LATTICE THEORETIC CHARACTERIZATION OF ABSTRACT GEOMETRIES

By

Fumitomo MAEDA

(Received Oct. 20, 1950)

Frink [1][1] and Prenowitz [1] have characterized the lattice of linear subspaces of a projective geometry, and the lattice of convex sets of a descriptive geometry, all these geometries being of arbitray dimensions, finite or infinite.

The purpous of this paper is to find the fundamental theorems which enable us to characterize all types of these geometries simultaneously. These theorems are as follows:

THEOREM I. *An abstract lattice L is isomorphic with the lattice of all subgeometries of a suitable abstract geometry G with finitary operations, if and only if it is a relatively atomic, upper continuous lattice.*

THEOREM II. *A relatively atomic, upper continuous lattice L is a direct sum of sublattices S_α $(\alpha \in I)$ of L. And any two points in the same S_α are connected, and two points which are contained in different S_α and S_β are not connected.*

Thus there exists a one-one correspondence between a relatively atomic, upper continuous lattice L and an abstract geometry G with finitary operations. For example, when L is a relatively atomic, upper continuous *distributive* lattice, that is, L is an atomic, complete Boolean algebra, then the associated geometry G is a point set and subgeometries mean the subsets of G. When L is a relatively atomic, upper continuous, *modular* lattice, that is, L is an atomic, upper continuous, complemented modular lattice, then the associated geometry G is a projective geometry. Similarly Prenowitz's [1] investigations are special cases of Theorem I[2]. In the last part of this paper, I generalize the Prenowitz's [1] results, introducing the concept "linear lattice".

Since a lattice L with 0 is relatively atomic, if and only if each element of L is the join of its contained points, Theorem I may be stated as follows:

1) The numbers in square brackets refer to the list of references at the end of the paper.
2) Prenowitz [1] did not considered about Theorem II.

THEOREM I'. *An abstract lattice L is isomorphic with the lattice of all subgeometries of a suitable abstract geometry G with finitary operations, if and only if*

(1') *L is upper continuous,*

(2') *every element of L is a join of points.*

Theorem I' is similar to the following theorem obtained by Birkhoff and Frink [1, 302 Theorem 2]:

THEOREM I''. *An abstract lattice L is isomorphic with the lattice of all subalgebras of a suitable abstract algebra A with finitary operations, if and only if*

(1'') *L is upper continuous,*

(2'') *every element of L is a join of ↑-inaccesible elements.*

The difference between (2') and (2'') arises from the following facts. In the abstract geometry, every subgeometry is a join of points but in the abstract abgebra, every subalgebra is a join of principal subalgebras, which are ↑-inaccessible elements.[1]

§ 1. Relatively Atomic, Upper Continuous Lattices.

DEFINITION 1·1. For any element a ($\neq o$) of a lattice L with 0, if there exists a point p such that $p \leqq a$, then L is called *atomic*. If $a < b$ implies $a < a \smile p \leqq b$ for some point p, then L is called *relatively atomic.*

LEMMA 1·1. *A lattice L with 0 is relatively atomic, if and only if each element of L is the join of its contained points.*

PROOF. Necessity.[2] Let $S = \{p\,;\, p \leqq a\}$.[3] Suppose $a \neq \bigvee(p\,;\, p \in S)$. Since a is an upper bound of S, there exists b an upper bound of S such that $a \quad b$. Thus $a \frown b < a$. Hence there exists a point p such that $a \frown b < (a \frown b) \smile p \leqq a$. Then p is an element of S and $p \leqq b$ since b is an upper bound of S. Thus $p \leqq a \frown b$ and this contradicts to $a \frown b < (a \frown b) \smile p$.

Sufficiency. If $a < b$ and $b = \bigvee(p\,;\, p \in T)$ where $T = \{p\,;\, p \leqq b\}$, then there exists a point $p \in T$ such that $p \nleqq a$, and $a < a \smile p \leqq b$.

DEFINITION 1·2. In a lattice L with 0, if p, q are points such that

$$q \leqq p \smile x, \qquad q \frown x = 0 \tag{1}$$

for some element $x \in L$, we say p is *perspective* to q.[4] If p, q are points

1) Cf. Birkhoff and Frink [1] 302.

2) This proof is due to Prenowitz [1] 673, Theorem 17.

3) $\{p\,;\, p \leqq a\}$ means the set of all points p such that $p \leqq a$.

4) Prenowitz [1, p. 661] restricted to the case where x is a point. (Cf. Remark 3.2 below.) Above definition includes the case where q is a point on a line through p and parallel to x.

such that there exists a sequence $p=p_1, \dots , p_n=q$ where p_i is perspective to p_{i+1} $(i=1,2, \dots , n-1)$, we say p is *projective* to q. If p, q are points such that there exists a sequence $p=p_1, \dots , p_n=q$ where p_i is perspective to p_{i+1} or p_{i+1} is prespective to p_i $(i=1,2,\dots,n-1)$, we say p and q are *connected*.

REMARK 1·1. In (1) $p\frown x=0$. For if $p\leqq x$, then $q\leqq x$ which contradicts to $q\frown x=0$. Especially when L is modular, since $q\smile x$ and $p\smile x$ cover x, we have $q\smile x=p\smile x$. Therefore, above defind perspectivity is equivalent to that of the continuous geometry. And when L is distributive, p is prespective to q if and only if $p=q$.

DEFINITION 1·3. By a *complete congruence relation* in a lattice L we mean an equivalence relation $\equiv$ in L which is preserved under arbitrary (finite or infinite) joins and meets, that is, if $a_\alpha\equiv b_\alpha$ where α ranges over an arbitrary set of indices I and $\bigvee(a_\alpha\,;\,\alpha\in I)$, $\bigvee(b_\alpha\,;\,\alpha\in I)$ exist, then $\bigvee(a_\alpha\,;\,\alpha\in I)\equiv\bigvee(b_\alpha\,;\,\alpha\in I)$ and similarly for the operation "meet".

We call a lattice L *quasi simple*, if there exist only trivial complete congruence relations in L.[1]

LEMMA 1·2. *If any two points of a relatively atomic lattice L are projective to each other, then L is quasi simple.*

PROOF.[2] Let $\equiv$ be any complete congruence relation in L. If there exist a, b in L such that $a\equiv b$ and $a\neq b$, then by Lemma 1·1, there exists a point p which is contained in just one of a, b. Let us say $p\leqq a$, $p \nleqq b$. Then $p\equiv o$, for $p=p\frown a\equiv p\frown b=0$.

Next suppose the point p is perspective to a point q. Then there exist $x\in L$ such that

$$q\leqq p\smile x, \qquad q\frown x = 0.$$

Since $p\equiv 0$ we have $p\smile x\equiv x$. Hence

$$q = q\frown(p\smile x) \equiv q\frown x = 0.$$

Therefore by the assumption, $q\equiv 0$ for every point $q\in L$. And by Lemma 1·1, $x\equiv 0$ for every element $x\in L$. That is, $\equiv$ is a trivial complete congruence relation.

DEFINITION 1·4. Let $\{a_\delta\,;\,\delta\in D\}$ be a directed set of a complete lattice L. When

$$a_\delta\uparrow a \qquad \text{implies} \qquad a_\delta\frown b\uparrow a\frown b,$$

we say that L is an *upper continuons lattice.*

1) Prenowitz [1] 660.
2) Cf. Prenowitz [1] 672.

LEMMA 1·3. *In a relatively atomic, complete lattice L, the following two propositions* (α) *and* (β) *are equivalent:*

(α) *L is upper continuous.*

(β) *Let p be a point and S a set of points in L. Then* $p \leqq \bigvee(q\,;q \in S)$ *implies* $p \leqq q_1 \smile \dots \smile q_n$ *where each* q_i *is in S.*

PROOF. (α)→(β). Let ν be a finite subset of S, and put $a_\nu = \bigvee(q\,;q \in \nu)$, $a = \bigvee(q\,;q \in S)$. Then $a_\nu \uparrow a$. Let $p \leqq a$. If $p \frown a_\nu = 0$ for every ν, then by (α) $p \frown a = 0$ which contradicts to $p \leqq a$. Hence there exists a finite subset $\nu = \{q_1, \dots, q_n\}$ such that $p \frown (q_1 \smile \dots \smile q_n) \neq 0$, that is $p \leqq q_1 \smile \dots \smile q_n$.

(β)→(α). For $a_\delta \uparrow a$ and b, put $S_\delta = \{p\,;p \leqq a_\delta\}$, $S = \{p\,;p \leqq a\}$, $T = \{p\,;p \leqq b\}$. Then by Lemma 1·1, $a_\delta = \bigvee(p\,;p \in S_\delta)$, $a = \bigvee(p\,;p \in S)$, $b = \bigvee(p\,;p \in T)$ and $a_\delta \frown b = \bigvee(p\,;p \in S_\delta \cdot T)$, $a \frown b = \bigvee(p\,;p \in S \cdot T)$. When $p \in S$, since

$$p \leqq a = \bigvee(a_\delta\,;\delta \in D) = \bigvee(\bigvee(q\,;q \in S_\delta)\,;\delta \in D) = \bigvee(q\,;q \in \textstyle\sum_{\delta \in D} S_\delta),$$

we have, by (β),

$$p \leqq q_1 \smile \dots \smile q_n, \qquad q_i \in \textstyle\sum_{\delta \in D} S_\delta \qquad (i = 1, \dots, n).$$

Since $\delta_1 < \delta_2$ implies $S_{\delta_1} \leqq S_{\delta_2}$, there exists δ_0 such that all $q_i \in S_{\delta_0}$. Hence $p \in S_{\delta_0}$ and $S = \sum_{\delta \in D} S_\delta$. Now

$$\begin{aligned}\bigvee(a_\delta \frown b\,;\delta \in D) &= \bigvee(\bigvee(p\,;p \in S_\delta \cdot T)\,;\delta \in D) = \bigvee(p\,;p \in \textstyle\sum_{\delta \in D} S_\delta \cdot T) \\ &= \bigvee(p\,;p \in S \cdot T) = a \frown b.\end{aligned}$$

Consequently $a_\delta \frown b \uparrow a \frown b$.

DEFINTION 1·5. In a lattice L with 0, by $a \nabla b$, it is meant that $a \frown b = 0$ and $(a \smile x) \frown b = x \frown b$ for every $x \in L$. If S is any subset of L, denote by S^∇ the set of a such that $a \nabla b$ for all $b \in S$.

LEMMA 1·4. *In a lattice with* 0, $a \nabla b$, $a_1 \leqq a$, $b_1 \leqq b$ *imply* $a_1 \nabla b_1$.

PROOF. $a_1 \frown b_1 \leqq a \frown b = 0$ and for all $x \in L$,

$$(a_1 \smile x) \frown b_1 = (a_1 \smile x) \frown (a \smile x) \frown b \frown b_1 = (a_1 \smile x) \frown x \frown b \frown b_1 = x \frown b_1.$$

DEFINTION 1·6. Let $\{S_\alpha\,;\alpha \in I\}$ be a family of subsets with 0 of an upper continuous lattice L. If

(1°) every $a \in L$ is expressible in the form

$$a = \bigvee(a_\alpha\,;\alpha \in I) \qquad \text{where } a_\alpha \in S_\alpha \ (\alpha \in I),$$

(2°) $$\alpha \neq \beta \ \text{ implies } \ S_\beta \leqq S_\alpha^\nabla$$

then we say that L is a *direct sum* of $S_\alpha (\alpha \in I)$, and we write $L = \sum(\oplus S_\alpha\,;\alpha \in I)$.[1] S_α is called the *component* of the direct sum.

1) In this case S_α ($\alpha \in$ I) are complete sublattices of L, and the expression in (1°) is unique. Cf. Maeda [1] 88, Lemma 2·1 (iii) and Lemma 2·2.

LEMMA 1·5. *If an upper continuous lattice L is a direct sum of more than one component, then L is not quasi simple.*

PROOF. Let $L=\sum(\oplus S_\alpha; \alpha \in I)$, then any elements a, b in L are expressed as

$$a = \bigvee(a_\alpha; \alpha \in I), \quad b = \bigvee(b_\alpha; \alpha \in I), \quad \text{where } a_\alpha,\ b_\alpha \in S\ (\alpha \in I).$$

Let β be a fixed element of I. And define $a \equiv b$ by the relation $a_\alpha = b_\alpha$ for all $\alpha \in I$ not β. Then $\equiv$ is a non-trivial complete congruence relation[1].

LEMMA 1·6. *In a relatively atomic, upper continuous lattice L, the following two propositions* (α) *and* (β) *are equivalent:*

(α) $a \nabla b$.

(β) $a \frown b = 0$, *and there exist no points* p, q *such that* $p \leqq a$, $q \leqq b$, p *is perspective to* q.

PROOF. (α)→(β). $a \frown b = 0$ is evident. Next assume that there exists points p, q such that $p \leqq a$, $q \leqq b$, p is perspective to q. Then there exists $x \in L$, such that $q \leqq p \smile x$, $q \frown x = 0$. Hence

$$(p \smile x) \frown q = q > 0 = x \frown q,$$

and $p \nabla q$ is false, which contradicts to $a \nabla b$ by Lemma 1·4.

(β)→(α). It is evident when $a=0$ or $b=0$. Hence assume that a, $b \neq 0$. If (α) is false, there exists $x \neq 0$ such that

$$(a \smile x) \frown b > x \frown b.$$

Since L is relatively atomic, there exists a point q such that

$$(a \smile x) \frown b \geqq (x \frown b) \smile q > x \frown b.$$

Then $(x \frown b) \frown q = 0$, $q \leqq b$ and we have $q \frown x = 0$. Since $q \leqq a \smile x$, by Lemma 1·1 $q \leqq \bigvee(p; p \leqq a) \smile \bigvee(q; q \leqq x)$. Hence by Lemma 1·3, there exist points $p_1, \dots, p_n$ contained in a such that

$$q \leqq p_1 \smile \dots \smile p_n \smile x. \tag{1}$$

Delete superfluous elements from $p_1, \dots p_n$. Then $q \nleqq p_2 \smile \dots \smile p_n \smile x$, that is $q \frown (p_2 \smile \dots \smile p_n \smile x) = 0$. Therefore (1) means that p_1 is perspective to q. This contradicts to (β).

THEOREM 1·1. *A relatively atomic, upper continuous lattice L is a direct sum of sublattices* S_α *of L, that is,* $L=\sum(\oplus S_\alpha; \alpha \in I)$. *And any two points in the same* S_α *are connected, and two points which are contained in diff-*

---

1) Cf. Maeda [1] 88, Theorem 2·1 (ii).

erent S_α *and* S_β *are not connected.*[1]

PROOF. Since the connectedness of two points in L is an equivalence relation, we can divide the set of all points into classes $P_\alpha(\alpha\in I)$, such that any two points in the same P_α are connected, and two points which are contained in different P_α and P_β are not connected. Denote by S_α the set of all elements of L which are expressed as the join of points in P_α and 0. For $a_\alpha\in S_\alpha$, $b_\beta\in S_\beta$ $(\alpha\neq\beta)$, let $p\leqq a_\alpha$, $q\leqq b_\beta$, then by Lemma 1·3,

$$p \leqq p_1\cup\ldots\cup p_n, \quad \text{where} \quad p_i\in P_\alpha \quad (i=1,\ldots,n),$$
$$q \leqq q_1\cup\ldots\cup q_m, \quad \text{where} \quad q_j\in P_\beta \quad (j=1,\ldots,m).$$

Delete superfluous elements from $p_1,\ldots,p_n$, $q_1,\ldots,q_m$. Then $p \not\leqq p_2\cup\ldots\cup p_n$, that is $p\cap(p_2\cup\ldots\cup p_n)=0$. Therefore p_1 is perspective to p, hence $p\in P_\alpha$. Similarly $q\in P_\beta$. Therefore p is not perspective to q, and by Lemma 1·6, $a_\alpha\nabla b_\beta$, that is $S_\alpha\leqq S_\beta^\nabla$.

For any element $a\in L$, denote by a_α the join of all points p, such that $p\leqq a$ and $p\in P_\alpha$, then by Lemma 1·1,

$$a = \bigvee(a_\alpha;\alpha\in I) \qquad \text{where} \qquad a_\alpha\in S_\alpha \quad (\alpha\in I).$$

Consequently $L=\sum(\oplus S_\alpha;\alpha\in I)$.

LEMMA 1·7. *If a relatively atomic, upper continuous lattice L is quasi simple, then any two points of L are connected.*

PROOF. By Lemma 1·5 and Theorem 1·1.

§ 2. Lattices of All Subgeometries of Abstract Geometries.

DEFINITION 2·1. Let G be a set of points. If for any finite points $p_1,\ldots,p_n$ of G, there exists a subset $p_1+\ldots+p_n$ of G containing p_i, which satisfies

(1°) $p_1=p_2$ implies $p_1+p_2+\ldots+p_n=p_2+\ldots+p_n$,

(2°) for any permutation $p_{i1},\ldots,p_{in}$, of $p_1,\ldots,p_n$,
$p_1+\ldots+p_n=p_{i1}+\ldots+p_{in}$,

(3°) $q_i\in p_1^{(i)}+\ldots+p_{n_i}^{(i)}$ $(i=1,\ldots,m)$ imply
$q_1+\ldots+q_m\leqq p_1^{(1)}+\ldots+p_{n_1}^{(1)}+\ldots+p_1^{(m)}+\ldots+p_{n_m}^{(m)}$,

then G is called an *abstract geometry with finitary operations.* Let A be a subset of G such that $p_1,\ldots,p_n\in A$ implies $p_1+\ldots+p_n\leqq A$. Then we say A is a *subgeometry* of G.

LEMARK 2·1. $p_1+\ldots+p_n$ is a subgeometry of G. For, if $q_1,\ldots,q_m\in p_1$

1) This theorem holds also in a relatively atomic, *conditionally* upper continuous lattice, where "subdirect sum" must be used instead of "direct sum". Cf. Maeda [1] 87-88.

$+\dots+p_n$, then

$$q_1+\dots+q_m \leqq p_1+\dots+p_n+\dots+p_1+\dots+p_n = p_1+\dots+p_n .$$

THEOREM 2·1. *The set of all subgeometries of an abstract geometry G with finitary operations, is a relatively atomic, upper continuous lattice.*

PROOF. The set L_G of all subgeometries of G satisfies the following conditions:

(1°) $G\in L_G$,

(2°) The intersection of any family of subgeometries belongs to L_G,

(3°) The set-union of any direct family of L_G belongs to L_G.

Hence L_G is an upper continuous lattice, partially ordered by set-inclusion.[1]

If $A<B$ in L_G, there exists a point p such that $p\notin A$, $p\in B$. Then in the lattice L_G

$$A<A\smile p \leqq B.$$

Hence L_G is relatively atomic.

REMARK 2·1 By Definition 1·2, if in L_G,

$$q\leqq p\smile A, \qquad q\frown A = 0,$$

then p is perspective to q. In this case we say p is perpective to q in G. Similarly for projectivity and connectedness.

THEOREM 2·2. *Let L be a relatively atomic, upper continuous lattice, and denote by G_L the set of all points of L. In G_L, if we define $p_1+\dots+p_n$ by the set of points contained in $p_1\smile\dots\smile p_n$, then G_L is an abstract geometry of finitary operations, and L is isomorphic to the lattice of all subgeometries of G_L.*

PROOF. (i) If we define $p_1+\dots+p_n$ as above, then G_L satisfies the conditions of Definition 2·1, and G_L is an abstract geometry with finitary operations.

(ii) Put $S(a)=\{p\,;p\leqq a\}$ for $a\in L$, and $a(X)=\bigvee(p\,;\,p\in X)$ for $X\leqq G_L$. Then $X=S(a(X))$ if and only if X is a set of all p such that $p\leqq \bigvee(q;q\in X)$. Hence by Lemma 1·3, X is a subgeometry of G_L if and only if $X=S(a(X))$.[2]

1) Cf. Birkhoff and Frink [1] 301, and Birkhoff [1] 64, Ex. 3 (*b*).

2) $X\to\overline{X}=S(a(X))$ is a closure operation in G_L. For it is evident that $\overline{X}\geqq X$, and $X\geqq Y$ implies $\overline{X}\geqq\overline{Y}$. And $\overline{\overline{X}}=S(a(\overline{X}))=\{p;\,p\leqq a(\overline{X})\}=\{p;\,p\leqq\bigvee(p\,;\,p\in\overline{X})\}=\{p;\,p\leqq\bigvee(p;\,p\leqq a(X))\}$. Since L is relatively atomic, by Lemma 1·1, we have $\overline{\overline{X}}=\{p;\,p\leqq a(X)\}=S\,(a(X))=\overline{X}$. Therefore the above means that with respect to this closure operation, X is closed if and only if X is a subgeometry of G_L.

Next, for any $a \in L$, by lemma 1·1,

$$a = \bigvee(p;p \leqq a) = \bigvee(p;p \in S(a)) = a(S(a)).$$

Therefore by $a \rightarrow S(a)$, $X \rightarrow a(X)$, there exists a one-one correspondence between L and the lattice of all subgemetries of G_L, which preserves the inclusion relation. Hence L is isomorphic to the lattice of all subgemetries of G_L.

§ 3. Lattices of All Additively Closed Sets of Generalized Linear Geometries.

In this section I generalize the linear geometries difined by Prenowitz [1, 662] as follows:

DEFINITION 3·1. Let G be a set of points, where 3-term relation *order*, indicated (pqr) is defined.[1]

01. If (pqr) then p, q, r are distinct.

02. If (pqr) then (rqp).

For any p, q the set consisting of p, q and all x for which (pxq) is called the *sum* of p and q and is denoted $p+q$. (When $p=q$, let $p+p=p$.)

06′. (Transversal property) If x is $y+r$ and y is in $p+q$, then there exists z such that x is in $p+z$ and z is in $q+r$.

Then G is called a *generalized linear geometry*.[2] A subset A of G such that $p, q \in A$ implies $p+q \leqq A$ is called an *additively closed set* of G.

LEMMA 3·1.[3] *In a generalized linear geometry G, let X, Y be two subsets of G. If* (1) *X, $Y \neq 0$, we define $X+Y$ to be the set-union* $\sum_{p \in X, q \in Y}(p+q)$. *If* (2) *$Y=0$ we define $X+Y=Y+X=X$. Then*

(1°) $X+Y \geqq X$,

(2°) $X+Y = Y+X$,

(3°) $(X+Y)+Z = X+(Y+X)$,

---

1) Relation $(p\,q\,r)$ is to be interpreted concretely to mean: (1) in a descriptive geometry that q is an interior point of the line interval which joins p and r; (2) in a projective geometry that p, q, r are distinct and belong to a line. Cf. Prenowitz [1] 661.

2) Prenowitz [1, p. 662] defined the *linear geometry* as a set G which satisfies 01, 02, 03, 04′, 06′, where

03. If $p \neq q$ there is a unique line pq contaning p and q, where *line* pq is the set consisting of p, q and all x for which $(x\,p\,q)$ or $(p\,x\,q)$ or $(p\,q\,x)$.

04′. If p, q are points, then p is projective to q.

A linear geometry which satisfies

07′₁. There exist p, q, r such that $(p\,q\,r)$ is true and $(q\,r\,p)$ is false,

is called a *descriptive geometry*, and a linear geometry which satisfies

07_2. If $(p\,q\,r)$ then $(q\,r\,p)$,

is a *projective geometry*.

3) Cf. Prenowitz [1] 665.

(4°) *X is an additively closed set if and only if* $X+X=X$,

(5°) *if* X, Y *are additively closed sets, then* $X+Y$ *is an additively closed set.*

RROOF. (i) (1°) and (2°) are evident.

(ii) From 06′, we have $(p+q)+r \leqq p+(q+r)$. By interchanging p and r we have $(r+q)+p \leqq r+(q+p)$. Hence we have

$$(p+q)+r = p+(q+r).$$

Let $p \in (X+Y)+Z$, then there exist $x \in X$, $y \in Y$, $z \in Z$ such that

$$p \in (x+y)+z = x+(y+z) \leq X+(Y+Z).$$

Hence $(X+Y)+Z \leq X+(Y+Z)$. As above, interchanging X and Z, we have (3°).

(iii) X is an additively closed set if and only if $p, q \in X$ impiles $p+q \leqq X$. This is equivalent to $X+X \leqq X$ and so by (1°) to $X+X=X$.

(iv) When X and Y are additively closed sets, by (2°), (3°) and (4°) we have $(X+Y)+(X+Y)=X+Y$ and $X+Y$ is an additively closed set.

REMARK 3·1 Let $p_1, \dots, p_n$ be points of a generalized linear geometry G. By induction, if we define $p_1+\dots+p_n$ as

$$p_1+\dots+p_n = (p_1+\dots+p_{n-1})+p_n,$$

then by Lemma 3·1, it satisfies (1°), (2°) and (3°) of Definition 1·2 and G is an abstract geometry with finitary operations, and a set A of G is a subgemetry if and only if A is an additively closed set.

DEFINITION 3·2 A lattice L with o is called a *linear lattice* when for a point p such that $p \leqq a \smile b$ $(a,b \neq 0)$, there exist points q, r such that $p \leqq q \smile r$ where $q \leqq a$, $r \leqq b$.[1)]

REMARK 3·2. In a linear lattice L, p is perspective to q if and only if there exists a point r such that

$$q \leqq p \smile r, \quad q \frown r = 0.$$

LEMMA 3·2 *If a relatively atomic, upper continuous, linear lattice L is quasi simple, then any two points of L are projective to each other.*

PROOF. We can prove as Prenowitz [1] 676 Theorem 22.

THEOREM 3·1. *The lattice* L_G *of all additively closed sets of a genera-*

1) As Prenowitz [1] 671, Theorem 13, we can prove that a relatively atomic, linear lattice is quasi modular. As Prenowitz [1] 674 Theorem 20, we can also prove that a relatively atomic, upper continuous lattice is a linear lattice if and only if for points p, q such that $p \leqq q \smile a$ $(a \neq 0)$ there exists a point r such that $p \leqq q \smile r$, $r \leqq a$.

lized linear geometry G is a relatively atomic, upper coutinuous, linear lattice.

PROOF. By Theorem 2·1 and Remark 3·1 L_G is a relatively atomic, upper continuous lattice,[1] and by Lemme 3·1 (5°) L_G is a linear lattice.

THEOREM 3·2. *Let L be a relatively atomic, upper continuous, linear lattice, and denote by G_L the set of all points of L. In G_L we define the relation order as follows: (pqr) means that $q \leqq p \smile r$ and p,q,r are distinct. Then G_L is a generalized linear geometry, and L is isomorphic to the lattice of all additively closed sets of G_L. And L is quasi simple if and only if any two points of G_L are projective to each other.*

PROOF. 01, 02 follow directly from the definition of the order (pqr), $p+q$ in G_L is a set of points x such that $x \leqq p \smile q$. Suppose that x, y are points in L such that $x \leqq y \smile r$ and $y \leqq p \smile q$. Then $x \leqq p \smile q \smile r$ and since L is a linear lattice, there exists a point z such that $z \leqq q \smile r$ and $x \leqq p \smile z$. Hence 06′ is valid in G_L. Therefore G_L is a generalized linear geometry, and by Theorem 2·2 and Remark 3·1, L is isomorphic to the lattice of all additively closed sets of L_L. The last part of the theorem follows from Lemma 1·2 and Lemma 3·2.

References.

G. Birkhoff [1], *Lattice theory*, Am. Math. Soc. Colloquium pub., second ed., 1948.

G. Birkhoff and O. Frink [1], *Representations of lattices by sets*, Trans. Am. Math. Soc. **64** (1948), 299–316.

O. Frink [1], *Complemented modular lattices and projective spaces of infinite dimension*, Trans. Am. Math. Soc. **60** (1946), 452–467.

F. Maeda [1], *Direct sums and normal ideals of lattices*, this journal **14** (1950) 85–92.

W. Prenowitz [1], *Total lattices of convex sets and of linear spaces*, Annals of Math. **49** (1948), 659–688.

FACULTY OF SCIENCE,
HIROSHIMA UNIVERSITY.

---

1) Of course we can prove directly by the method used in the proof of Theorem 2·1.

Added in proof. MacLane defined *exchange lattices* as relatively atomic, upper continuons lattices wlich satisfy the following exchange axiom (η'):

(η') If p,q are points, and $a < a \smile q \geqq a \smile p$, then $a \smile p = a \smile q$.

If we define *exchange geometries* as abstract geometries with finitary operations which satisfy the following condition (4°):

(4°) If $q \in p_1 + \cdots + p_n$, then $p_n \in p_1 + \cdots + p_{n-1} + q$,

Then between exchange lattices and exchange geometries, there are close connections as Theorems 2.1 and 2.2. Cf. S. MacLane, *A lattice formulation for transcencence degrees and p-bases*, Duke math. Jonr. 4 (1938), 455-468.