# EQUATIONS OF SCHRÖDER

By

Minoru URABE

(Received December, 25, 1950)

## Preface.

By means of the iteration method, Schröder[1] has solved the functional equation as follows: $f[\varphi(x)]=\lambda f(x)$, where $\varphi(x)$ is a given regular function such that the expansion of $\varphi(x)$ is as follows: $\varphi(x)=\lambda x+\cdots$, and $|\lambda|>1$ or $0<|\lambda|<1$. In 1945, Fukuhara[2] has extended the equation into those of many variables. For the given functions $\varphi^{\mu}(x)$ such that $\varphi^{\mu}(x)=a^{\mu}_{\nu}x^{\nu}+\cdots$, (det. $|a^{\mu}_{\nu}|\neq 0$), he has considered the functional equations as follows:

$$f_i[\varphi(x)] = \lambda_i f_i(x)+\delta f_{i-1}(x)+\Psi_i(x)\ ,$$

where $\lambda_i$ are the eigen values of $\| a^{\mu}_{\nu}\|$ and $\delta$ is an arbitrary number such that $|\delta|$ is sufficiently small and $\Psi_i(x)$ are the suitable polynomials of the functions $f_i$. Under the condition that the absolute values of all the eigen values of $\|a^{\mu}_{\nu}\|$ are less or greater than unity, he has solved the equations.

In this paper, we study the extended equations of Schröder from different point of view.

In Chapter I, we add some remarks on Fukuhara's paper. In Chapter II, we consider the transformation $\mathfrak{T}$: $'x^{\mu}=\varphi^{\mu}(x)$. Assuming the existence of one parameter group of the transformations containing $\mathfrak{T}$, we show, that the equations of Schröder resemble closely to the finite forms of the characteristic equations[3] of a linear homogeneous partial differential equation. Thus, when some more conditions are satisfied, by making use of the results of the previous paper,[4] we easily get the solutions of the equations newly obtained. In Chapter III, we study the fundamental theorem which plays an important roll for the subsequent discussions. In Chapter IV, we consider the conditions assumed in Chap. II, and we find that, when the absolute values of all the eigen values of $\| a^{\mu}_{\nu} \|$ are less or greater than unity, there exists a one parameter group containing the

1) Schröder, Math. Ann., 1871.
2) Fukuhara, Kyūshū-Teikoku-Daigaku Rigaku-Hōkoku, Vol. 1, No.2 (1945).
3) Urabe, This Journal, Vol. 15, No. 1 (p.25).
4) do.

given transformation. When the $L$-th determinant divisor of $\mathfrak{M}$ is unity,[1] there exists a group satisfying all the conditions assumed in Chap. II and the equations obtained in Chap. II completely coincide with the equations of Schröder. Thus, we see that, in this case, the conditions assumed in Chap. II, although apparently complicated, are weaker than Fukuhara's. When the $L$-th determinant divisor of $\mathfrak{M}$ is not unity, the conditions in Chap. I and II, slip out each other and the equations obtained also do, although they resemble closely. In Chapter V, assuming that the $L$-th determinant divisor of $\mathfrak{M}$ is unity, we show that there exists no other group possessing the regular operator functions than the groups obtained in Chap. IV.

Apart from consideration of the equations of Schröder, the results obtained in Chap. IV and V seem to be of much interest of themselves. From our results, it seems possible to study some characters of continuous groups of transformations by means of the finite transformations.

Weakening of our conditions and application of our results are problems remained unsolved.

## Chapter I. Equations of Schröder of many variables.

### § 1. Preliminaries.

Given a set of $n$ functions $\varphi^{\mu}(x)$ of $n$ variables $x^{\nu}$ such that $\varphi^{\mu}(x)$ are regular in the vicinity of $x^{\nu}=0$ and the expansions of these functions are as follows:

(1.1) $$\varphi^{\mu}(x) = a^{\mu}_{\nu}x^{\nu}+\cdots\cdots ,^{(2)}$$

where $\det.|a^{\mu}_{\nu}|\neq 0$ and the unwritten terms are those of the second and higher orders.[3] Let the Jordan's form of the matrix $A=\| a^{\mu}_{\nu} \|$ be $A$. We write $\tilde{A}$ as follows:

(1.2) $$\tilde{A} = TAT^{-1} = \sum_{i=1}^{R}\sum_{\iota=1}^{L_i}\oplus \tilde{A}^{i}_{\iota} ,$$

where $\sum\oplus$ denotes a direct sum of matrices and $\tilde{A}^{i}_{\iota}$ is a matrix of $P_{\iota}^{i}$-th order which has the form as follows: $\tilde{A}^{i}_{\iota}=\begin{pmatrix} \lambda_i & 0 & \cdots\cdots & 0 \\ \delta & \lambda_i & & \vdots \\ 0 & \delta & \lambda_i & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots\cdots & \delta & \lambda_i \end{pmatrix}$[4]. Here $\delta$ is an

1) Cf. Chap. III, § 3.

2) $a^{\mu}_{\nu}x^{\nu}$ means $\sum_{\nu=1}^{n} a^{\mu}_{\nu}x^{\nu}$ . In the following, as here, we use the convention of tensor calculus.

3) In the following, we agree that the unwritten terms in the expansion formulae denote the terms of the higher orders than those written explicitly.

4) $\lambda_1, \lambda_2, \cdots\cdots, \lambda_R$ are distinct from one another.

arbitrary number which is not zero, and $\lambda_i$ are the eigen values of the maxrix $A$. From det.$|A|\neq 0$, $\lambda_i \neq 0$.

After Fukuhara, we assume that the absolute values of all the eigen values are less than unity, namely

$$(1.3) \qquad 0<|\lambda_i|<1.$$

We consider the following relations

$$(1.4) \qquad \lambda_i = \lambda_1^{p_1}\lambda_2^{p_2}\ldots\ldots\lambda_R^{p_R} ,$$

for non-negative integers $p_1, p_2, \ldots, p_R$ such that $p_1+p_2+\cdots+p_R \geqq 2$. We arrange $\lambda_i$ so that $\lambda_a (a=1, 2, \ldots, S)$ are not expressed as (1.4) and $\lambda_x$ $(x=S+1, \ldots, R)$ are expressed as (1.4) and moreover $|\lambda_{S+1}|\geqq|\lambda_{S+2}|\geqq\ldots\geqq|\lambda_R|$. Then the relations (1.4) which really hold are of the forms as follows:

$$(1.5) \qquad \lambda_x = \lambda_1^{p_1}\lambda_2^{p_2}\ldots\ldots\lambda_{x-1}^{p_{x-1}} .$$

We denote the eigen value by $\lambda_{lp}^i$, which is a $(p, p)$-element of $\tilde{A}_l^i$, and we express the relations (1.5) as follows:

$$(1.6) \qquad \lambda_x = \prod_{i=1}^{x-1}\prod_{l=1}^{L_i}\prod_{p=1}^{P_l^i}\lambda_{lp}^{i\ \overset{x}{p}{}_{lp}^{i}} ,$$

where $\overset{x}{p}{}_{lp}^i$ are non-negative integers such that $\sum_{i=1}^{x-1}\sum_{l=1}^{L_i}\sum_{p=1}^{P_l^i}\overset{x}{p}{}_{lp}^i \geqq 2$.

Then the equations of Schröder of many variables which Fukuhara has obtained are written as follows:

$$(1.7) \qquad f_{lp}^i[\varphi(x)] = \lambda_i f_{lp}^i(x)+\delta f_{lp-1}^i(x)+\Psi_{lp}^i(x) ,^{(1)}$$

where $\Psi_{lp}^i(x)$ is a suitable linear combination with constant coefficients of $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P_m^j} f_{mq}^j(x)^{\overset{i}{p}{}_{mq}^j}$ for all sets of $\overset{i}{p}{}_{mq}^j$ satisfying $\lambda_i=\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P_m^j}\lambda_{mq}^{j\ \overset{i}{p}{}_{mq}^j}$. Assuming (1.3), Fukuhara has solved the equations (1.7), making use of transformation of the equations themselves. In this chapter, first, modifying his method, we solve the equations (1.7) directly, and next we add some remarks.

## § 2. Formal solutions.

In (1.2), put $T=\|t_\nu^\mu\|$. We consider the transformation of the variables $x^\nu$ and the functions $\varphi^\nu$ as follows: $\bar{x}^\mu=t_\nu^\mu x^\nu$, $\bar{\varphi}^\mu=t_\nu^\mu\varphi^\nu$. Put $f_{lp}^i(x)=f_{lp}^i(T^{-1}\bar{x})=\bar{f}_{lp}^i(\bar{x})$, then the equations (1.7) are transformed into the equations

1) $f_{l0}^i\equiv 0$ . Hereafter we use this convention.

as follows:

$$\bar{f}^i_{ip}[\bar{\varphi}(x)] = \lambda_i \bar{f}^i_{ip}(\bar{x}) + \delta \bar{f}^i_{ip-1}(\bar{x}) + \bar{\Psi}^i_{ip}(\bar{x}) . \tag{2.1}$$

These are of the same forms and moreover the constant coefficients in $\bar{\Psi}^i_{ip}(\bar{x})$ are the same as those in $\Psi^i_{ip}(x)$. Put $T^{-1} = \| T^\mu_\nu \|$, then $\bar{\varphi}^\mu(x) = t^\mu_\nu \varphi^\nu(x) = t^\mu_\nu a^\nu_\omega x^\omega + \cdots = t^\mu_\nu a^\nu_\omega T^\omega_\sigma \bar{x}^\sigma + \cdots = \bar{\varphi}^\mu(\bar{x})$. By (1.2), $\| t^\mu_\nu a^\nu_\omega T^\omega_\sigma \| = TAT^{-1} = \hat{A}$. Therefore we can write $\bar{\varphi}^\mu(\bar{x})$ as follows:

$$\bar{\varphi}^\mu(\bar{x}) = \bar{\varphi}^i_{ip}(\bar{x}) = \lambda^i_{ip} \bar{x}^i_{ip} + \delta \bar{x}^i_{ip-1} + \cdots\cdots .^{(1)} \tag{2.2}$$

We seek for the formal solutions of (2.1). In the following,[2] for brevity, we drop the bars. After having differentiated both sides of (2.1) with respect to $x^\nu$, we put $x^\nu = 0$. Then we have recurring formulae determining the values of the derivatives of $f^i_{ip}$ for $x^\nu = 0$ and the coefficients of $\Psi^i_{ip}$. Thus we see that the equations (2.1) have the formal solutions as follows:

$$f^i_{ip}(x) = x^i_{ip} + \cdots\cdots , \tag{2.3}$$

where the coefficients of $\prod_{j=1}^{i-1} \prod_{m=1}^{L_j} \prod_{q=1}^{P^j_m} x^j_{mq}{}^{\overset{i}{p}{}^j_{mq}}$ are put zero.[3] For these solutions, the coefficients of $\Psi^i_{ip}$ are uniquely determined, consequently the form of the equations of Schröder are uniquely determined.

## § 3. Convergence of the formal solutions.

Modifying Fukuhara's proof, we directly prove the convergence of the formal solutions obtained in § 2.

We take positive numbers $\Lambda_i$ such that $1 > \Lambda_i > |\lambda_i|$. Let the maximum values of $\Lambda_i$ be $\Lambda$. Since $\Lambda < 1$, if we take a sufficiently great number $N$, then $\Lambda^N < |\lambda_i|$. In (1.2), $\delta$ is arbitrary, therefore we take $|\delta|$ so small that $|\lambda_i| + |\delta| < \Lambda_i$. If we take a sufficiently small $r$, then, from (2.2), for $|x^\mu| \leq r$,

$$|\varphi^i_{ip}(x)| < \Lambda_i |x| , \tag{3.1}$$

where $|x| = \max. |x^\mu|$. We denote the sum of all the terms of at most

1) For $p=1$, there does not appear the term $\bar{x}^i_{i0} = \bar{x}^i_{ip-1}$. Hereafter we use this convention

2) In §2 and §3 of this chapter.

3) The process of determination of the values of the derivatives is quite analogous to that in the paper: Urabe, ibid. The coefficients of $\prod_j \prod_m \prod_q x^j_{mq}{}^{\overset{i}{p}{}^j_{mq}}$ can be taken arbitrarily, but, for simplicity, in this paper we put these zero.

$(N-1)$-th order in the formal solutions $f^i_{ip}(x)$ by $P^i_{ip}(x)$. Then, for $|x|\leq r$, there exists a positive number $A$ such that

$$(3.2)\qquad |P^i_{ip}(\varphi)-\lambda_i P^i_{ip}(x)-\delta P^i_{ip-1}(x)-\Psi^i_{ip}(P)|\leq A|x|^N,$$

where $\Psi^i_{ip}(P)$ are the functions which are obtained from $\Psi^i_{ip}(x)$ by substituting $P^j_{mq}$ for $f^j_{mq}$.

We consider the family $\mathfrak{F}^i_{ip}$ of the functions $f^i_{ip}$ which are regular for $|x|\leq r$ and satisfy the following conditions:

$$(3.3)\qquad |f^i_{ip}(x)-P^i_{ip}(x)|\leq K^i_{ip}|x|^N.$$

Now, from (3.1), for $|x|\leq r$, the following functions are also regular:

$$(3.4)\qquad 'f^i_{ip}(x)=\frac{1}{\lambda_i}\Big[f^i_{ip}\{\varphi(x)\}-\delta f^i_{ip-1}(x)-\Psi^i_{ip}(f)\Big].$$

Now it follows that

$$\begin{aligned}(3.5)\quad 'f^i_{ip}(x)-P^i_{ip}(x)&=\frac{1}{\lambda_i}\Big[f^i_{ip}(\varphi)-\delta f^i_{ip-1}(x)-\Psi^i_{ip}(f)-\lambda_i P^i_{ip}(x)\Big]\\ &=\frac{1}{\lambda_i}\Big[f^i_{ip}(\varphi)-P^i_{ip}(\varphi)\Big]+\frac{1}{\lambda_i}\Big[P^i_{ip}(\varphi)-\lambda_i P^i_{ip}-\delta P^i_{ip-1}-\Psi^i_{ip}(P)\Big]\\ &\quad+\frac{1}{\lambda_i}\Big[\delta P^i_{ip-1}-\delta f^i_{ip-1}+\Psi^i_{ip}(P)-\Psi^i_{ip}(f)\Big].\end{aligned}$$

Now, there exist the positive numbers $B^i_{ip}$ such that, for $|x|\leq r$,

$$(3.6)\qquad |\delta P^i_{ip-1}-\delta f^i_{ip-1}+\Psi^i_{ip}(P)-\Psi^i_{ip}(f)|\leq B^i_{ip}\,'K^i_{ip}|x|^N,$$

where $'K^i_{ip}=\max\limits_{j\leq i,\, m\leq l,\, q<p} K^j_{mq}$. From (3.3) and (3.1), we have:

$$(3.7)\qquad |f^i_{ip}(\varphi)-P^i_{ip}(\varphi)|\leq K^i_{ip}|\varphi|^N<K^i_{ip}\Lambda^N|x|^N.$$

Thus, from (3.5), (3.2), (3.6) and (3.7), we have:

$$|'f^i_{ip}-P^i_{ip}|\leq\frac{K^i_{ip}\Lambda^N+A+B^i_{ip}\,'K^i_{ip}}{|\lambda_i|}|x|^N.$$

Thus, in order that $'f^i_{ip}(x)$ also belongs to the family $\mathfrak{F}^i_{ip}$, it is sufficient if

$$(3.8)\qquad K^i_{ip}\Lambda^N+A+B^i_{ip}\,'K^i_{ip}\leq K^i_{ip}|\lambda_i|.$$

Now $\Lambda^N<|\lambda_i|$, therefore we can take $K^i_{ip}$ successively so great that (3.8) hold. For such $K^i_{ip}$, $'f^i_{ip}(x)$ belongs to the family $\mathfrak{F}^i_{ip}$.

Then, applying successively the theorem of existence of fixed points on $\mathfrak{F}^i_{ip}$, we see that there exist regular solutions of (2.1). Now, for sufficiently great $N$, the formal solutions of (2.1) are uniquely determined for given $P^i_{lp}(x)$. Thus we can conclude that the formal solutions (2. 3) converge for $|x|\leqq r$, namely they express the regular solutions of (2. 1), consequently the regular solutions of (1. 7).

## § 4. Modification of the equations.

For the simplicity, we transform the equations (2. 1) into those of the same forms where $\delta=1$. For this purpose, we put as follows:

$$\text{(4. 1)} \qquad \tilde{x}^i_{lp} = \delta^{P^i_l - p}\,\bar{x}^i_{lp}\,, \quad \tilde{\varphi}^i_{lp} = \delta^{P^i_l - p}\,\bar{\varphi}^i_{lp}\,, \quad \tilde{f}^i_{lp} = \delta^{P^i_l - p}\,\bar{f}^i_{lp}\,,$$

Substituting these into (2. 1), we have:

$$\text{(4. 2)} \qquad \tilde{f}^i_{lp}[\bar{\varphi}(x)] = \lambda_i \tilde{f}^i_{lp}(\bar{x}) + \tilde{f}^i_{lp-1}(\bar{x}) + \tilde{\Psi}^i_{lp}(\bar{x})\,,$$

where $\tilde{\Psi}^i_{lp}$ are of the same forms with respect to $\tilde{f}^i_{lp}$ as $\bar{\Psi}^i_{lp}$ with respect to $\bar{f}^i_{lp}$, and differ only in the coefficients. Now, from (4. 1), it follows that

$$\text{(4. 3)} \qquad \tilde{\varphi}^i_{lp} = \delta^{P^i_l - p}(\lambda_i \bar{x}^i_{lp} + \delta \bar{x}^i_{lp-1} + \cdots) = \lambda_i \tilde{x}^i_{lp} + \tilde{x}^i_{lp-1} + \cdots\,,$$

and, for the solutions $\bar{f}^i_{lp}$ given by (2. 3),

$$\text{(4. 4)} \qquad \tilde{f}^i_{lp}(x) = \delta^{P^i_l - p}(\bar{x}^i_{lp} + \cdots) = \tilde{x}^i_{lp} + \cdots\,,$$

where all the coefficients of $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P^j_m} \tilde{x}^{j\;p^j_{mq}}_{mq}$ vanish. Thus, if we regard the functions $\tilde{f}^i_{lp}$ and $\tilde{\varphi}^i_{lp}$ as the functions of the variables $\tilde{x}^i_{lp}$, then, from (4. 2), we have:

$$\text{(4. 5)} \qquad \tilde{f}^i_{lp}(\tilde{\varphi}) = \lambda_i \tilde{f}^i_{lp}(\tilde{x}) + \tilde{f}^i_{lp-1}(\tilde{x}) + \tilde{\Psi}^i_{lp}(\tilde{x})\,.$$

Thus we see that there exist regular solutions $\tilde{f}^i_{lp}$ of the equations (4. 5), where $\tilde{\varphi}^i_{lp}$ and $\tilde{f}^i_{lp}$ are expanded as (4. 3) and (4. 4) in the vicinity of $\tilde{x}^i_{lp}=0$. These solutions $\tilde{f}^i_{lp}$ are formally determined directly from (4. 5) in the same way as $\bar{f}^i_{lp}$ from (2. 1), and, for these solutions, the forms of the equations (4. 5) are uniquely determined.

Put $\sum_{i=1}^{R}\sum_{l=1}^{L_i}\oplus\, \Delta^i_l = \Delta = \|\delta^u_v\|$, where $\Delta^i_l$ is a matrix of $P^i_l$-th order which has the form as follows: $\Delta^i_l = \begin{pmatrix} \delta^{P^i_l-1} & 0 & \cdots & 0 \\ 0 & \delta^{P^i_l-2} & & \vdots \\ \vdots & & \ddots & \\ \vdots & & \delta & 0 \\ 0 & \cdots\cdots & 0 & 1 \end{pmatrix}$. Then it is evident that

$\Delta^i_l \tilde{A}^i_l (\Delta^i_l)^{-1} = A^i_l = \begin{pmatrix} \lambda_i & 0 & \cdots & 0 \\ 1 & \lambda_i & & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 1 & \lambda_i \end{pmatrix}$. Put $\Delta T = S = \|s^u_v\|$, then it is readily seen that

(4.6) $$\overset{\circ}{A} = SAS^{-1} = \sum_{i=1}^{R}\sum_{l=1}^{L_i}\oplus A_i^l .$$

From (4.1), we have;

(4.7) $$\tilde{x}^\mu = s_\nu^\mu x^\nu , \quad \tilde{\varphi}^\mu = s_\nu^\mu \varphi^\nu \quad \text{and} \quad \tilde{f}^\mu = \delta_\nu^\mu f^\nu .$$

The last formulae give the relations between the solutions of the initial equations (1.7) and those of the transformed equations (4.5).

## § 5. The case where the absolute values of all the eigen values are greater than unity.

In this paragraph, we discuss the case where $|\lambda_i|>1$. Fukuhara has not discussed this case sufficiently, but he has only showed the existence of the solutions. As the remark to his paper, we give explicit forms of the solutions and we show that the same results really hold even in this case as in the former case.

In the following, we study the equations (4.5) instead of (1.7). Consequently, hereafter we drop the tilde on $f_{ip}^l$ in (4.5).

Put

(5.1) $$'x^\mu = \varphi^\mu(x) = a_\nu^\mu x^\nu + \cdots\cdots ,$$

then, from $\det.|a_\nu^\mu| \neq 0$, we can solve the above equations with respect to $x^\nu$ as follows:

(5.2) $$x^\nu = \psi^\nu('x) = b_\mu^\nu{}'x^\mu + \cdots\cdots .$$

Then it is evident that $B=\|b_\nu^\mu\|=A^{-1}$. Let the Jordan's form of $B$ be $\overset{\circ}{B}$, then it is readily seen that, for a suitable matrix $U=\|u_\nu^\mu\|$,

(5.3) $$\overset{\circ}{B} = UBU^{-1} = \sum_{i=1}^{R}\sum_{l=1}^{L_i}\oplus B_i^l ,$$

where $B_i^l$ is a matrix of $P_i^l$-th order which has the form as follows:

$B_i^l=\begin{pmatrix}\lambda_i' & 0 & \cdots & 0 \\ 1 & \lambda_i' & & \vdots \\ 0 & \ddots & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 1 & \lambda_i'\end{pmatrix}$. Here $\lambda_i'$ are eigen values of the matrix $B$ and $\lambda_i'=1/\lambda_i$, consequently $|\lambda_i'|<1$.

By § 4, if we put $'\bar{x}^\mu=u_\nu^\mu{}'x^\nu$, then, for the equations

(5.4) $$'f_{ip}^l(\psi) = \lambda_i'\,'f_{ip}^l('x) + 'f_{ip-1}^l('x) + '\Psi_{ip}^l('x) ,$$

there exist regular solutions $'f_{ip}^l$, which are expanded as follows:

(5.5) $$'f_{ip}^l = '\bar{x}_{ip}^l + \cdots\cdots ,$$

where all the coefficients of the terms $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P_m^j} {'\bar{x}_{mq}^j}^{'\overset{i}{p}{}_{mq}^j}$ vanish. Here $'\Psi_{lp}^i('x)$ is a suitable linear combination of $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P_m^j} {'f_{mq}^j('x)}^{'\overset{i}{p}{}_{mq}^j}$ for all sets of $'\overset{i}{p}{}_{mq}^j$ satisfying $\lambda_i'=\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P_m^j} {'\lambda_{mq}^j}^{'\overset{i}{p}{}_{mq}^j}$. Since $\lambda_i'=1/\lambda_i$, the set of $'\overset{i}{p}{}_{mq}^j$ is the same as the set of $\overset{i}{p}{}_{mq}^j$ satisfying $\lambda_i=\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P_m^j} {\lambda_{mq}^j}^{\overset{i}{p}{}_{mq}^j}$.

In order to solve the equations (5.4) with respect to $'f_{lp}^i('x)$, we use a lemma. We attach the non-zero numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ to the variables $u_1, u_2, \dots, u_n$ respectively. For the monomial $u_1^{p_1}u_2^{p_2}\dots u_n^{p_n}$, we call the number $\lambda_1^{p_1}\lambda_2^{p_2}\dots\lambda_n^{p_n}$ the order of the monomial. When a polynomial $f(u)$ is a sum of the monomials of the same order, we call the common order of each monomial the order of the polynomial. Then we have the following

**Lemma.** *When $f_i(u)$ $(i=1, 2, \dots)$ are polynominals of the order $\mu_i$, then $\prod_i [f_i(u)]^{p_i}$ is a polynomial of the order $\prod_i \mu_i^{p_i}$.*

Proof. For the monomial $u_1^{p_1}\cdots u_n^{p_n}$ of the order $\mu$,

$$(5.6)\qquad (\lambda_1 u_1)^{p_1}(\lambda_2 u_2)^{p_2}\dots(\lambda_n u_n)^{p_n} = \mu u_1^{p_1}\dots u_n^{p_n},$$

and conversely, when (5.6) holds, the order of $u_1^{p_1}\dots u_n^{p_n}$ is $\mu$. In order that the polynomial $f(u)$ be a polynomial of the order $\mu$, it is necessary and sufficient that $f(\lambda u)=\mu f(u)$. Then we have:

$$\prod_i [f_i(\lambda u)]^{p_i} = \prod_i [\mu_i f_i(u)]^{p_i} = \prod_i \mu_i^{p_i}\cdot\prod_i [f_i(u)]^{p_i}.$$

From this, the order of $\prod_i [f_i(u)]^{p_i}$ is $\prod_i \mu_i^{p_i}$. Q. E. D.

Solving (5.4) with respect to $'f_{lp}^i('x)$, we have:

$$(5.7)\qquad 'f_{lp}^i('x) = \lambda_i 'f_{lp}^i(x)-(\lambda_i)^2\, 'f_{lp-1}^i(x)+\dots+(-1)^{p-1}(\lambda_i)^p\, 'f_{l1}^i(x)+'\Psi_{lp}^i(x),$$

where $'\Psi_{lp}^i$ is a linear combination of $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P_m^j} {'f_{mq}^j(x)}^{\overset{i}{p}{}_{mq}^j}$, for all sets of $\overset{i}{p}{}_{mq}^j$ satisfying $\lambda_i=\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P_m^j} {\lambda_{mq}^j}^{\overset{i}{p}{}_{mq}^j}$.

By induction, we prove (5.7). For $i=l=p=1$, since $'\Psi_{11}^1('x)=0$, (5.7) is valid evidently. We assume that (5.7) are valid for $j, m, q$ such that that $j\leqq i$, $m\leqq l$, $q<p$. Then, if we attach the number $\lambda_j$ to $'f_{mq}^j(x)$, then, by the lemma, it is readily seen that $'f_{mq}^j('x)$ is a polynomial of the order $\lambda_j$. Thus, by the lemma, we see that $'\Psi_{lp}^i('x)$ is a polynomial of $'f_{mq}^j(x)$ of the order $\lambda_i$. From (5.4), it follows that

$$(5.8)\qquad 'f_{lp}^i('x) = \lambda_i['f_{lp}^i(x)-'f_{lp-1}^i('x)-'\Psi_{lp}^i('x)].$$

By our assumption,

$'f^i_{ip-1}('x)=\lambda_i 'f^i_{ip-1}(x)-(\lambda_i)^2\,'f^i_{ip-2}(x)+\cdots+(-1)^{p-2}(\lambda_i)^{p-1}\,'f^i_{i1}(x)+'\Psi^i_{ip-1}(x)$.
Substituting this into (5.8), from the above mentioned, we have (5.7) for $i, l, p$. Thus (5.7) is valid for any $i, l, p$.

We write (5.7) briefly as follows:

$$(5.9)\qquad 'f^\mu('x) = e^\mu_\nu\,'f^\nu(x)+'\Psi^\mu(x)\ .$$

By the linear transformation $'f^\mu=l^\mu_\nu f^\nu$, (5.9) are transformed as follows:

$$(5.10)\qquad f^\mu('x) = L^\mu_\nu e^\nu_\omega l^\omega_\sigma f^\sigma(x)+\Psi^\mu(x)\ ,$$

where $\Psi^\mu(x)$ is of the same form with respect to $f^\mu(x)$ as $'\Psi^\mu(x)$ with respect to $'f^\mu(x)$. Here $L=\|l^\mu_\nu\|$ is a direct sum of the matrices of $P^i_l$-th order corresponding to the block of $\|e^\mu_\nu\|=E$ and $\|L^\mu_\nu\|=L^{-1}$. Now, it is easily proved that there exists such matrix $L$ that $L^{-1}EL=\mathring{A}$. If we take such $L$, then (5.10) becomes the equations of the forms (4.5). Now, since $E=\mathring{B}^{-1}$, putting $L^{-1}U=V=\|v^\mu_\nu\|$, we have: $VU^{-1}\mathring{B}^{-1}UV^{-1}=\mathring{A}$. From (5.3), it follows that $VB^{-1}V^{-1}=\mathring{A}$, namely

$$(5.11)\qquad VAV^{-1} = \mathring{A}\ .$$

Now, from (5.5), it follows that $'f^\mu(x)=\bar{x}^\mu+\cdots=u^\mu_\nu x^\nu+\cdots$, consequently $f^\mu(x)=L^\mu_\nu\,'f^\nu(x)=L^\mu_\nu u^\nu_\omega x^\omega+\cdots$. If we transform the variables $x^\mu$ to $\tilde{x}^\mu$ by the substitution $\tilde{x}^\mu=v^\mu_\nu x^\nu$, then it follows that

$$(5.12)\qquad f^\mu(x) = \tilde{x}^\mu+\cdots\cdots.$$

If we transform the given functions $\varphi^\mu$ to $\tilde{\varphi}^\mu$ by the substitution $\tilde{\varphi}^\mu=v^\mu_\nu\varphi^\nu$, then, from (5.11), the expansions of $\tilde{\varphi}^\mu$ are as follows:

$$(5.13)\qquad \tilde{\varphi}^i_{ip} = \lambda_i\tilde{x}^i_{ip}+\tilde{x}^i_{ip-1}+\cdots\cdots\ .$$

Since $\tilde{x}^\mu=v^\mu_\nu x^\nu=L^\mu_\omega\bar{x}^\omega$, making use of the lemma, it is readily seen that all the coefficients of $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P^j_m}\tilde{x}^{j\ \ t_j}_{mq\ p_{mq}}$ in $f^i_{ip}(x)$ vanish, because all the corresponding coefficients in $'f^i_{ip}(x)$ vanish. Then, the solutions given by (5.12) are quite the same in the form as the solutions in the case where $|\lambda_i|<1$.

Thus we see that the results in § 4 are valid also in the case where $|\lambda_i|>1$.

Summarizing the results in this chapter, we have

**Theorem I.** *For the given regular functions* $\varphi^\mu(x)=a^\mu_\nu x^\nu+\cdots$, $(det.|a^\mu_\nu|\neq0)$, *we assume that the absolute values of all the eigen values* $\lambda_i$*'s of the matrix* $\|a^\mu_\nu\|$ *are either greater or less than unity. We consider the functional equations as follows:*

(*S*) $f^i_{i\nu}[\varphi(x)] = \lambda_i f^i_{i\nu}(x) + f^i_{i\nu-1}(x) + \Psi^i_{i\nu}(x)$ ,

*where* $\Psi^i_{i\nu}(x)$ *is a suitable linear combination with constan coefficients of* $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P^j_m} f^j_{mq}(x)^{p^{ij}_{mq}}$ *for all sets of non-negative integers* $p^{ij}_{mq}$ *satisfying* $\lambda_i = \prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P^j_m} \lambda^{j}_{mq}{}^{p^{ij}_{mq}}$, $(\sum_{j=1}^{i-1}\sum_{m=1}^{L_j}\sum_{q=1}^{P^j_m} p^{ij}_{mq} \geqq 2)$. *We consider the linear transformations* $\tilde{x}^\mu = t^\mu_\nu x^\nu$ *and* $\tilde{\varphi}^\mu = t^\mu_\nu \varphi^\nu$ *such that the expansions of* $\tilde{\varphi}^\mu$ *become as follows:*

$$(\varphi) \qquad \tilde{\varphi}^i_{i\nu} = \lambda_i \tilde{x}^i_{i\nu} + \tilde{x}^i_{i\nu-1} + \cdots\cdots .^{(1)}$$

*Then, for the equations* (*S*), *there exist regular solutions* $f^i_{i\nu}$ *which are expanded as follows:*

$$(f) \qquad f^i_{i\nu} = \tilde{x}^i_{i\nu} + \cdots\cdots ,$$

*where all the coefficients of* $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P^j_m} \tilde{x}^j_{mq}{}^{p^{ij}_{mq}}$ *vanish. Moreover, these solutions are formally determined directly from the equations* (*S*), *consequently the solutions of such forms are unique, and, for these solutions, the form of the equations* (*S*) *is uniquely determined.*

## Chapter II. Equations of Schröder and characteristic equations.

### § 1. Preliminaries.

For the given functions $\varphi^\mu(x)$, we consider the transformation

$$(1.1) \qquad \mathfrak{T}: \ 'x^\mu = \varphi^\mu(x) = a^\mu_\nu x^\nu + \cdots\cdots , \quad \text{det.}|a^\mu_\nu| \neq 0 .$$

We assume that there exists a one parameter group $\mathfrak{G}$ of transformations which contains the given transformation $\mathfrak{T}$. We take a canonical parameter $t$[2] and assume that, for $t=t_0$, the transformation of $\mathfrak{G}$ becomes the given transformation. Let the operator of $\mathfrak{G}$ be $X \equiv \xi^\mu \frac{\partial}{\partial x^\mu}$. We assume that $\xi^\mu$ are regular and the expansions of $\xi^\mu$ in the vicinity of $x^\nu=0$ are as follows:

$$(1.2) \qquad \xi^\mu(x) = c^\mu_\nu x^\nu + \cdots\cdots .$$

When all the eigen values of $\|c^\mu_\nu\| = C$ lie in a convex domain which does not contain the origin, in the previous paper[3], we have solved the characteristic equations of the linear differential equation $Xf=0$. In this chapter, from the characteristic equations of a linear homogeneous differential equation, we deduce the equations of the analogous form as the equations

1) This is always possible.
2) We assume that $t$ is real
3) Urabe, ibid.

of Schröder $(S)$ and, making use of this result, we solve them. Besides, by means of the results in the previous paper[1], we get a new form of the functional equations simpler than those in Theorem I.

## § 2. The relation of the eigen values of $A$ and $C$.

The finite transformations of $\mathfrak{G}$ are obtained by integrating the following differential equations and by putting $'x^{\mu}=x^{\mu}$ for $t=0$,

$$\frac{d'x^{\mu}}{dt} = \xi^{\mu}('x) . \tag{2.1}$$

The results integrated are as follows: $'x^{\mu}=e^{tX}(x^{\mu})$, therefore the finite transformations of $\mathfrak{G}$ are expanded in the vicinity of $x^{\nu}=0$ as follows:

$$'x^{\mu} = a^{\mu}_{\nu}(t)x^{\nu}+\cdots\cdots . \tag{2.2}$$

By our assumption,

$$a^{\mu}_{\nu}(t_0) = a^{\mu}_{\nu} , \quad \cdots\cdots . \tag{2.3}$$

Substituting (2.2) into (2.1)[2], by comparison of the terms of the first order of $x^{\nu}$, we have $\frac{da^{\mu}_{\nu}(t)}{dt}=c^{\mu}_{\omega}a^{\omega}_{\nu}(t)$. Put $\|a^{\mu}_{\nu}(t)\|=A(t)$, $\|a^{\mu}_{\nu}\|=A$, then

$$\frac{dA(t)}{dt} = CA(t) . \tag{2.4}$$

Integrating (2.4), we have $A(t)=e^{tC}$. From (2.3), it follows that

$$A = A(t_0) = e^{t_0C} . \tag{2.5}$$

Let the Jordan's form of $C$ be $\mathring{C}$, and we write $\mathring{C}$ as follows:

$$\mathring{C} = SCS^{-1} = \sum_{i=1}^{R}\sum_{l=1}^{L_i}\oplus C^i_l , \quad C^i_l = \begin{pmatrix} \mu_i & 0 & \cdots & 0 \\ 1 & \mu_i & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 1 & \mu_i \end{pmatrix}^{(3)} , \tag{2.6}$$

where $C^i_l$ is a matrix of $P^i_l$-th order. Then, from (2.5), it is readily seen that the Jordan's form of $A$ is written as follows:

$$\mathring{A} = \sum_{i=1}^{R}\sum_{l=1}^{L_i}\oplus A^i_l , \quad A^i_l = \begin{pmatrix} \lambda_i & 0 & \cdots & 0 \\ 1 & \lambda_i & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 1 & \lambda_i \end{pmatrix}^{(4)} , \tag{2.7}$$

1) Urabe, ibid.

2) When the number of the paragraph denotes that of the same chapter, we omit the number of the chapter.

3) $\mu_1, \mu_2, \cdots\cdots, \mu_R$ are distinct from one another.

4) $\lambda_1, \lambda_2, \cdots\cdots \lambda_R$ are not necessarily distinct from one another.

where $A_i^l$ is a matrix of $P_i^l$-th order and

$$\lambda_i = e^{t_0 \mu_i} . \tag{2.8}$$

Now, from (2.6) and (2.8), it follows that

$$e^{t_0 C_i^l} = \begin{pmatrix} \lambda_i & 0 \cdots\cdots & 0 & 0 \\ \lambda_i t_0 & \lambda_i & & \vdots \\ \lambda_i \frac{t_0^2}{2!} & \lambda_i t_0 & \ddots & \vdots \\ \vdots & \vdots & \lambda_i & 0 \\ \lambda_i \frac{t_0^{P_i^l-1}}{(P_i^l-1)!} & \lambda_i \frac{t_0^{P_i^l-2}}{(P_i^l-2)!} \cdots & \lambda_i t_0 & \lambda_i \end{pmatrix} . \tag{2.9}$$

Therefore there exists a matrix $K = \sum_{i=1}^{R} \sum_{l=1}^{L_i} \oplus K_i^l$ such that $K_i^l e^{t_0 C_i^l} (K_i^l)^{-1} = A_i^l$ and $K_i^l$ are of the form $\begin{pmatrix} \times & 0 \cdots & 0 \\ \times & \times & \vdots \\ \vdots & \ddots & 0 \\ \times & \cdots\cdots & \times \end{pmatrix}$. Then evidently it follows that

$$K e^{t_0 \mathring{C}} K^{-1} = \mathring{A} . \tag{2.10}$$

Consequently, from (2.6), we have: $KSe^{t_0 C} S^{-1} K^{-1} = \mathring{A}$. From (2.5), putting $KS = T$, we have

$$TAT^{-1} = \mathring{A} . \tag{2.11}$$

By our assumptions on $\mu_i$, there exists a line $L$ passing through the origin, in one side of which all the eigen values $\mu_i$ lie. By $\rho(\mu_i)$, we denote the distance from the point $\mu_i$ to the line $L$. By $\mu_a$ $(a=1, 2, \ldots, S)$ we denote $\mu_i$ which is not expressed as follows:

$$\mu_i = \mu_1 p_1 + \mu_2 p_2 + \cdots + \mu_R p_R , \tag{2.12}$$

for non-negative integers $(p_1, p_2, \ldots, p_R)$ such that $p_1 + p_2 + \cdots + p_R \geqq 2$. By $\mu_x$ $(x=S+1, \ldots, R)$, we denote $\mu_i$ which is expressed as (2.12), and we arrange $\mu_x$ so that $\rho(\mu_{S+1}) \leqq \rho(_{S+2}) \leqq \cdots \leqq \rho(\mu_R)$. Then the relations (2.12) which really hold, are as follows:

$$\mu_i = \mu_1 p_1 + \mu_2 p_2 + \cdots + \mu_{i-1} p_{i-1} . \tag{2.13}$$

When the relation (2.12) holds, from (2.8), the following relation among $\lambda_i$ holds:

$$\lambda_i = \lambda_1^{p_1} \lambda_2^{p_2} \cdots\cdots \lambda_R^{p_R} . \tag{2.14}$$

However, when (2.14) holds, (2.12) does not necessarily hold. By $\mu_{ip}^l$, we denote the eigen values of $C$ which is a $(p, p)$-element of $C_i^l$, and we write the relation (2.13) as follows:

(2. 15) $$\mu_i=\sum_{j=1}^{i-1}\sum_{m=1}^{L_j}\sum_{q=1}^{P_m^j}\mu_{mq}^j p_{mq}^i .$$

## § 3. Integration of the characteristic equations.

By the definition, the characteristic equations of the linear homogeneous differential equation $Xf=0$ are written as follows:

(3. 1) $$Xg_{ip}^i=\mu_i g_{ip}^i+g_{ip-1}^i+\Phi_{ip}^i ,$$

where $g_{i0}^i\equiv 0$ and $\Phi_{ip}^i$ is a suitable linear combination of $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P_m^j} g_{mq}^{j\,p_{mq}^{i\,j}}$ for all sets of $p_{mq}^{j}$ such that $\mu_i=\sum_{j=1}^{i-1}\sum_{m=1}^{L_j}\sum_{q=1}^{P_m^j}\mu_{mq}^j p_{mq}^i$. By the previous paper,[1] if we transform the variables $x^\mu$ and the functions $\xi^\mu$ by the substitutions $\bar{x}^\mu=s_\nu^\mu x^\nu$ and $\bar{\xi}^\mu=s_\nu^\mu\xi^\nu$ where $\|s_\nu^\mu\|=S$ given by (2. 6), then there exist solutions $g_{ip}^i$ which are expanded as follows:

(3. 2) $$g_{ip}^i=\bar{x}_{ip}^i+\cdots\cdots ,$$

where all the coefficients of $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P_m^j}\bar{x}_{mq}^{j\,p_{mq}^{i\,j}}$ vanish. By these substitutions, from (2. 6), it is evident that the functions $\bar{\xi}^\mu$ are expanded as follows:

(3. 3) $$\bar{\xi}_{ip}^i=\mu_i\bar{x}_{ip}^i+\bar{x}_{ip-1}^i+\cdots\cdots .$$

After having substituted $'x^\mu$ for $x^\mu$ in (3. 1), making use of (2. 1), we integrate the equations (3. 1). The results are as follows:

(3. 4)
$$g_{ip}^i('x)=e^{\mu_i t}\left[\frac{t^{p-1}}{(p-1)!}g_{i1}^i(x)+\frac{t^{p-2}}{(p-2)!}g_{i2}^i(x)+\cdots+tg_{ip-1}^i(x)+g_{ip}^i(x)+\Psi_{ip}^i(x,t)\right]$$

where $\Psi_{ip}^i(x,t)$ is a linear combination of $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P_m^j} g_{mq}^j(x)^{p_{mq}^{i\,j}}$ for all sets of $p_{mq}^{j}$ such that $\mu_i=\sum_{j=1}^{i-1}\sum_{m=1}^{L_j}\sum_{q=1}^{P_m^j}\mu_{mq}^j p_{mq}^j$ and its coefficients are polynomials of $t$ having no constant terms.

In order to prove these results, we use a lemma. We attach the numbers $\mu_1,\mu_2,\dots,\mu_n$ to $u_1,u_2,\dots,u_n$ respectively, and we call the number $\mu_1 p_1+\mu_2 p_2+\cdots+\mu_n p_n$ the exponential order of the monomial $u_1^{p_1}u_2^{p_2}\cdots\cdots u_n^{p_n}$. If a polynomial $f(u)$ is a sum of the monomials of the same exponential order $\mu$, then we call $\mu$ the exponential order of the polynomial $f(u)$. Then we can easily prove the following

**Lemma.** *If $f_i(u)$ $(i=1,2\dots)$ are polynomials of the exponential order $\nu_i$, then $\prod_i[f_i(u)]^{p_i}$ is a polynomial of the exponential order $\sum_i\nu_i p_i$.*

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1) Urabe, ibid.

Proof. For a monomial of the exponential order $\mu$,

$$(e^{\mu_1}u_1)^{p_1}\ldots\ldots(e^{\mu_n}u_n)^{p_n} = e^{\mu}u_1^{p_1}\ldots\ldots u_n^{p_n}\ , \tag{3.5}$$

and conversely, when (3.5) holds, the formal exponential index $\mu$ is the exponential order of the monomial. For a polynomial $f(u)$, in order that the polynomial $f(u)$ be a polynomial of the exponential order $\mu$, it is necessary and sufficient that $f(e^{\mu_1}u_1, \ldots, e^{\mu_n}u_n)=e^{\mu}f(u)$, where $\mu$ is a formal exponential index. Then it follows that

$$\prod_i [f_i(e^{\mu}u)]^{p_i} = \prod_i [e^{\nu_i}f_i(u)]^{p_i} = e^{\sum_i \nu_i p_i}\cdot\prod_i [f_i(u)]^{p_i}\ .$$

Thus the lemma is proved. Q.E.D.

Now we shall prove (3.4). For $i=l=p=1$, substituting $'x^\nu$ for $x^\nu$ in (3.1), we have: $\xi^\nu('x)\frac{\partial g_{11}^1('x)}{\partial' x^\nu}=\mu_1 g_{11}^1('x)$. By means of (2.1), this is written as follows: $\frac{dg_{11}^1('x)}{dt}=\mu_1 g_{11}^1('x)$. Integrating this differential equation, we have: $g_{11}^1('x)=ce^{\mu_1 t}$. From the initial condition that $'x^\nu=x^\nu$ for $t=0$, we have:

$$g_{11}^1('x) = e^{\mu_1 t}g_{11}^1(x)\ . \tag{3.6}$$

Namely (3.4) is valid for $i=l=p=1$. We assume that (3.4) are valid for $j, m, q$ such that $j\leqq i$, $m\leqq l$, $q<p$. Substituting $'x^\nu$ for $x^\nu$ in (3.1), by means of (2.1), we have:

$$\frac{dg_{ip}^i('x)}{dt} = \mu_i g_{ip}^i('x)+g_{ip-1}^i('x)+\Phi_{ip}^i('x)\ . \tag{3.7}$$

If we attach the numbers $\mu_j$ to $g_{mq}^j(x)$, by our assumption and the lemma, $g_{mq}^j('x)$ is a polynomial of $g_{mq}^k(x)$ $(k\leqq j)$ of the exponential order $\mu_j$. Therefore, by the lemma, $\Phi_{ip}^i('x)$ is a polynomial of $g_{mq}^j(x)$ of the exponential order $\mu_i$, therefore $\Phi_{ip}^i('x)$ is of the form as follows: $\Phi_{ip}^i('x)=e^{\mu_i t}\,'\Psi_{ip}^i(x, t)$, where $'\Psi_{ip}^i(x, t)$ is a linear combination of $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P_m^j} g_{mq}^j(x)^{p_{mq}^{ij}}$ and its coefficients are polynomials of $t$. Then, by our assumption, from (3.4), (3.7) is written as follows:

$$\frac{dg_{ip}^i('x)}{dt} = \mu_i g_{ip}^i('x)+e^{\mu_i t}\left[\frac{t^{p-2}}{(p-2)!}g_{i1}^i(x)+\cdots+tg_{ip-2}^i(x)+g_{ip-1}^i(x)\right]$$
$$+e^{\mu_i t}\left[\Psi_{ip-1}^i(x, t)+'\Psi_{ip}^i(x, t)\right]\ .$$

Integrating this differential equation, by means of the initial condition, we have (3.4) for $i, l, p$. Thus (3.4) is valid for any $i, l, p$.

For $t=t_0$, $'x^\nu=\varphi^\nu(x)$ and, from (2.8), $e^{\omega_i t_0}=\lambda_i$. Thus, for $t=t_0$, (3.4) are written as follows:

(3.8)

$$g^i_{ip}[\varphi(x)]=\lambda_i\left[\frac{t_0^{p-1}}{(p-1)!}g^i_{i1}(x)+\frac{t_0^{p-2}}{(p-2)!}g^i_{i2}(x)+\cdots+t_0 g^i_{ip-1}(x)+g^i_{ip}(x)+'\Psi^i_{ip}(x)\right]$$

where $'\Psi^i_{ip}(x)=\Psi^i_{ip}(x,t_0)$.

If we put $f^\mu=k^\mu_\nu g^\nu$ where $\|k^\mu_\nu\|=K$ defined in § 2, then, by the same reasonings as on Chap. I (5.7), making use of (2.9) and (2.10), we have:

(3.9) $$f^i_{ip}(\varphi)=\lambda_i f^i_{ip}(x)+f^i_{ip-1}(x)+\Psi^i_{ip}(x)\ ,$$

where $\Psi^i_{ip}(x)$ is a linear combination of $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P^j_m} f^j_{mq}(x)^{p^{i\,j}_{mq}}$. For the solutions $g^i_{ip}$ given by (3.2) of the characteristic equations (3.1), from $f^\mu=k^\mu_\nu g^\nu$, we have: $f^\mu=k^\mu_\nu\bar{x}^\nu+\cdots=k^\mu_\nu s^\nu_\omega x^\omega+\cdots$. Put $\|k^\mu_\nu s^\nu_\omega\|=KS=T=\|t^\mu_\nu\|$ and $\tilde{x}^\mu=t^\mu_\nu x^\nu$, $\tilde{\varphi}^\mu=t^\mu_\nu\varphi^\nu$. Then

(3.10) $$f^i_{ip}=\tilde{x}^i_{ip}+\cdots\cdots\ ,$$

and, from (2.11),

(3.11) $$\tilde{\varphi}^i_{ip}=\lambda_i\tilde{x}^i_{ip}+\tilde{x}^i_{ip-1}+\cdots\cdots\ .$$

From $\tilde{x}^\mu=t^\mu_\nu x^\nu$ and $\bar{x}^\mu=s^\mu_\nu x^\nu$, $\tilde{x}^\mu=k^\mu_\nu\bar{x}^\nu$, consequently, from the lack of the terms $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P^j_m}\bar{x}^{j}_{mq}{}^{p^{i\,j}_{mq}}$ in $g^i_{ip}$, there do not appear the terms $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P^j_m}\tilde{x}^{j}_{mq}{}^{p^{i\,j}_{mq}}$ in $f^i_{ip}$.

Summarizing the results we have

**Theorem II.** *For the given functions $\varphi^\mu(x)$ such that $\varphi^\mu(x)=a^\mu_\nu x^\nu+\cdots\cdots$, $det.|a^\mu_\nu|\neq 0$, we assume:*

(i) *there exists a one parameter group of transformations which contains the transformation $'x^\mu=\varphi^\mu(x)$,*

(ii) *the operator functions $\xi^\mu$ of that group are expanded as follows: $\xi^\mu=c^\mu_\nu x^\nu+\cdots\cdots$,*

(iii) *all the eigen values of the matrix $\|c^\mu_\nu\|$ lie in a convex domain which does not contain the origin.*

*We consider the functional equations as follows:*

$(S')$ $$f^i_{ip}(\varphi)=\lambda_i f^i_{ip}(x)+f^i_{ip-1}(x)+\Psi^i_{ip}(x)\ ,$$

*where $\Psi^i_{ip}(x)$ is a suitable linear combination of $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P^j_m} f^j_{mq}{}^{p^{i\,j}_{mq}}$ for all sets* *of* $\overset{i}{p}{}^j_{mq}$ *such that* $\mu_i=\sum_{j=1}^{i-1}\sum_{m=1}^{L_j}\sum_{q=1}^{P^j_m}\mu^j_{mq}\overset{i}{p}{}^j_{mq}$. *When we transform the variables* $x^\mu$ *and the functions* $\varphi^\mu$ *by a suitable transformation as follows:* $\tilde{x}^\mu=t^\mu_\nu x^\nu$, $\tilde{\varphi}^\mu=t^\mu_\nu\varphi^\nu$, *then*

$$(\varphi') \qquad \tilde{\varphi}^i_{lp} = \lambda_i\tilde{x}^i_{lp}+\tilde{x}^i_{lp-1}+\cdots\cdots ,$$

*and there exist regular solutions of* $(S')$ *which are expanded as follows:*

$$(f') \qquad f^i_{lp} = \tilde{x}^i_{lp}+\cdots\cdots ,$$

*where all the coefficients of the terms* $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P^j_m}\tilde{x}^j_{mq}{}^{\overset{i}{p}{}^j_{mq}}$ *vanish.*

The equations $(S')$ obtained by integration of the characteristic equations (3.1) resemble closely to the equations $(S)$ in Theorem I. The equations $(S')$ are the finite forms of the characteristic equations of a linear homogeneous partial differential equation. The solution $f^i_{lp}$ given by $(f')$ are linear combination of the solution $g^i_{lq}$ of the characteristic equations, consequently, by the result of the previous paper[1], $f^i_{lp}$ are easily obtained. The equations $(S)$ and $(S')$ are same except one difference. The difference is only that, in the equations $(S)$ the set of numbers $\overset{i}{p}{}^j_{mq}$ satisfy $\lambda_i=\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P^j_m}\lambda^j_{mq}{}^{\overset{i}{p}{}^j_{mq}}$, while in the equations $(S')$ the set of numbers $\overset{i}{p}{}^j_{mq}$ satisfy $\mu_i=\sum_{j=1}^{i-1}\sum_{m=1}^{L_j}\sum_{q=1}^{P^j_m}\mu^j_{mq}\overset{i}{p}{}^j_{mq}$. As noticed in § 2 in this chapter, the sets of $\overset{i}{p}{}^j_{mq}$ satisfying the latter relations are in general less in number than the sets of $\overset{i}{p}{}^j_{mq}$ satisfying the former relations. Consequently, the number of the terms of $\Psi^i_{lp}$ is less in $(S')$ than in $(S)$, and the lack of the terms of the forms $\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{P^j_m}x^j_{mq}{}^{\overset{i}{p}{}^j_{mq}}$ in the solutions is less in $(S')$ than in $(S)$. The formal determination of the solutions of the form $(f)$ or $(f')$ is possible for $\Psi^i_{lp}$ in $(S)$[2], but, in general, for $\Psi^i_{lp}$ in $(S')$, this is not always possible. However, Theorem II asserts the existence of the solutions of the forms $(f')$. This means that, under the assumptions of Theorem II, the formal determination of the solutions of the forms $(f')$ is possible also for $\Psi^i_{lp}$ in $(S')$.

## § 4. Simpler form of the equations.

By the previous paper[3], if we operate $X\text{-}\mu_i$ successively on the solu-

1) Urabe, ibid.

2) Up to the present, the coefficients of $\prod_j\prod_m\prod_q x^j_{mq}{}^{\overset{i}{p}{}^j_{mq}}$ are put zero merely for the sake of simplicity. If necessary, we may give them arbitrary values. Thus, the formal determination of the solutions of the form $(f')$ is possible for $\Psi^i_{lp}$ in $(S)$.

3) Urabe, ibid.

tions $g^i_{lP^i_l}$ of the characteristic equations (3.1), then, after certain times, they become identically zero. By $(X-\mu_i)^{Q^i_l} g^i_{lP^i_l}$, we denote the first which identically vanish. Then $Q^i_l \geqq P^i_l$. Put

$$(4.1) \qquad G^i_{lp} = (X-\mu_i)^{Q^i_l - p} g^i_{lP^i_l}, \qquad (p = 1, 2, \dots, Q^i_l).$$

Then, from (3.1), we have:

$$(4.2) \quad \begin{cases} \text{for } i=a, & G^a_{lp} = g^a_{lp}; \\ \text{for } i=x, & \begin{cases} G^x_{lQ^x_l} = g^x_{lP^x_l}, \\ G^x_{lQ^x_l-1} = g^x_{lP^x_l-1} + \Phi^x_{lP^x_l}, \\ \quad \vdots \\ G^x_{lQ^x_l-P^x_l+1} = g^x_{l1} + \Phi^x_{l2} + (X-\mu_x)\Phi^x_{l3} + \dots + (X-\mu_x)^{P^x_l-2}\Phi^x_{lP^x_l}, \\ G^x_{lQ^x_l-P^x_l} = \Phi^x_{l1} + (X-\mu_x)\Phi^x_{l2} + \dots + (X-\mu_x)^{P^x_l-1}\Phi^x_{lP^x_l}, \\ \quad \vdots \\ G^x_{l1} = (X-\mu_x)^{Q^x_l-P^x_l-1}\Phi^x_{l1} + \dots + (X-\mu_x)^{Q^x_l-2}\Phi^x_{lP^x_l}. \end{cases} \end{cases}$$

Thus it is readily seen that $G^a_{lp}$ and $G^x_{lQ^x_l-P^x_l+1}, \dots, G^x_{lQ^x_l}$ are independent from one another and $G^x_{l1}, \dots, G^x_{lQ^x_l-P^x_l}$ are polynominals of $G^j_{mq}$ $(j \leqq x-1)$.[1] From (4.1), we have:

$$(4.3) \qquad XG^i_{lp} = \mu_i G^i_{lp} + G^i_{lp-1}, \quad (p = 1, 2, \dots, Q^i_l).$$

These are of the same form as (3.1) in which $\Phi^i_{lp}$ vanish. Therefore, performing the same process on (4.3) as on (3.1), we have:

$$(4.4) \quad G^i_{lp}[\varphi(x)] = \lambda_i\left[\frac{t_0^{p-1}}{(p-1)!}G^i_{l1}(x) + \frac{t_0^{p-2}}{(p-2)!}G^i_{l2}(x) + \dots + t_0 G^i_{lp-1}(x) + G^i_{lp}(x)\right].$$

In the same way as we have deduced (3.9) from (3.8), by the suitable linear transformation $F^\mu = {}'k^\mu_\nu G^\nu$, we can transform (4.4) into the equations of the following forms:

$$(4.5) \qquad F^i_{lp}[\varphi(x)] = \lambda_i F^i_{lp}(x) + F^i_{lp-1}(x), \qquad (p = 1, 2, \dots, Q^i_l).$$

Here $\|{}'k^\mu_\nu\| = K'$ are such that $K' = \sum_{i=1}^{R}\sum_{l=1}^{L_i} \oplus {}'K^i_l$, where ${}'K^i_l$ are the matrices of $Q^i_l$-th order such that

$$(4.6) \quad {}'K^i_l \begin{pmatrix} \lambda_i & 0 & \cdots & \cdots & \cdots & 0 \\ \lambda_i t_0 & & & & & \vdots \\ \vdots & & \lambda_i & & & \vdots \\ & & & & \lambda_i & 0 \\ \lambda_i \frac{t_0^{Q^i_l-1}}{(Q^i_l-1)!} & \cdots & \lambda_i \frac{t_0^{P^i_l-1}}{(P^i_l-1)!} & \cdots & \lambda_i t_0 & \lambda_i \end{pmatrix} ({}'K^i_l)^{-1} = \begin{pmatrix} \lambda_i & 0 & \cdots & 0 \\ 1 & \lambda_i & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & \lambda_i \end{pmatrix}.$$

1) Urabe, ibid.

Here we can assume that $'K^i_l$ have the forms as follows: $'K^i_l=\begin{pmatrix} \overset{1}{K}{}^i_l & 0 \\ \overset{2}{K}{}^i_l & K^i_l \end{pmatrix}$,

where $\overset{1}{K}{}^i_l$ and $K^i_l$ are of the forms $\begin{pmatrix} \times & 0 & \cdots & 0 \\ \vdots & \times & & \vdots \\ \vdots & & \ddots & 0 \\ \times & \cdots & \cdots & \times \end{pmatrix}$ and $K^i_l$ are of $P^i_l$-th order.

Then it is evident that

$$(4.7)\quad K^i_l\begin{pmatrix} \lambda_i & 0 & \cdots & \cdots & 0 \\ \lambda_i t_0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ & & t_0^{P^i_l-1}\ddots & & \vdots \\ \lambda_i\dfrac{t_0^{P^i_l-1}}{(P^i_l-1)!} & \cdots & & \ddots & 0 \\ & & & \cdots & \lambda_i \end{pmatrix}(K^i_l)^{-1}=\begin{pmatrix} \lambda_i & 0 & \cdots & \cdots & 0 \\ 1 & \lambda_i & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_i \end{pmatrix}.$$

If we write $F^\mu='k^\mu_\nu G^\nu$ as follows: $F^\mu='k^\mu_\alpha G^\alpha+'k^\mu_\omega G^\omega$, where $\alpha=\binom{i}{11}$, $\cdots$, $\binom{i}{lQ^i_l-P^i_l}$ and $\omega=\binom{i}{lQ^i_l-P^i_l+1},\cdots,\binom{i}{lQ^i_l}$, then, for the solutions $g^i_{lp}$ of the forms (3.2), we have: $F^\mu='k^\mu_\omega\bar{x}^{\bar{\omega}}+\ldots$, where $\bar{\omega}=\binom{i}{lp-(Q^i_l-P^i_l)}$ for $\omega=\binom{i}{lp}$, and, for $\mu=\binom{i}{lQ^i_l-P^i_l+1}$, $\cdots\cdots\cdots$, $\binom{i}{lQ^i_l}$, from (4.7), we can identify $\|'k^\mu_\omega\|=K^i_l$ with $K^i_l$ in §2. Therefore, for the variables $\tilde{x}^\mu$, in the same way as (3.10) is deduced, the following is concluded:

$$(4.6)\quad \begin{cases} F^\alpha_{lp}=\tilde{x}^\alpha_{lp}+\cdots\cdots, & (p=1,2,\cdots\cdots,P^\alpha_l=Q^\alpha_l), \\ F^x_{lp}=\tilde{x}^x_{lp-Q^x_l+P^x_l}+\cdots, & (p=Q^x_l-P^x_l+1,\cdots\cdots,Q^x_l). \end{cases}$$

Moreover, from the properties of $G^i_{lp}$, it is easily seen that $F^x_{l1},\cdots,F^x_{lQ^x_l-P^x_l}$ are polynominals of $F^i_{mq}$ $(i\leq x-1)$.

Thus we have

**Teorem III.** *We assume the same conditions as in Theorem II. We consider the fuuctional equations as follows:*

$$(S'_1)\qquad F^i_{lp}(\varphi)=\lambda_i F^i_{lp}(x)+F^i_{lp-1},\quad (p=1,2,\cdots,Q^i_l).$$

*When we transform the variables $x^\mu$ and the functions $\varphi^\mu$ by the suitable transformation as follows: $\tilde{x}^\mu=t^\mu_\nu x^\nu$, $\tilde{\varphi}^\mu=t^\mu_\nu\varphi^\nu$, then*

$$(\varphi')\qquad \tilde{\varphi}^i_{lp}=\lambda_i\tilde{x}^i_{lp}+\tilde{x}^i_{lp-1}+\ldots,$$

*and there exist regular solutions of $(S_1')$ such that*

$$(F_1') \quad \begin{cases} \textit{for } p = Q_i^i - P_i^i + 1, \cdots Q_i^i, & F_{ip}^i = \tilde{x}_{ip-Q_i^i+P_i^i}^i + \cdots, \\ \textit{for } p = 1, 2, , \ldots, Q_i^i - P_i^i, & F_{ip}^i \textit{ is a polynomial of } F_{mq}^j \textit{ for} \end{cases}$$

$j \leqq i-1$.

In the form, the equations $(S_1')$ are simpler than the equations $(S)$ or $(S')$. However, the number of the equations $(S_1')$ is in general greater than $n$ the number of the variables, and their solutions are not independent from one another on the whole.

(To be continued in our next)