operator physically significant and to have its eigenvalue μ through the equation (3), it must be true that $N\equiv 0$.

In almost all the spaces proposed in Wave Geometry, however, $N\mathrel{\not\equiv}0$,

(1) Y. Mimura; This Journal **5** (1935), 102 (W.G. No. 1).
(2) T. Sibata; This Journal **9** (1939), 179 (W.G. No. 34).

with an exception of cosmolygy in which it is possible to make $N \neq 0$ by choosing initial values of ψ, (for, in this case generally $N \equiv const.$). This fact suggests the unsuitability of considering the eigenvalue problem in general through the equation

$$ds\psi = \mu I\psi .$$

So, to realize the eigenvalue problem of the operator $ds(=\gamma_i dx^i)$ in general, some modifications must necessarily to be expected; but we have not yet arrived at the answer, so we will leave it for future solution.

We shall here, therefore, confine ourselves to investigating the meaning of signs of the eigenvalue $+\sqrt{g_{ij}dx^i dx^j}$ and $-\sqrt{g_{ij}dx^i dx^j}$ when $N \equiv 0$.

If we take coordinates such that

$$ds^2 = -\sum_{a,b=1}^{3} g_{ab} dx^a dx^b + g_{44}(dx^4)^2 ,$$

and identify $x^4 = t$, we have, from (4),

$$\frac{dt}{(\psi^\dagger \psi)/\sqrt{g_{44}}} = \frac{\mu}{M} .$$

From this we see that:

i) when $M > 0$, μ and dt have the same sign;

ii) when $M < 0$, μ and dt have opposite signs. This may be physically interpreted as follows:

In the state $M > 0$, the eigenvalue of $ds(=\gamma_i dx^i)$, to have positive sign $+\sqrt{g_{ij}dx^i dx^j}$, shows that the metric-operations ds is performed in the positive direction of the coordinate time; and contrarily, in the state $M < 0$, the eigenvalue of ds to have negative sign $-\sqrt{g_{ij}dx^i dx^j}$ means that the metric-operations ds is done in the positive direction of the coordinate time.

We shall investigate these results more concretely by applying them to cosmology.

In cosmology in terms of Wave Geometry, we have in fact[1]

$$M \doteq (-pe^{-kt} + qe^{kt})\sqrt{1-k^2 r^2} , \qquad N \equiv 0 ,$$

$$(p, q > 0) .$$

Therefore, for $M \gtrless 0$, we have $t \gtrless \frac{1}{k} \log \sqrt{\frac{p}{q}}$ respectively. But from Itimaru's researches,[2] we know that according as $t > \frac{1}{k} \log \sqrt{\frac{p}{q}}$ or $t < \frac{1}{k} \log \sqrt{\frac{p}{q}}$, the whole universe belongs to the generation of red or violet shifts. So that the state $M > 0$ indicates that the universe belongs

(1) H. Takeno; This Journal, **8** (1938), 229 (W. G. No. 30).

(2) K. Itimaru; This Journal **8** (1938), 244 (W. G. No. 31).

to the generation of red shifts, and contrarily, $M<0$ indicates that the universe belongs to that of violet shifts. Therefore, if we put together this result and the relation between M and the signs of eigenvalues, we have a conclusion: In the epoch of red shifts (or violet shifts), the eigenvalue of ds $(=\gamma_i dx^i)$, to have $+$ sign (or $-$ sign) shows that the metric-operation has taken place in the positive sense of coordinate time.

This is also interpretable as follows:

In universe, if we take the positive direction of coordinate time so as always to coincide with the sense of the actual time, we have to take the eigenvalue with + sign in the epoch of red shifts and oppositely in the epoch of violet shifts.

Remark: It must be noted that "dt" in the previous section and "dt" in this have essentially different meanings, the former indicating a time-duration of the phenomenon u^i as seen from the equation: $\rho \dfrac{dt}{ds}=u^4$, the latter expressing a mere time interval measured by the operator ds.

4. Cosmological time.

In Wave Geometry it has been shown[1] that the space-time with de Sitter's line element proves well fitted to explain the construction of universe by regarding $u^i=\psi^\dagger A\gamma^i\psi$ as smoothed-out momentum density vector of nebulae, where ψ is a solution of the fundamental equation, provided the line element is

$$ds^2=-\frac{dr^2}{1-k^2r^2}-r^2d\theta^2-r^2\sin^2 d\varphi^2+(1-k^2r^2)dt^2.$$

The actual value of the fourth component of u^i in this case is calculated as follows[2]:

$$u^4=\frac{pe^{-kt}+qe^{kt}}{\sqrt{1-k^2r^2}}, \qquad p,q>0; \tag{5}$$

but on the other hand, since

$$u^4=\frac{1}{\sqrt{g_{44}}}(\psi^\dagger\psi)=\frac{1}{\sqrt{1-k^2r^2}}(\psi^\dagger\psi),$$

from this, and (5), we have

$$\psi^\dagger\psi=pe^{-kt}+qe^{kt}.$$

Solving for t, we have

(1) T. Sibata; This Journal **8** (1938), 199 (W. G. No. 29).
H. Takeno; This Journal **8** (1938), 223 (W. G. No. 30).
K. Itimura; This Journal **8** (1938), 239 (W. G. No. 31).
(2) H. Takeno; This Journal **8** (1938), 229 (W. G. No. 30).

$$t=\frac{1}{k}\log\frac{1}{2q}\left\{(\psi^\dagger\psi)\pm\sqrt{(\psi^\dagger\psi)^2-4pp}\right\}. \tag{6}$$

Specially, when $p \doteqdot 0 \quad q \not\doteqdot 0$,

$$t-t_0=\frac{1}{k}\log(\psi^\dagger\psi) \tag{7}$$

where $t_0=\frac{1}{k}\log q$; the case $(p\doteqdot 0)$ being considered, as Itimaru concluded,[1] to indicate the model of the present universe.

Relation (6) or (7) may be interpreted as follows:
Cosmological time is determined only by the density of existing matter;
or in other words:
One cannot experience the lapse of cosmological time before the density of existing matter undergoes change.

Furthermore, turning back to the expression for time:

$$t-t_0=\frac{1}{k}\log(\psi^\dagger\psi),$$

and also taking into account the irreversibility of time in the phenomenon, we find there a very close analogy of time and entropy which has the expression:

$$S=k\log W$$

and has an irreversible character with respect to phenomena, suggesting some possibilities of the identification of time and entropy.

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(1) K. Itimaru; This Journal **8** (1938), 247 (W. G. No. 31).