Wave Geometry No. 38

Wave Geometry Unifying Einstein's Law of Gravitation and Born's Theory of Electrodynamics. II.

By

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Introduction.

In Wave Geometry No. 25,[1] in which the wave geometry unifying Einstein's law of gravitation and Born's theory of electrodynamics was developed, we obtained, under some geometrical assumptions with respect to spinor ψ, the fundamental differential equations for ψ which supply Einstein's law of gravitation and Born's theory of electrodynamics as the conditions for integrability. Involuntarily, however, we were guilty of some incomplete treatments in obtaining the conditions for integrability.

Recently, in connection with our theory, J. Haantjes and W. Wrona[2] attempted to obtain Einstein's law of gravitation as the condition for integrability of a certain system of differential equations; but with the same incomplete treatments as in our theory.

The purpose of this paper, therefore, is to clear up the unsatisfactory points[3] of our theory (§ 2—§ 5), and to show the relations between our theory and that of J. Haantjes and W. Wrona (§ 6).

§ 1. Preparatory Statements.[4]

First, we shall outline the first part of **I.**[5] For the fundamental tensor g_{ij} $(i, j=1, \ldots, 4)$ we take 4–4 matrices γ_i such that

$$g_{ij}=\gamma_{(i}\gamma_{j)}\,, \tag{1.1}$$

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(1) T. Sibata; This Journal, **8** (1938), 51–79 (W. G. No. 25).

(2) J. Haantjes und W. Wrona; Über konformeuklidische und Einsteinsche Räume gerader Dimension. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42** (1939), 626–636.

(3) In this paper we shall deal with only the first part of the previous paper; the second part, concerning Born's theory of electrodynamics, will be revised in a subsequent paper.

(4) Throughout this paper the same notations are used as in Wave Geometry No. 25.

(5) Hereafter we denote Wave Geometry No. 25 by the letter **I.**

the form of γ_i being given in I.[1] For any 1–4 matrices $\phi=\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix}$ and $\psi=\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix}$ let us consider the following matrices:

$$\bar{\phi}\equiv\begin{pmatrix}\phi_1\\\phi_2\\0\\0\end{pmatrix}\qquad \bar{\bar{\phi}}\equiv\begin{pmatrix}0\\0\\\phi_3\\\phi_4\end{pmatrix}\qquad \bar{\psi}\equiv\begin{pmatrix}\psi_1\\\psi_2\\0\\0\end{pmatrix}\qquad \bar{\bar{\psi}}\equiv\begin{pmatrix}0\\0\\\psi_3\\\psi_4\end{pmatrix} \tag{1.2}$$

$$=\frac{1}{2}(1+\mathring{\gamma}_5)\phi\,,\quad =\frac{1}{2}(1-\mathring{\gamma}_5)\phi\,,\quad =\frac{1}{2}(1+\mathring{\gamma}_5)\psi\,,\quad =\frac{1}{2}(1-\mathring{\gamma}_5)\psi\,;$$

then there exist two independent vectors $\bar{\rho}^i_\xi\,(\xi=1,2)$ $\left(\text{or } \bar{\bar{\rho}}^i_\xi\,(\xi=1,2)\right)$ which satisfy the relations[2]:

$$\bar{\rho}^i_\xi\gamma_i\bar{\phi}=0 \qquad \text{or} \qquad \bar{\bar{\rho}}^i_\xi\gamma_i\bar{\bar{\phi}}=0\,. \qquad (\xi=1,2) \tag{1.3}$$

Such $\bar{\rho}^i_\xi$ (or $\bar{\bar{\rho}}^i_\xi$) $(\xi=1,2)$ generate a surface when, and only when,[3]

$$\bar{\rho}^j_{[\eta}\bar{\rho}^i_{\xi]}\gamma_i\nabla_j\bar{\phi}=0\,, \qquad (\xi,\eta=1,2) \tag{1.4a}$$

or

$$\bar{\bar{\rho}}^j_{[\eta}\bar{\bar{\rho}}^i_{\xi]}\gamma_i\nabla_j\bar{\bar{\phi}}=0\,, \qquad (\xi,\eta=1,2) \tag{1.4b}$$

where

$$\nabla_j\equiv\frac{\partial}{\partial x^j}-\Gamma_j\,, \qquad (j=1\,,\ldots\ldots,4)$$

Γ_j being 4–4 matrices defined by

$$\frac{\partial\gamma_i}{\partial x^j}-\left\{{h\atop ij}\right\}\gamma_h-\Gamma_j\gamma_i+\gamma_i\Gamma_j=0\,.$$

Now we put forward the following assumptions[4]: by the parallel displacement of vectors v^i, whose coefficients of connection are Riemannian, i. e. $\left\{{h\atop ij}\right\}$,

(a) $v^i\gamma_i\bar{\psi}=0$ is invariant along the surface generated by $\bar{\rho}^i_\xi\,(\xi=1,2)$ satisfying the equation $\bar{\rho}^i_\xi\gamma_i\bar{\phi}=0$ (for given $\bar{\phi}$), and

(b) $v^i\gamma_i\bar{\bar{\psi}}=0$ is invariant along the surface generated by $\bar{\bar{\rho}}^i_\xi\,(\xi=1,2)$ satisfying the equation $\bar{\bar{\rho}}^i_\xi\gamma_i\bar{\bar{\phi}}=0$ (for given $\bar{\bar{\phi}}$).

Under such assumptions the following relations are obtained[5];

$$\bar{\rho}^i_\xi\{\nabla_i-R_i-T^5_i\mathring{\gamma}_5\}\bar{\psi}=0\,, \qquad (\xi=1,2) \tag{1.5a}$$

and

$$\bar{\bar{\rho}}^i_\xi\{\nabla_i-R_i-T^5_i\mathring{\gamma}_5\}\bar{\bar{\psi}}=0\,, \qquad (\xi=1,2) \tag{1.5b}$$

which are equivalent to[6]

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(1) I, 52.
(2) Cf. Theorem 2 in I, 53.
(3) Cf. Theorem 3 in I, 54.
(4) I, 55.
(5) Lemma in I, 55.
(6) Theorem 4 in I, 57.

$$\{\nabla_i - R_i - T_i^{5}\gamma_5\}\psi = \Lambda\gamma_i\phi\,, \tag{1.6}$$

where
$$\psi \equiv \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}, \qquad \phi \equiv \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix},$$

and Λ is any 4–4 matrix of the form $\left(\begin{array}{cc|cc} \times & \times & 0 & \\ \times & \times & & \\ \hline & 0 & \times & \times \\ & & \times & \times \end{array}\right)$, R_i, T_i^5 being arbitrary vectors.

In the particular case

(A) $$R_i = T_i^5 = 0\,,$$

(1.5a, b) and (1.6) become, respectively,

$$\rho_\xi^i \nabla_i \bar{\psi} = 0 \quad (1.7a), \qquad \bar{\rho}_\xi^i \nabla_i \bar{\psi} = 0 \quad (1.7b),$$

and
$$\nabla_i \psi = \Lambda\gamma_i\phi\,, \tag{1.8}$$

which are equivalent. And if

(B) ϕ and ψ can take any initial values,

the condition for integrability of the fundamental equation for ψ (1.8) or (1.7a, b) is given by(1)

$$K_{ijkl} = \frac{1}{4}\epsilon_{ijpq}\,\epsilon_{klrs}\,g K^{pqrs}\,, \tag{1.9}$$

which is identical with Einstein space(2)

$$K_{jk} = \frac{K}{4}g_{jk} \qquad (K = \text{constant}). \tag{1.10}$$

Next, we shall indicate in what respects the statements given above are incomplete. In these statements we have considered only the conditions of integrability for ψ; investigation of the conditions for ϕ was not taken into account. In point of fact, ϕ must satisfy equations (1.4a) and (1.4b) in order that ρ_ξ^i $(\xi=1,2)$ and $\bar{\rho}_\xi^i$ $(\xi=1,2)$ may generate surfaces. Hence, by equation (1.8) or (1.7a, b), besides the conditions of integrability for ψ, we must take into account the conditions of integrability for ϕ of the equations (1.4a) and (1.4b). So these conditions shall be examined.

§ 2. The differential equations for $\bar{\phi}$ when the vectors $\bar{\rho}_\xi^i$ $(\xi=1,2)$ defined by $\bar{\rho}_\xi^i\gamma_i\bar{\phi}=0$ generate a surface.

In order that the vectors $\bar{\rho}_\xi^i$ defined by

$$\bar{\rho}_\xi^i \gamma_i \bar{\phi} = 0\,, \qquad (\xi = 1, 2) \tag{2.1a}$$

(1) Theorem 5 in **I**, 59.
(2) Theorem 6 in **I**, 63.

may generate a surface, it is necessary and sufficient that the alternant of the operators $\bar{\rho}^i_\xi \dfrac{\partial}{\partial x^i}$ $(\xi=1, 2)$ be expressed by a linear combination of $\bar{\rho}^i_\xi \dfrac{\partial}{\partial x^i}$ $(\xi=1, 2)$, i. e.

$$\bar{\rho}^j_{[\eta} \frac{\partial \bar{\rho}^i_{\xi]}}{\partial x^j} = \sigma^\zeta_{\eta\xi} \bar{\rho}^i_\zeta . \qquad \begin{pmatrix} \xi, \eta, \zeta=1, 2 \\ i, j=1, \dots, 4 \end{pmatrix} \tag{2.2}$$

But, from (2.1a), iterating $\bar{\rho}^j_\eta \nabla_j$ on both sides of (2.1a), we have

$$\bar{\rho}^j_\eta (\nabla_j \bar{\rho}^i_\xi) \gamma_i \bar{\phi} + \bar{\rho}^j_\eta \bar{\rho}^i_\xi \gamma_i \nabla_j \bar{\phi} = 0 . \tag{2.3}$$

Therefore, equation (2.2) holds good when, and only when,[1]

$$\bar{\rho}^j_\eta \bar{\rho}^i_\xi \gamma_i \nabla_j \bar{\phi} - \bar{\rho}^j_\xi \bar{\rho}^i_\eta \gamma_i \nabla_j \bar{\phi} = 0 . \tag{2.4}$$

Now we shall reduce this equation to a simpler form. First, we can show that $\bar{\rho}^i_\xi$ $(\xi=1, 2)$ are orthogonal zero vectors i. e. satisfy the relations:

$$\bar{\rho}^l_\xi \bar{\rho}_{\eta l} = 0 . \qquad \begin{pmatrix} \xi, \eta=1, 2 \\ l=1, \dots, 4 \end{pmatrix} \tag{2.5a}$$

For, iterating $\bar{\rho}^j_\eta \gamma_j$ on both sides of (2.1a), we have

$$\bar{\rho}^j_\eta \bar{\rho}^i_\xi \gamma_j \gamma_i \bar{\phi} = 0 , \qquad (\xi, \eta=1, 2)$$

from which, taking the symmetric part with respect to ξ and η, it follows that

$$\bar{\rho}^j_\eta \bar{\rho}^i_\xi g_{ji} \bar{\phi} = 0 ,$$

which gives (2.5a), since $\bar{\phi} \neq 0$. Iterating $\bar{\rho}^h_\xi \gamma_h$ on both sides of (2.4), and using (2.5a), we have

$$\bar{\rho}^h_\xi \bar{\rho}^j_\xi \bar{\rho}^i_\eta \gamma_h \gamma_i \nabla_j \bar{\phi} = 0 ,$$

or putting $P_\xi \equiv \bar{\rho}^j_\xi \nabla_j \bar{\phi}$, we have

$$\bar{\rho}^h_\xi \bar{\rho}^i_\eta \gamma_{[h} \gamma_{i]} P_\xi = 0 . \tag{2.6}$$

The solution P_ξ, above, has the form:

$$P_\xi = \sigma_\xi \bar{\phi} , \tag{2.7}$$

where σ_ξ is any scalar. For it is evident that (2.7) satisfies (2.6); and if (2.6) has independent solution other than (2.7), it must follow that

$$\bar{\rho}^h_\xi \bar{\rho}^i_\eta (\gamma_{[h} \gamma_{i]} + \gamma_{[h} \gamma_{i]} \overset{\circ}{\gamma}_5) = 0 , \tag{2.8}$$

from which, since $\qquad \gamma_{[h} \gamma_{i]} \overset{\circ}{\gamma}_5 = -\dfrac{1}{2} \epsilon_{hijk} D \gamma^j \gamma^k ,$

we get $\qquad \bar{\rho}^h_{[\xi} \bar{\rho}^i_{\eta]} - \dfrac{1}{2} \epsilon^{hijk} \dfrac{1}{D} \bar{\rho}_{\xi j} \bar{\rho}_{\eta k} = 0 ;$

(1) Cf. Theorem 3 in I, 54.

but, on the other hand, $\bar{\rho}^i_\xi$ satisfy the relation[1]:

$$\bar{\rho}^h_{[\xi}\bar{\rho}^i_{\eta]}+\frac{1}{2}\epsilon^{hijk}\frac{1}{D}\bar{\rho}_{\xi j}\bar{\rho}_{\eta k}=0\,, \tag{2.9}$$

hence it follows that $\bar{\rho}^h_{[\xi}\bar{\rho}^i_{\eta]}=0\,,$

which contradicts the independence of $\bar{\rho}^i_\xi\ (\xi=1,2)$. So the solution P_ξ of (2.6) is uniquely given by (2.7). Moreover, it is easily seen that if P_ξ has the form (2.7), i. e.

$$\bar{\rho}^i_\xi \nabla_i\bar{\phi}=\sigma_\xi\bar{\phi}\,, \tag{2.7$'$}$$

(2.4) is satisfied identically. In other words, (2.4) is equivalent to (2.7)$'$. The same applies to the vectors $\bar{\bar{\rho}}^i_\xi$. So we have the result; *the orthogonal zero vectors* $\bar{\rho}^i_\xi$ *or* $\bar{\bar{\rho}}^i_\xi$ *defined by*

$$\bar{\rho}^i_\xi\gamma_i\bar{\phi}=0 \quad (2.10a) \qquad \textit{or} \qquad \bar{\bar{\rho}}^i_\xi\gamma_i\bar{\bar{\phi}}=0\,,\quad (\xi=1,2) \quad (2.10b)$$

generate surfaces when, and only when,

$$\bar{\rho}^i_\xi\nabla_i\bar{\phi}=\sigma_\xi\bar{\phi} \quad (2.11a) \qquad \text{or} \qquad \bar{\bar{\rho}}^i_\xi\nabla_i\bar{\bar{\phi}}=\tau_\xi\bar{\bar{\phi}}\,. \quad (2.11b)$$

N. B. Equations (2.11a, b) are equivalent to saying that $\bar{\rho}^j_\eta\nabla_j\bar{\rho}^i_\xi$ or $\bar{\bar{\rho}}^j_\eta\nabla_j\bar{\bar{\rho}}^i_\xi$ are expressed by linear combinations of $\bar{\rho}^i_\zeta$ or $\bar{\bar{\rho}}^i_\zeta\ (\zeta=1,2)$ i. e.

$$\bar{\rho}^j_\eta\nabla_j\bar{\rho}^i_\xi=\sigma^\zeta_{\xi\eta}\bar{\rho}^i_\zeta \quad (2.12a) \qquad \text{or} \qquad \bar{\bar{\rho}}^j_\xi\nabla_i\bar{\bar{\rho}}^i_\eta=\tau^\zeta_{\xi\eta}\bar{\bar{\rho}}^i_\zeta\,. \quad (2.12b)$$

For, using (2.11a), (2.3) becomes

$$\bar{\rho}^j_\eta(\nabla_j\bar{\rho}^i_\xi)\gamma_i\bar{\phi}=0\,,$$

which gives (2.12a). The same applies to $\bar{\bar{\rho}}^i_\xi\ (\xi=1,2)$. So we can say that *the necessary and sufficient condition for* $\bar{\rho}^i_\xi$ *or* $\bar{\bar{\rho}}^i_\xi$ *defined by* (2.10a) *or* (2.10b) *to generate surface is given by* (2.12a) *or* (2.12b).

§ 3. The conditions for integrability of the differential equations for $\bar{\phi}$ or $\bar{\bar{\phi}}$ when $\bar{\rho}^i_\xi$ or $\bar{\bar{\rho}}^i_\xi\ (\xi=1,2)$ generate surface.

We shall obtain the conditions for integrability of (2.11a, b) together with (2.10a, b). Iterating $\bar{\rho}^j_\eta\nabla_j$ on both sides of (2.11a), and taking the antisymmetric part of the resulting equation with respect to ξ and η, we have

$$\frac{1}{4}\bar{\rho}^j_\eta\bar{\rho}^i_\xi K_{jipq}\gamma^p\gamma^q\bar{\phi}=\left(\bar{\rho}^j_\eta\frac{\partial\sigma_\xi}{\partial x^j}-\bar{\rho}^j_\xi\frac{\partial\sigma_\eta}{\partial x^j}\right)\bar{\phi}\,, \tag{3.1}$$

because of (2.12a) and[2]

$$\nabla_{[j}\nabla_{i]}\bar{\phi}=\frac{1}{8}K_{jipq}\gamma^p\gamma^q\bar{\phi}\,. \tag{3.2}$$

(1) Equation (4.5a) in **I**, 61.
(2) **I**, 60.

But, since the right hand side of (3.1) is a multiple of $\bar{\phi}$, necessarily

$$\bar{\rho}^h_\zeta \gamma_h \cdot \bar{\rho}^j_\eta \bar{\rho}^i_\xi K_{jipq} \gamma^p \gamma^q \bar{\phi} = 0\,, \tag{3.3}$$

or, in consequence of

$$\gamma_h \gamma^{[p} \gamma^{q]} = 4\delta^{[p}_h \gamma^{q]} + \gamma^{[p} \gamma^{q]} \gamma_h\,, \tag{3.4}$$

(3.3) becomes

$$\bar{\rho}^h_\zeta \bar{\rho}^j_\eta \bar{\rho}^i_\xi K_{jihq} \gamma^q \bar{\phi} = 0\,. \tag{3.5}$$

Since (3.5) shows that $\bar{\rho}^h_\zeta \bar{\rho}^j_\eta \bar{\rho}^i_\xi K_{jihq}$ are expressed by linear combinations of $\bar{\rho}_{\xi q}$ $(\xi = 1, 2)$, (3.5) is rewritten in the form

$$\bar{\rho}^k_\tau \bar{\rho}^h_\zeta \bar{\rho}^j_\eta \bar{\rho}^i_\xi K_{jihk} = 0\,, \tag{3.6a}$$

which is the condition for integrability of (2.11a). Moreover, iterating $\bar{\rho}^l_\sigma \nabla_l$ on both sides of (3.6a), we have

$$\bar{\rho}^l_\sigma \bar{\rho}^k_\tau \bar{\rho}^h_\zeta \bar{\rho}^j_\eta \bar{\rho}^i_\xi K_{jihk;\,l} = 0\,. \tag{37.a}$$

where $K_{jihk;\,l}$ denotes the covariant derivative of K_{jihk} with respect to $\{^{\,j}_{lk}\}$. Similarly, from (2.11b) we have

$$\bar{\rho}^k_\tau \bar{\rho}^h_\zeta \bar{\rho}^j_\eta \bar{\rho}^i_\xi K_{jihk} = 0\,, \tag{3.6b}$$

$$\bar{\rho}^l_\sigma \bar{\rho}^k_\tau \bar{\rho}^h_\zeta \bar{\rho}^j_\eta \bar{\rho}^i_\xi K_{jihk;\,l} = 0\,. \tag{3.7b}$$

$$\cdots\cdot$$

So we have the result: *The conditions for integrability of* (2.11a, b) *are given by* (3.6a, b), (3.7a, b)....

As the simplest case, we now consider in detail what results when (2.11a) are *completely integrable.* Here (3.6a) must hold good identically for all values of $\bar{\phi}$. Using (2.9), (3.6a) is rewritten as

$$\{\bar{\rho}^k_{[\tau} \bar{\rho}^h_{\zeta]} - \bar{\rho}^k_{[\tau} \bar{\rho}^{h*}_{\zeta]}\} \{\bar{\rho}^j_{[\eta} \bar{\rho}^i_{\xi]} - \bar{\rho}^j_{[\eta} \bar{\rho}^{i*}_{\xi]}\} K_{jihk} = 0\,, \qquad (\xi, \eta, \zeta, \tau = 1, 2)\,,$$

or

$$\bar{\rho}^k_\tau \bar{\rho}^h_\zeta \bar{\rho}^j_\eta \bar{\rho}^i_\xi \{K_{jihk} - K_{jihk*} - K_{ji*hk} + K_{ji*hk*}\} = 0\,, \tag{3.8}$$

where

$$\bar{\rho}^k_{[\tau} \bar{\rho}^{h*}_{\zeta]} \equiv \frac{1}{2} \epsilon^{khlm} \frac{1}{D} \bar{\rho}_{\tau l} \bar{\rho}_{\zeta m}\,,$$

$$K_{jihk*} \equiv \frac{1}{2} \epsilon_{hklm} D K_{ji}{}^{lm}\,, \qquad \text{etc.}$$

Expressing $\bar{\rho}^j_{[1} \bar{\rho}^i_{2]}$ of (3.8) in terms of ϕ_1 and ϕ_2, and taking the condition that the resulting equation may hold good for all values of ϕ_1 and ϕ_2, we have[1]

$$K_{jihk} - K_{jihk*} - K_{ji*hk} + K_{ji*hk*} = R\{g_{jh} g_{ik} - g_{jk} g_{ih} - \epsilon_{jihk} D\}\,, \tag{3.9}$$

where R is any scalar. When the fundamental tensor g_{ij} is real and the determinant $\|g_{ij}\|$ is negative, separating the foregoing equation into real and imaginary parts, we have

(1) See Note I.

$$K_{jihk}+K_{ji*hk*}=\overset{1}{R}(g_{jh}g_{ik}-g_{jk}g_{ih})-i\overset{2}{R}\,\epsilon_{jihk}D\,, \tag{3.10}$$

and

$$-K_{jihk*}-K_{ji*hk}=i\overset{2}{R}(g_{jh}g_{ik}-g_{jk}g_{ih})-\overset{1}{R}\,\epsilon_{jihk}D\,, \tag{3.11}$$

where $\overset{1}{R}$ and $\overset{2}{R}$ are real and imaginary parts of R respectively, i. e.

$$R=\overset{1}{R}+i\overset{2}{R}\,.$$

But we can show that (3.10) is identical with (3.11). For, multiplying (3.10) by ϵ^{hklm} and contracting them by h and k, we have (3.11). From (3.10), taking alternation with respect to suffixes $[jih]$, we have $\overset{2}{R}=0$, and, contracting (3.10) with respect to j, h and i, k, it follows that

$$\overset{1}{R}=\frac{1}{6}K_{hk}{}^{hk}\,.$$

So that (3.10) becomes

$$K_{ji}{}^{hk}+K_{ji*}{}^{hk*}=R\delta^h_{[j}\delta^k_{i]}\,, \qquad R=\frac{1}{3}K_{hk}{}^{hk}\,, \tag{3.12}$$

which shows that the space is conformally flat.[(1)]

So we have the result: *The necessary and sufficient condition for the complete integrability of* (2.11a) *is given by* (3.12), *i. e. the space is conformally flat.*

The same result is obtained for the equation (2.11b).

When (2.11a) is not completely integrable, the conditions for integrability (not necessarily complete) of (2.11a) are given by (3.6a), (3.7a), ... But we can show that (3.6a) is 4th degree with respect to ϕ_1 and ϕ_2.[(2)] So we have the result: *When* (2.11a) *has five, or more than five, independent solutions of* $\overline{\Psi}_1$, $\overline{\Psi}_2$, *the conditions of integrability become* (3.12) *and necessarily completely integrable.*

§ 4. The conditions for integrability of the fundamental equation for ψ.

In this section we shall investigate the conditins for integrability of (1.8) together with (2.11a, b). Equation (1.8) is identical with (1.7a, b) and the conditions of integrability for ψ of (1.7a, b) were given by equations (4.4a, b) in I,[(3)] i. e.

$$\bar{\rho}^j_\eta\bar{\rho}^i_\xi K_{ij}{}^{kl}\gamma_k\gamma_l\overline{\overline{\Psi}}=0\,, \tag{4.1a}$$

and

$$\bar{\bar{\rho}}^j_\eta\bar{\bar{\rho}}^i_\xi K_{ij}{}^{kl}\gamma_k\gamma_l\overline{\overline{\Psi}}=0\,. \tag{4.1b}$$

Iterating $\bar{\rho}^h_\zeta\nabla_h$ and $\bar{\bar{\rho}}^h_\zeta\nabla_h$ on both sides of (4.1a) and (4.1b) respectively, we have

(1) J. Haantjes and W. Wrona have proved that (3.12) defines conformally flat space; loc. cit., 635.

(2) See (N.4) in Note **I.**

(3) I, 60.

$$\bar{\rho}^h_\zeta \bar{\rho}^j_\eta \bar{\rho}^i_\xi K_{ij}{}^{kl}{}_{;h} \gamma_k \gamma_l \overline{\Psi} = 0\,, \tag{4.2a}$$

and
$$\bar{\rho}^h_\zeta \bar{\rho}^j_\eta \bar{\rho}^i_\xi K_{ij}{}^{kl}{}_{;h} \gamma_k \gamma_l \Psi = 0\,. \tag{4.2b}$$

So, combining these equations with the first result of § 3, we have the result: *When* $\bar{\rho}^i_\xi$ *and* $\bar{\rho}^i_\xi$ $(\xi=1,2)$ *defined by* (1.3) *generate surfaces respectively, the conditions for integrability of* (1.8) *are given by* (3.6a, b), (3.7a, b), *and* (4.1a, b), (4.2a, b)....

Now, we shall consider two particular cases in detail:

(I) ψ can take any initial values.

(II) ψ and ϕ can take any initial values.

Case I. Here (4.1a, b) and (4.2a, b) must hold good identically for all values of ψ. Moreover, in consequence of

$$2\gamma_{[k}\gamma_{l]}\overline{\Psi} = (\gamma_{[k}\gamma_{l]} - \gamma_{[k}\gamma_{l]}\mathring{\gamma}_5)\overline{\Psi}$$
$$= (\gamma_{[k}\gamma_{l]} + \gamma_{[k}\gamma_{l]*})\overline{\Psi}\,,$$

and
$$2\gamma_{[k}\gamma_{l]}\Psi = (\gamma_{[k}\gamma_{l]} - \gamma_{[k}\gamma_{l]*})\Psi\,,$$

from (4.1a, b) and (4.2 a, b), we have

$$\bar{\rho}^j_\eta \bar{\rho}^i_\xi (K_{ij}{}^{kl} + K_{ij}{}^{kl*}) = 0\,, \tag{4.3a}$$

$$\bar{\rho}^j_\eta \bar{\rho}^i_\xi (K_{ij}{}^{kl} - K_{ij}{}^{kl*}) = 0\,, \tag{4.3b}$$

$$\bar{\rho}^h_\zeta \bar{\rho}^j_\eta \bar{\rho}^i_\xi (K_{ij}{}^{kl} + K_{ij}{}^{kl*})_{;h} = 0\,, \tag{4.4a}$$

$$\bar{\rho}^h_\zeta \bar{\rho}^j_\eta \bar{\rho}^i_\xi (K_{ij}{}^{kl} - K_{ij}{}^{kl*})_{;h} = 0\,. \tag{4.4b}$$

So we have the result: *When* (1.8) *together with* (2.11a, b) *is completely integrable for* ψ, *equations* (3.6a, b), (3.7a, b), (4.3a, b), *and* (4.4a, b) *are compatible.*

Case II. Here (4.3a, b) become[(1)]

$$K_{ji}{}^{hk} = K_{ji*}{}^{hk*}\,; \tag{4.5}$$

accordingly, the conditions are given by (4.5) and (3.12), which are equivalent to

$$K_{ji}{}^{hk} = \frac{R}{2}\delta^h_{[j}\delta^k_{i]}\,, \qquad R = \frac{1}{3}K_{hk}{}^{hk}\,. \tag{4.6}$$

So we have the result: *When* $\bar{\rho}^i_\xi$ *and* $\bar{\rho}^i_\xi$ *defined by* (1.3) *generate surfaces respectively, and* ϕ *and* ψ *can take any initial values, the necessary and sufficient condition for integrability of* (1.8) *is given by* (4.6).

N. B. In Wave Geometry No. 25, as the condition for complete integrability of (1.8) we obtained (4.5), or[(2)]

$$K_{ij} = \lambda g_{ij}\,.$$

(1) Cf. Theorem 5 in **I**, 59–61.
(2) Cf. Theorem 5 and 6 in **I**, 59–63.

But this equation gives only a necessary, but not a sufficient, condition. To obtain the sufficient conditions, equation (3.12) must be added.

§ 5. The metric g_{ij} satisfying the conditions for integrability of the fundamental equation for ψ.

We shall find the forms of line-elements in the two cases stated in § 4, i. e. find the solutions of g_{ij} which satisfy the equations

$$\text{(I)} \quad (3.6a, b), (3.7a, b), (4.3a, b), (4.4a, b), \ldots \tag{5.1}$$

or

$$\text{(II)} \quad (4.6). \tag{5.2}$$

In case (II), as shown by H. Takeno,(1) (4.6) admits only the line element of de Sitter form:

$$ds^2 = -(1-k^2r^2)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 + (1-k^2r^2)dt^2, \tag{5.3}$$

where

$$k^2 = -\frac{R}{4}.$$

So we have the result: *When ϕ and ψ can take any initial values, the necessary and sufficient condition for integrability of* (1.8) *and* (2.11a, b) *is given by* (4.6), *which supplies only the line-element of de Sitter form* (5.3).

In case I, we shall find the line-elements under the assumption that the metric g_{ij} has a signature of the form $- - - +$, and the space determined by g_{ij} is static and spherically symmetric in polar coordinates r, θ, φ and time coordinate t. Under this assumption the general form of the line element is written as follows

$$ds^2 = -e^{\mu}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 + e^{\nu}dt^2, \tag{5.5}$$

where μ and ν are certain functions of r only. We shall determine μ and ν such that (5.5) satisfies (5.1). By actual calculations, we see that for the line element of the form (5.5) the components of $K_{ij}{}^{kl}$ have the forms(2)

$$\left.\begin{aligned} &K_{12}{}^{12} = K_{13}{}^{13} = -\frac{1}{2}r^{-1}\mu' e^{-\mu}, \quad K_{14}{}^{14} = -e^{-\mu}\left(-\frac{1}{2}\nu'' - \frac{1}{4}\nu'^2 + \frac{1}{4}\mu'\nu'\right), \\ &K_{23}{}^{23} = -r^{-2}(1-e^{-\mu}), \qquad K_{24}{}^{24} = K_{34}{}^{34} = \frac{1}{2}r^{-1}\nu' e^{-\mu}, \\ &K_{ij}{}^{kl} = 0, \quad (i, j, k, l \neq), \qquad K_{ij}{}^{il} = 0, \quad (i, j, l \neq), \end{aligned}\right\} \tag{5.6}$$

where the accents denote differentiation with respect to r. Substituting (5.6) into (3.6a), we have,(3) unless (5.5) does not satisfy (3.12) i. e. (5.5) is not conformally flat,

(1) H. Takeno; This Journal, **7** (1937), 42 (W. G. No. 11).
(2) Cf. T. Sibata; This Journal, **8** (1938), 211 (W. G. No. 29).
(3) See Note **II**.

$$\phi_1=0 \quad \text{or} \quad \phi_2=0; \tag{5.7a}$$

similarly, from (3.6b),

$$\phi_3=0 \quad \text{or} \quad \phi_4=0. \tag{5.7b}$$

Then, corresponding to (5.7a), from (2.10a) $\bar{\rho}^i_\xi$ are determined as follows[1]

$$\left.\begin{array}{lllll} \bar{\rho}^i_1: & 0, & -1, & -\dfrac{i}{\sin\theta}, & 0, \\ \bar{\rho}^i_2: & e^{-\frac{\mu}{2}}, & 0, & 0, & e^{-\frac{\nu}{2}}, \end{array}\right\} \tag{5.8a}$$

or

$$\left.\begin{array}{lllll} \bar{\rho}^i_1: & 0, & -1, & \dfrac{i}{\sin\theta}, & 0, \\ \bar{\rho}^i_2: & e^{-\frac{\mu}{2}}, & 0, & 0, & -e^{-\frac{\nu}{2}}, \end{array}\right\} \tag{5.9a}$$

and, corresponding to (5.7b), from (2.10b), we have[2]

$$\left.\begin{array}{lllll} \bar{\bar{\rho}}^i_1: & 0, & 1, & \dfrac{i}{\sin\theta}, & 0, \\ \bar{\bar{\rho}}^i_2: & e^{-\frac{\mu}{2}}, & 0, & 0, & e^{-\frac{\nu}{2}}, \end{array}\right\} \tag{5.8b}$$

or

$$\left.\begin{array}{lllll} \bar{\bar{\rho}}^i_1: & 0, & 1, & -\dfrac{i}{\sin\theta}, & 0, \\ \bar{\bar{\rho}}^i_2: & e^{-\frac{\mu}{2}}, & 0, & 0, & -e^{-\frac{\nu}{2}}. \end{array}\right\} \tag{5.9b}$$

By actual calculation, it is easily shown that each of the above four sets constitutes a complete system. So that for the line element of the form (5.5) (not conformally flat) the four sets (5.8a)–(5.9b) give the general forms of $\bar{\rho}^i_\xi$ and $\bar{\bar{\rho}}^i_\xi$ defined by (2.10a, b) which generate surfaces. We can see that for each of these four sets (4.3a) and (4.3b) hold good simultaneously when, and only when,

$$\left.\begin{array}{l} (K_{12}{}^{kl}+K_{12}{}^{kl*})-(K_{34}{}^{kl}+K_{34}{}^{kl*})=0, \\ (K_{13}{}^{kl}+K_{13}{}^{kl*})-(K_{42}{}^{kl}+K_{42}{}^{kl*})=0, \end{array}\right\} \tag{5.10a}$$

and

$$\left.\begin{array}{l} (K_{12}{}^{kl}-K_{12}{}^{kl*})+(K_{34}{}^{kl}-K_{34}{}^{kl*})=0, \\ (K_{13}{}^{kl}-K_{13}{}^{kl*})+(K_{42}{}^{kl}-K_{42}{}^{kl*})=0. \end{array}\right\} \tag{5.10b}$$

By (5.6), these equations are equivalent to

$$K_{12}{}^{12}-K_{34}{}^{34}=0, \qquad K_{13}{}^{13}-K_{24}{}^{24}=0, \tag{5.11}$$

which give

$$\mu'+\nu'=0. \tag{5.12}$$

Moreover, if (5.12) holds good, equations (4.3a, b) vanish identically. So we have the result: *For the line element of the form of* (5.5) (*not conformally*

(1) Cf. (2.6a) in **I**, 53.
(2) Cf. (2.6b) in **I**, 54.

flat) there exists two sets of solutions for each of $\bar{\rho}^i_\xi$ *and* $\bar{\bar{\rho}}^i_\xi$ *satisfying* (2.10a) *and* (2.10b) *and each generating a surface; these two sets are given by* (5.8a), (5.9a), *and* (5.8b), (5.9b). *And for each set of* $\bar{\rho}^i_\xi$ *and* $\bar{\bar{\rho}}^i_\xi$ *the fundamental equations* (1.7a, b) *are completely integrable when, and only when,* $\mu'+\nu'=0$.

N. B. From this result we see that in Case I, for Schwarzschild's line-element:

$$ds^2=-\frac{dr^2}{1-\frac{2m}{r}}-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+\left(1-\frac{2m}{r}\right)dt^2,$$

the fundamental equations for ψ (1.7a, b) are completely integrable for ψ.

§ 6. Comparison of the theory of J. Haantjes and W. Wrona with ours.

To compare our theory with that of J. Haantjes and W. Wrona, we shall translate the fundamental equations for ψ into differential equations of tensor form, as used by them.

From $\bar{\psi}$ and $\bar{\bar{\psi}}$ we construct the following tensors:

$$\bar{\omega}^{ij}\equiv\tilde{\bar{\psi}}C\gamma^{[i}\gamma^{j]}\bar{\psi} \quad \text{and} \quad \bar{\bar{\omega}}^{ij}\equiv\tilde{\bar{\bar{\psi}}}C\gamma^{[i}\gamma^{j]}\bar{\bar{\psi}}, \tag{6.1}$$

where $\tilde{\bar{\psi}}$ and $\tilde{\bar{\bar{\psi}}}$ denote the transposed matrices of $\bar{\psi}$ and $\bar{\bar{\psi}}$ respectively, and C is the matrix which makes $C\gamma_i\,(i=1,\ldots\ldots,4)$ symmetric,[1] i. e.

$$C\gamma_i=\tilde{\gamma}_i\tilde{C}. \tag{6.2}$$

Then we see that[2] $\bar{\omega}^{ij}$ and $\bar{\bar{\omega}}^{ij}$ are proportional to $\bar{\rho}^i_{[1}\bar{\rho}^j_{2]}$ and $\bar{\bar{\rho}}^i_{[1}\bar{\bar{\rho}}^j_{2]}$ respectively. Therefore $\bar{\omega}^{ij}$ and $\bar{\bar{\omega}}^{ij}$ are bivectors which satisfy the relations

$$\bar{\omega}^{ij}=-\bar{\omega}^{ij*}, \quad (6.3) \qquad \text{and} \qquad \bar{\bar{\omega}}^{ij}=+\bar{\bar{\omega}}^{ij*}. \quad (6.4)$$

Conversely, when bivectors $\bar{\omega}^{ij}$ and $\bar{\bar{\omega}}^{ij}$ are given satisfying (6.3) and (6.4), from (6.1) $\bar{\psi}$ and $\bar{\bar{\psi}}$ are uniquely determined.[3] So that the determination of $\bar{\psi}$ and $\bar{\bar{\psi}}$ is equivalent to the determination of bivectors $\bar{\omega}^{ij}$ and $\bar{\bar{\omega}}^{ij}$ satisfying (6.3) and (6.4). Using such bivectors (6.1), the fundamental differential equations for ψ (1.7a, b) are written, because of $\nabla_l C=0$,[4] in the form

$$\bar{\rho}^l_\xi\nabla_l\bar{\bar{\omega}}^{ij}=0, \quad (6.5a) \qquad \bar{\bar{\rho}}^l_\xi\nabla_l\bar{\omega}^{ij}=0, \quad (\xi=1,2) \quad (6.5b)$$

which are the differential equations of tensor form translated from (1.7a, b).

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(1) Such a matrix C was defined in Spinor Calculus, This Journal **8** (1938), 177 (W. G. No. 26).

(2) See Note **III.**

(3) See Note **IV.**

(4) T. Sibata; This Journal, **9** (1939), 182 (W. G. No. 34).

In Haantjes and Wrona's paper(1) a bivector $f^{k\lambda}$ satisfying the relation $f^{k\lambda}=+f^{k\lambda *}$ was called the bivector of the first kind, and a bivector $\varphi^{\nu\mu}$ satisfying the relation $\varphi^{\nu\mu}=-\varphi^{\nu\mu *}$ the bivector of the second kind; and concerning these bivectors the following theorem was obtained; the differential equation:

$$\varphi^{\nu\mu}\nabla_{\mu}f^{k\lambda}=0 \tag{6.6}$$

for bivector $f^{k\lambda}$ of the first kind, where $\varphi^{k\lambda}\equiv p^{[k}q^{\lambda]}$ is a bivector of the second kind and p^k and q^λ are " X_2 bildend," is completely integrable when, and only when,

$$K_{\mu\nu}=\lambda g_{\mu\nu}\,. \tag{6.7}$$

But if we make the corrospondence

$$\bar{p}_1^{\lambda}\longrightarrow p^{\lambda}\,,\qquad \bar{p}_2^{\lambda}\longrightarrow q^{\lambda}\,,\qquad \bar{\omega}^{k\lambda}\longrightarrow f^{k\lambda}\,,$$

we see that (6.6) is identical with (6.5b). Hence (6.6) is implied in (1.7b). Therefore, if we apply to (6.6) the last result obtained in § 4, we have the theorem: *When the bivectors of the first and second kinds* $\varphi^{\nu\mu}\equiv p^{[\nu}q^{\mu]}$ *and* $f^{k\lambda}$ *can have any initial values, and* p^ν *and* q^μ *are " X_2 bildend," the necessary and sufficient conditions for integrability of* (6.6) *are given by* (4.5) *and* (3.12), *which are equivalent to*

$$K_{ji}{}^{hk}=\frac{R}{2}\delta^{h}_{[j}\delta^{k}_{i]}\,,\qquad R=\frac{1}{3}K_{hk}{}^{hk}\,. \tag{6.8}$$

These conditions do not coincide with (6.7). In Haantjes and Wrona's theorem, like my previous paper, it was overlooked that the condition for p^ν and q^μ satisfying the relation $p^{[\nu}q^{\mu]}=p^{[\nu}q^{\mu]*}$ to be " X_2 bildend " imposes further restriction on the metric g_{ij}. Since the condition is given by (3.12), we must add (3.12) to (6.7); then it becomes (6.8). So that, as the condition of integrability, we must take (6.8) in place of (6.7).(2) However, if we remove the restriction that $\varphi^{\nu\mu}$ can have any initial values, as the conditions for integrability of (6.6), we shall have results similar to those stated in § 4 and 5.

Note I.

(3.8) is written as

$$\bar{p}_{\tau}^{k}\bar{p}_{\zeta}^{h}\bar{p}_{\eta}^{i}\bar{p}_{\xi}^{i}L_{jihk}=0\,, \tag{N.1}$$

where
$$L_{jihk}=K_{jihk}-K_{jihk*}-K_{ji*hk}+K_{ji*hk*}\,. \tag{N.2}$$

If we put
$$\bar{p}_{\xi}^{i}\overset{a}{h}_{i}=\bar{p}_{\xi}^{a}\,,\qquad L^{jihk}\overset{a}{h}_{j}\overset{b}{h}_{i}\overset{c}{h}_{h}\overset{d}{h}_{k}=\overset{0}{L}_{abcd}\,,$$

where $\overset{a}{h}_{i}$ are defined by

(1) J. Hantjes und W. Wrona; loc. cit., 7.

(2) (6.7) is a necessary, but not sufficient, condition.

$$\sum_{a=1}^{4} \overset{a}{h}_i \overset{a}{h}_j = g_{ij},$$

(N.1) becomes
$$\bar{p}^a_\tau \bar{p}^b_\zeta \bar{p}^c_\eta \bar{p}^d_\xi \overset{0}{L}_{abcd} = 0. \tag{N.3}$$

Since in I[1] $\bar{p}^a_{[1}\bar{p}^b_{2]}$ are expressed in terms of ϕ_1 and ϕ_2, substituting these values into (N.3), we have, after some calculations,

$$\begin{aligned}\frac{1}{4}(\phi_1)^4\{\overset{0}{L}_{1212}-\overset{0}{L}_{1313}-2i\overset{0}{L}_{1213}\}+(\phi_1)^3\phi_2\{-\overset{0}{L}_{1214}+i\overset{0}{L}_{1314}\}\\ +\frac{1}{2}(\phi_1)^2(\phi_2)^2\{-\overset{0}{L}_{1212}-\overset{0}{L}_{1313}+2\overset{0}{L}_{1414}\}+\phi_1(\phi_2)^3\{\overset{0}{L}_{1214}+i\overset{0}{L}_{1314}\}\\ +\frac{1}{4}(\phi_2)^4\{\overset{0}{L}_{1212}-\overset{0}{L}_{1313}+2i\overset{0}{L}_{1213}\}=0.\end{aligned} \tag{N.4}$$

In order that this may hold good for all values of ϕ_1 and ϕ_2, each coefficient of ϕ_1, ϕ_2 must vanish identically; hence,

$$\left.\begin{aligned}&\overset{0}{L}_{1213}=0, \quad \overset{0}{L}_{1212}-\overset{0}{L}_{1313}=0,\\ &\overset{0}{L}_{1214}=0, \quad \overset{0}{L}_{1314}=0, \quad \overset{0}{L}_{1313}=\overset{0}{L}_{1414}.\end{aligned}\right\} \tag{N.5}$$

Taking account of the identity:

$$\overset{0}{L}_{ijkl}=-\overset{0}{L}_{ij*kl}, \qquad \overset{0}{L}_{ijkl}=\overset{0}{L}_{klij},$$

the equations above are expressible in a single equation:

$$\overset{0}{L}_{ijkl}=R(2\delta^k_{[i}\delta^l_{j]}-\epsilon_{ijkl}), \tag{N.6}$$

which is identical with (3.9).

Note II.

In this case we take $\overset{a}{h}_i$ $(\sum_{a=1}^{4}\overset{a}{h}_i\overset{a}{h}_j=g_{ij})$ as

$$\overset{1}{h}_1=ie^{\frac{\mu}{2}}, \quad \overset{2}{h}_2=ir, \quad \overset{3}{h}_3=ir\sin\theta, \quad \overset{4}{h}_4=e^{\frac{\nu}{2}}, \quad \overset{a}{h}_i=0 \ (i\neq a),$$

and put
$$L_{ij}{}^{kl}\overset{a}{h}{}^i\overset{b}{h}{}^j\overset{c}{h}_k\overset{d}{h}_l=\overset{0}{L}_{abcd}.$$

Then, substituting (5.6) into (N.4), we have

$$(\phi_1)^2(\phi_2)^2\{-\overset{0}{L}_{1212}+\overset{0}{L}_{1414}\}=0. \tag{N.7}$$

But
$$-\overset{0}{L}_{1212}+\overset{0}{L}_{1414}\neq 0,$$

unless (5.5) is not conformally flat. For, if

$$-\overset{0}{L}_{1212}+\overset{0}{L}_{1414}=0,$$

(1) Cf. footnote on p. 61 in I.

then (N.5) holds good and (5.5) is necessarily conformally flat. Hence, from (N.7), we have (5.7a).

Note III. $\bar{\omega}^{ij}$ and $\bar{\bar{\omega}}^{ij}$ can be determined from (6.6) in Spinor Calculus[1] as follows: if we put

$$\overset{\circ}{\bar{\omega}}_{ab}=\bar{\omega}^{ij}\overset{a}{h}_i\overset{b}{h}_j\,,\qquad \overset{\circ}{\bar{\bar{\omega}}}_{ab}=\bar{\bar{\omega}}^{ij}\overset{a}{h}_i\overset{b}{h}_j\,,$$

$\overset{\circ}{\bar{\omega}}_{ab}$ and $\overset{\circ}{\bar{\bar{\omega}}}_{ab}$ are expressible as

$$\left.\begin{aligned}\overset{\circ}{\bar{\omega}}_{12}=-\overset{\circ}{\bar{\omega}}_{34}&=\phi_1\phi_1-\phi_2\phi_2\,,\\ \overset{\circ}{\bar{\omega}}_{13}=-\overset{\circ}{\bar{\omega}}_{42}&=-i(\phi_1\phi_1+\phi_2\phi_2)\,,\\ \overset{\circ}{\bar{\omega}}_{14}=-\overset{\circ}{\bar{\omega}}_{23}&=-2\phi_1\phi_2\,,\end{aligned}\right\}\tag{N.8}$$

$$\left.\begin{aligned}\overset{\circ}{\bar{\bar{\omega}}}_{12}=\overset{\circ}{\bar{\bar{\omega}}}_{34}&=\phi_3\phi_3-\phi_4\phi_4\,,\\ \overset{\circ}{\bar{\bar{\omega}}}_{13}=\overset{\circ}{\bar{\bar{\omega}}}_{42}&=-i(\phi_3\phi_3+\phi_4\phi_4)\,,\\ \overset{\circ}{\bar{\bar{\omega}}}_{14}=\overset{\circ}{\bar{\bar{\omega}}}_{23}&=-2\phi_3\phi_4\,.\end{aligned}\right\}\tag{N.9}$$

Comparing (N.8) with the actual forms of $\bar{\rho}^{i}_{[1}\bar{\rho}^{j}_{2]}$[2], we see that $\bar{\omega}^{ij}$ is proportional to $\bar{\rho}^{i}_{[1}\bar{\rho}^{j}_{2]}$. The same holds good for $\bar{\bar{\omega}}^{ij}$.

Note IV. When $\bar{\omega}^{ij}$ are bivectors which satisfy the relation

$$\bar{\omega}^{ij}=-\bar{\omega}^{ij*}\,,\tag{N.10}$$

we can show that $\underset{1}{a}^i$ and $\underset{2}{a}^i$ defined by

$$\bar{\omega}^{ij}=\underset{1}{a}^{[i}\underset{2}{a}^{j]}\,,\tag{N.11}$$

must satisfy the relation

$$\underset{1}{a}^i\underset{1}{a}_i=0\,,\qquad \underset{2}{a}^i\underset{2}{a}_i=0\,,\qquad \underset{1}{a}^i\underset{2}{a}_i=0\,.\tag{N.12}$$

For, from (N.10), we have

$$\underset{a}{a}^j\bar{\omega}_{jk}=-\frac{1}{2}\epsilon_{jklm}D\underset{a}{a}^j\bar{\omega}^{lm}\,.\qquad (a=1,2)\tag{N.13}$$

But the right-hand side of (N.13) must vanish identically because of (N.11). Hence (N.13) becomes

$$\underset{a}{a}^j\underset{[1}{a}_j\underset{2]}{a}_k=0\,,\qquad (a=1,2)$$

from which it follows that

$$\underset{a}{a}^j\underset{1}{a}_j=0\,,\qquad \underset{a}{a}^j\underset{2}{a}_j=0\,,\qquad (a=1,2)$$

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(1) T. Sibata; This Journal, **8** (1938), 180 (W. G. No. 26).

(2) The actual forms of $\bar{\rho}^{i}_{[1}\bar{\rho}^{j}_{2]}$ are given in the footnote of **1**, 61.

which is identical with (N.12). Therefore $\bar{\omega}^{ij}$ is the form of $\bar{\rho}^{i}_{[1}\bar{\rho}^{j}_{2]}$; hence $\bar{\Psi}$ can be determined from

$$\bar{\omega}^{ij}=\tilde{\bar{\Psi}}C\gamma^{[i}\gamma^{j]}\bar{\Psi}.$$

The same applies to $\bar{\bar{\omega}}^{ij}$.

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