DIRECT AND SUBDIRECT FACTORIZATIONS OF LATTICES

By

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Let a lattice L be a product[1] $L_1 \dots L_n$ of n lattices L_i $(i=1, \dots, n)$. If L has the null element 0 and the unit element 1, then L_i has the null element 0_i and the unit element 1_i. The element z_i which is expressed in $L_1 \dots L_n$ as $[0_1, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_n]$ is an element of the center of L[2]. The center of L is a Boolean algebra, and using this center we can easily solve the factorization problem of lattices[3]. But for the lattices L without 0 or 1, the centers of L do not exist. Hence for the factorization problem of such lattices, we must seek Boolean algebras. From this point of view, I investigated the direct factorizations and the subdirect factorizations of lattices without the assumption that 0 and 1 exist.

§ I. Direct Factorizations of Lattices.

By a *direct factorization* of a lattice L we mean the system of lattices L_i $(i=1, \dots, u)$, when L is isomorphic to the product $\Pi(L_i; i=1, \dots, n) = L_1 \dots L_n$. Let $\Theta(L)$ denote the set of all congruence relations on L. Funayama and Nakayama proved that $\Theta(L)$ is an upper continuous, distributive lattice by defining $\theta \leqq \phi$ if and only if $x \equiv y\,(\theta)$ implies $x \equiv y\,(\phi)$[4]. Two congruence relations θ and ϕ are called *permutable* if $a \equiv x\,(\theta)$ and $x \equiv b\,(\phi)$ for some x imply $a \equiv y\,(\phi)$ and $y \equiv b\,(\theta)$ for some y. The set of all congruence relations which are permutable with θ for all $\theta \in \Theta(L)$ is denoted by $\Gamma(L)$. And the center of $\Theta(L)$ is denoted by $\Theta_z(L)$. Since $\Theta(L)$ is distributive, $\theta \in \Theta_z(L)$ if and only if θ has its complement θ'. If $L \cong L_1 L_1$, the mapping $[x_1, x_2] \to x_1$ is a homomorphism of L onto L_1 and hence generates a congruence relation θ_1, which we call a *decomposition congruence relation*. If we denote by $\Theta_0(L)$ the set of all decomposition

1) Cardinal product in Birkhoff's [1, p. 25] sense. The numbers in square brackets refer to the list at the end of this paper.

2) Center in Birkhoff's [1, p. 27] sense.

3) Cf. Birkhoff [1] 26.

4) Cf. Birkhoff [1] 24. A complete lattice L is called *upper continuous* when $a_\delta \uparrow a$ implies $a_\delta \frown b \uparrow a \frown b$. When L is distributive, this is equivalent to $V(a; a \in S) \frown b = V(a \frown x; a \in S)$ for all $S \leqq L$. We use also 0 and 1 for the zero element and the unit element of $\Theta(L)$ respectively.

congruence relations on L, Dilworth [1, p. 351-352] proved that

$$\Theta_0(L) = \Gamma(L) \wedge \Theta_z(L),$$

and $\theta \in \Theta_0(L)$ if and only if $\theta \in \Theta_z(L)$ and θ and its complement θ' are permutable. Therefore if $\theta \in \Theta_0(L)$ then $\theta' \in \Theta_0(L)$. If we denote by L_θ the homomorphic image of L which is generated by θ, then $L \simeq L_\theta L_{\theta'}$, if and only if $\theta \in \Theta_0(L)$ and $\theta \frown \theta' = 0$, $\theta \smile \theta' = 1$.

Now we have the following theorems.

THEOREM 1·1. *$\Theta_0(L)$ is a Boolean algebra as a sublattice of $\Theta(L)$.*

PROOF. If $\theta_1, \theta_2 \in \Theta_0(L)$, then since $\theta_1, \theta_2 \in \Gamma(L)$ and $\theta_1, \theta_2 \in \Theta_z(L)$, by the properties of $\Gamma(L)$ and $\Theta_z(L)$, we have $\theta_1 \smile \theta_2 \in \Gamma(L)$[1] and $\theta_1 \smile \theta_2 \in \Theta_z(L)$, that is $\theta_1 \smile \theta_2 \in \Theta_0(L)$.

Since the complements θ_1', θ_2' of $\theta_1, \theta_2 \in \Theta_0(L)$ belong to $\Theta_0(L)$, by above $\theta_1' \smile \theta_2' \in \Theta_0(L)$. Being $\theta_1 \frown \theta_2$ the complement of $\theta_1' \smile \theta_2'$ in $\Theta(L)$, we have $\theta_1 \frown \theta_2 \in \Theta_0(L)$, completing the proof.

THEOREM 1·2. *In order that $L \simeq L_{\theta_1} \dots L_{\theta_n}$ it is necessary and sufficient that $\theta_i \in \Theta_0(L)$ $(i=1, \dots, n)$ and*

$$\theta_1 \frown \dots \frown \theta_n = 0, \qquad \theta_i \smile \theta_j = 1, \qquad (i \neq j).$$

PROOF. (i) Necessity. When $L \simeq L_{\theta_1} \dots L_{\theta_n}$, since $L \simeq L_{\theta_i} L_{\phi_i}$ where $L_{\phi_i} \simeq L_{\theta_1} \dots L_{\theta_{i-1}} L_{\theta_{i+1}} \dots L_{\theta_n}$, we have $\theta_i, \phi_i \in \Theta_0(L)$ and

$$\theta_i \frown \phi_i = 0, \qquad \theta_i \smile \phi_i = 1.$$

When $i \neq j$, being $\theta_j \geqq \phi_i$, we have

$$\theta_i \smile \theta_j \geqq \theta_i \smile \phi_i = 1.$$

If $x \equiv y$ $(\theta_1 \frown \dots \frown \theta_n)$, then $x \equiv y$ (θ_i) for all i. And the i-th component of x in $L_{\theta_1} \dots L_{\theta_n}$ is equall to the i-th component of y. Therefore $x = y$. That is

$$\theta_1 \frown \dots \frown \theta_n = 0.$$

(ii) Sufficiency. Put $\phi_1 = \theta_2 \frown \dots \frown \theta_n$, then since

$$\theta_1 \frown \phi_1 = 0, \qquad \theta_1 \smile \phi_1 = (\theta_1 \smile \theta_2) \frown \dots \frown (\theta_1 \smile \theta_n) = 1,$$

we have $L \simeq L_{\theta_1} L_{\phi_1}$.

Considering that $\theta_2, \dots, \theta_n$ are congruence relations on L_{ϕ_1}, from

$$\theta_2 \frown \dots \frown \theta_n = \phi_1, \qquad \theta_i \smile \theta_j = 1, \qquad (i \neq j),$$

as above, we have $L_{\phi_1} \simeq L_{\theta_2} L_{\phi_2}$ where $\phi_2 = \theta_3 \frown \dots \frown \theta_n$. Continuing this

1) Cf. Dilworth [1] 351, Lemma. 3.1.

process, we have

$$L \cong L_{\theta_1} \dots L_{\theta_n}.$$

THEOREM 1·3. *Associated with two direct factorizations of a lattice* L:

$$L \cong \prod(L_{\theta_i};\ 1, \dots, m), \qquad L \cong \prod(L_{\phi_j};\ j = 1, \dots, n),$$

there exists a direct factorization of L:

$$L \cong \prod(L_{\psi_{ij}};\ i = 1, \dots, m;\ j = 1, \dots, n)$$

such that

$$L_{\theta_i} \cong \prod(L_{\psi_{ij}};\ j = 1, \dots, n), \qquad L_{\phi_j} \cong \prod(L_{\psi_{ij}};\ i = 1, \dots, m)^{1)}.$$

PROOF. If we put $\psi_{ij} = \theta_i \smile \phi_j$ $(i = 1, \dots, m;\ j = 1, \dots, n)$, then $\psi_{ij} \in \Theta_0(L)$, and

$$\Lambda(\psi_{ij};\ i=1, \dots, m;\ j=1, \dots, n) = \Lambda(\theta_i \smile \phi_j;\ i=1, \dots, m;\ j=1, \dots, n)$$
$$= \Lambda(\theta_i;\ i=1, \dots, m) \smile \Lambda(\phi_j;\ j=1, \dots, n) = 0,$$

and when $i \neq k$ or $j \neq l$,

$$\psi_{ij} \smile \psi_{kl} = \theta_i \smile \phi_j \smile \theta_k \smile \phi_l = 1.$$

Hence $$L \cong \prod(L_{\psi_{ij}};\ i = 1, \dots, m;\ j = 1, \dots, n).$$

Since $$\Lambda(\psi_{ij};\ j = 1, \dots, n) = \Lambda(\theta_i \smile \phi_j;\ j = 1, \dots, n)$$
$$= \theta_i \smile \Lambda(\phi_j;\ j = 1, \dots, n) = \theta_i,$$

and $$\psi_{ij} \smile \psi_{il} = \theta_i \smile \phi_j \smile \phi_l = 1 \qquad (j \neq l),$$

we have $$L_{\theta_i} \cong \prod(L_{\psi_{ij}};\ j = 1, \dots, n).$$

Similarly for L_{ϕ_i}.

§ 2. Subdirect Factorizations of Lattices.

By a *subdirect factorization* of a lattice L we mean the system of lattices $L_\alpha (\alpha \in I)$, when L is isomorphic to the subdirect union of $L_\alpha (\alpha \in I)$ in Birkhoff's [1, p. 91] sense. If we denote by θ_α the congruence relation introduced by the homomorphism $L \rightarrow L_\alpha$, then $L_\alpha \cong L_{\theta_\alpha}$ and $\Lambda(\theta_\alpha;\ \alpha \in I) = 0$. Conversely, when $\{\theta_\alpha;\ \alpha \in I\}$ is a subset of $\Theta(L)$ such that $\Lambda(\theta_\alpha;\ \alpha \in I) = 0$, the system of lattices $L_{\theta_\alpha}(\alpha \in I)$ is a subdirect factorization of L[2].

Since $\Theta(L)$ is an upper continuous, distributive lattice, it is pseudo-complemented, and the correspondence $\theta \rightarrow \theta^{**}$ is a closure operation in

1) This theorem is already proved by Nakayama [1, p.72], without using the decomposition congruence relations on L.

2) Cf. Birkhoff [1] 92.

$\Theta(L)$. The closed elements of $\Theta(L)$ satisfying $\theta=\theta^{**}$ from a complete Boolean algebra $\Theta_*(L)$, in which join is given by the new operation $\theta\bigvee\phi$ $=(\theta\smile\phi)^{**}$, while the meet operation is the same as in $\Theta(L)$[1].

If $\theta\in\Theta_z(L)$, θ has a complement θ' which is equal to θ^*. Hence for $\theta, \phi\in\Theta_z(L)$,

$$\theta\bigvee\phi=(\theta\smile\phi)^{**}=(\theta\smile\phi)''=\theta\smile\phi.$$

Therefore $\Theta_z(L)$ is a sublattice of $\Theta_*(L)$.

Thus we have the following relations:

$$\Theta(L)\supset\Theta_*(L)\supset\Theta_z(L)\supset\Theta_0(L),$$

where $\Theta(L)$ is an upper continous, distributive lattice with join $\smile$ and meet $\frown$, and $\Theta_*(L)$ is a complete Boolean algebra with join $\bigvee$ and meet $\frown$, and $\Theta_z(L)$ is a Boolean algebra where two joins $\smile$ and $\bigvee$ coincide, and it is a sublattice of both $\Theta(L)$ and $\Theta_*(L)$, and $\Theta_0(L)$ is a Boolean subalgebra of $\Theta_z(L)$.

When a system of lattices $L_{\theta_\alpha}(\alpha\in I)$ is a subdirect factorization of L then $\Lambda(\theta_\alpha;\ \alpha\in I)=0$ in $\Theta(L)$. For fixed α, if we put $\phi_\alpha=\Lambda(\theta_\beta;\ \beta\in I,\ \beta\neq\alpha)$, then $\theta_\alpha\frown\phi_\alpha=0$ and $\theta_\alpha\leq\phi_\alpha^*$. Since $\phi_\alpha^*\frown\phi_\alpha=0$, if we use ϕ_α^* insted of θ_α for this fixed α, we have also a subdirect factorization of L, where $L_{\phi_\alpha^*}$ is smaller than L_{θ_α} in some sense, and ϕ_α^* is the greatest elements which can be used instead of θ_α. In this case $\phi_\alpha^*\in\Theta_*(L)$. Hence we have the following definition:

When $\{\theta_\alpha;\ \alpha\in I\}$ be a subset of $\Theta_*(L)$ such that

$$\theta_\alpha^*=\Lambda(\theta_\beta;\ \beta\in I,\ \beta\neq\alpha)\quad\text{for all}\quad\alpha\in I,$$

the system of lattices $L_{\theta_\alpha}(\alpha\in I)$ is called a *canonical subdirect factorization* of L. And we write $L\simeq\prod^*(L_{\theta_\alpha}:\ \alpha\in I)$.

THEOREM 2·1. *In order that $L\simeq\prod^*(L_{\theta_\alpha}:\ \alpha\in I)$, it is necessary and sufficient that $\theta_\alpha\in\Theta_*(L)$ for all $\alpha\in I$, and*

$$\Lambda(\theta_\alpha;\ \alpha\in I)=0,\qquad \theta_\alpha\bigvee\theta_\beta=1\qquad(\alpha\neq\beta).\qquad(1)$$

PROOF. (i) Necessity. By the definition, $\theta_\alpha\in\Theta_*(L)$ and $\theta_\alpha^*=\Lambda(\theta_\beta;\ \beta\in I,\ \beta\neq\alpha)$. Hence $\Lambda(\theta_\beta;\ \beta\in I)=\theta_\alpha^*\frown\theta_\alpha=0$. And when $\alpha\neq\beta$, since $\theta_\alpha^*\leq\theta_\beta$, we have $\theta_\alpha\bigvee\theta_\beta\geq\theta_\alpha\bigvee\theta_\alpha^*=1$.

(ii) Sufficiency. If we put $\phi_\alpha=\Lambda(\theta_\beta;\ \beta\in I,\ \beta\neq\alpha)$, from (1) we have

$$\theta_\alpha\frown\phi_\alpha=0,\qquad\theta_\alpha\bigvee\phi_\alpha=\Lambda(\theta_\alpha\bigvee\theta_\beta;\ \beta\in I,\ \beta\neq\alpha)=1.$$

1) Cf. Birkhoff [1] 148.

Hence ϕ_α is a complement of θ_α in $\Theta_*(L)$. That is $\phi_\alpha=\theta_\alpha^*$.

THEOREM 2·2. *Associated with any two canonical subdirect factorizations of a lattice L:*

$$L \cong \Pi^*(L_{\theta_\alpha};\ \alpha \in A), \qquad L \cong \Pi^*(L_{\phi_\beta};\ \beta \in B),$$

there exists a canonical subdirect factorization of L:

$$L \cong \Pi^*(L_{\psi_{\alpha\beta}};\ \alpha \in A,\ \beta \in B),$$

such that $\quad L_{\theta_\alpha} \cong \Pi^*(L_{\psi_{\alpha\beta}};\ \beta \in B), \qquad L_{\phi_\beta} \cong \Pi^*(L_{\psi_{\alpha\beta}};\ \alpha \in A).$

PROOF. Since $\Theta_*(L)$ is a complete Boolean algebra, putting

$$\psi_{\alpha\beta} = \theta_\alpha \vee \phi_\beta \qquad (\alpha \in A,\ \beta \in B),$$

we can prove as Theorem 1·3.

THEOREM 2·3. *Let $\theta_1, \dots, \theta_n$ be elements of $\Theta(L)$, such that*

$$\theta_1 \frown \dots \frown \theta_n = 0, \qquad \theta_i \smile \theta_j = 1 \qquad (i \neq j), \tag{1}$$

then the system $\{L_{\theta_1}, \dots, L_{\theta_n}\}$ is a canonical subdirect factorization of the lattice L.

PROOF. Put $\phi_1=\theta_2 \frown \dots \frown \theta_n$, then

$$\theta_1 \frown \phi_1 = 0, \qquad \theta_1 \smile \phi_1 = (\theta_1 \smile \theta_2) \frown \dots \frown (\theta_1 \smile \theta_n) = 1.$$

Hence θ_1 has a complment ϕ_1, and $\theta_1 \in \Theta_z(L)$. Similarly for $\theta_i (i=2, \dots, n)$. Therefore (1) are the relations on elements of $\Theta_z(L)$, which is a Boolean subalgebra of $\Theta_*(L)$. Consequently by Theorem 2·1, $\{L_{\theta_1}, \dots, L_{\theta_n}\}$ is a canonical subdirect factorization of L.

COROLLARY. *The direct factorization of a lattice L is a canonical subdirect factorization of L.*

PROOF. If $\{L_{\theta_1}, \dots, L_{\theta_n}\}$ is a direct factorization of L, then by Theorem 1·2, $\theta_i \in \Theta_0(L)$ and

$$\theta_1 \frown \dots \frown \theta_n = 0, \qquad \theta_i \smile \theta_j = 1 \qquad (i \neq j).$$

Hence by Theorem 2·3 $\{L_{\theta_1}, \dots, L_{\theta_n}\}$ is a canonical subdirect factorization of L.

THEOREM 2·4. *If a lattice L has a canonical subdirect factorization with subdirectly irreducible factors, then $\Theta_*(L)$ is an atomic complete Boolean algebra. And any canonical subdirect factorization of L is obtainable by grouping these subdirectly irreducible factors into subfamilies.*

PROOF. Let $L \cong \Pi^*(L_{\psi_\alpha};\ \alpha \in I)$ be the canonical subdirect factorization where L_{ψ_α} are subdirectly irreducible. When $\psi_\alpha=1$, since L_{ψ_α} is an one element lattice, we may omit ψ_α. Hence $\psi_\alpha < 1$ for all $\alpha \in I$. If there exist $\phi \in \Theta_*(L)$ such that $\psi_\alpha < \phi < 1$, then there exists ϕ' such that

$$\phi \frown \phi' = \psi_\alpha, \qquad \phi \vee \phi' = 1,$$

and L_{ψ_α} is a subdirect union of L_ϕ and $L_{\phi'}$, which contradicts to the fact that L_{ψ_α} is subdirectly irreducible. Hence ψ_α are maximal elements of $\Theta_*(L)$ for all $\alpha \in I$, and since $\wedge(\psi_\alpha;\ \alpha \in I)=0$, $\Theta_*(L)$ is an atomic complete Boolean algebra. Then the last part of the theorem is evident from Theorem 2·2.

REMARK. Birkhoff[1] has proved that every lattice L can be represented as a subdirect union of subdirectly irreducible lattices. From this can we deduce the canonical subdirect factorization with subdirectly irreducible factors?

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1) Bf Birkhoff [1] 92.