On the Logarithmic Functions of Matrices. II.
(On Some Properties of Local Lie Groups)

By

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§ 1. Logarithmic functions of real matrices.

In the preceding paper[1] we have obtained the following results about the logarithmic functions of complex matrices. We consider the complex matrices of order n. Let $\mathfrak{M}$ be the totality of regular matrices, $\tilde{\mathfrak{M}}$ the totality of regular matrices whose all characteristic values are not negative, $\mathfrak{A}_{(a)}$ the totality of matrices whose different characteristic values μ_i have the imaginary part $I(\mu_i)$ such that $a-\pi \leqq I(\mu_i) < a+\pi$ (a is any real number), $\tilde{\mathfrak{A}}_{(a)}$ the totality of matrices such that $a-\pi < I(\mu_i) < a+\pi$ (a is any real number), and $\mathfrak{A}^*$ the totality of matrices such that $\mu_i \not\equiv \mu_j$ ($mod.\ 2\pi\sqrt{-1}$) for all different characteristic values μ_i.[2] Then we have the following properties:

(1) There exisits in $\mathfrak{A}_{(a)}$ one and only one matrix such that $exp\,A = M$ for a given matrix $M \in \mathfrak{M}$.

(2) The exponential mapping $A \rightarrow exp\,A = M$ is a topological mapping from $\tilde{\mathfrak{A}}_{(a)}$ onto $\tilde{\mathfrak{M}}$

(3) Let $A, B \in \mathfrak{A}^*$. $AB = BA$ if and only if $exp\,A\,exp\,B = exp\,B\,exp\,A$.

(4) Let $A \in \mathfrak{A}^*$. $A = \begin{pmatrix} U & W \\ O & V \end{pmatrix}$ if and only if $exp\,A = \begin{pmatrix} H & L \\ O & K \end{pmatrix}$, where U and V are the matrices of the same order as H and K respectively.

In this section we shal consider the logarithmic functions of real matrices. We denote by $\mathfrak{M}_{real}$, $\tilde{\mathfrak{M}}_{real}$, $\mathfrak{A}_{(0)real}$, $\tilde{\mathfrak{A}}_{(0)real}$ and $\mathfrak{A}^*_{real}$ the totality of the real matrices belonging to $\mathfrak{M}$, $\tilde{\mathfrak{M}}$, $\mathfrak{A}_{(0)}$, $\tilde{\mathfrak{A}}_{(0)}$ and $\mathfrak{A}^*_{real}$ respectively. Then it is obvious that the above properties (3) and (4) hold for $\mathfrak{A}^*_{real}$.

Next we shall investigate the properties (1) and (2) in the case of real matrices.

1) K. Morinaga and T. Nôno : On the Lograrithmic Functions of Matrices I, Journal of Science of the Hiroshima University, Ser, A, Vol. 14, No. 2, 1950.

2) $\mathfrak{A}^* = Log(\mathfrak{M})$ and $\mathfrak{A}^* \subset \mathfrak{A}_{(a)}$.

Let M be a matrix belonging to $\mathfrak{M}_{real}$, and we shall transform M into the Jordan's canonical form M_P by a matrix P. Since $\overline{M}_P$[1] is a canonical form of M, the whole of blocks of M_P must coincide with that of $\overline{M}_P$. Therefore we have

$$M=PM_PP^{-1} \tag{1,1}$$

and

$$M_P=M_1^{(+)}\dot{+}\dots\dot{+}M_u^{(+)}\dot{+}M_1^{(-)}\dot{+}\dots\dot{+}M_v^{(-)}\dot{+}(M_1^{(c)}\dot{+}\overline{M_1^{(c)}})\dot{+}\dots\dot{+}(M_w^{(c)}\dot{+}\overline{M_w^{(c)}}),^{2)} \tag{1,2}$$

where the upper indices $(+)$, $(-)$ and (c) denote that their characteistic values are positive, negative and complex numbers respectively, and

$$M_i^{(+)}=M_{i1}^{(+)}\dot{+}\dots\dot{+}M_{ip_i}^{(+)},\quad M_{i\alpha}^{(+)}=\begin{pmatrix}\sigma_i & 1 & & 0\\ & \ddots & \ddots & \\ 0 & & \ddots & 1\\ & & & \sigma_i\end{pmatrix},\quad \begin{pmatrix}\sigma_i>0 \\ i=1,\dots,u;\ \alpha=1,\dots,p_i\end{pmatrix}, \tag{1,3}$$

$$M_j^{(+)}=M_{j1}^{(+)}\dot{+}\dots\dot{+}M_{jq_j}^{(-)},\quad M_{j\beta}^{(-)}=\begin{pmatrix}\tau_j & 1 & & 0\\ & \ddots & \ddots & \\ 0 & & \ddots & 1\\ & & & \tau_j\end{pmatrix},\quad \begin{pmatrix}\tau_j<0 \\ j=1,\dots,v;\ \beta=1,\dots,q_j\end{pmatrix}, \tag{1,4}$$

$$M_k^{(c)}=M_{k1}^{(c)}\dot{+}\dots\dot{+}M_{kr_k}^{(c)},\quad M_{kr}^{(c)}=\begin{pmatrix}\zeta_k & 1 & & 0\\ & \ddots & \ddots & \\ 0 & & \ddots & 1\\ & & & \zeta_k\end{pmatrix},\quad \begin{pmatrix}\zeta_k=\xi_k+\sqrt{-1}\eta_k,\ \eta_k\neq 0 \\ k=1,\dots,w;\ r=1,\dots,r_k\end{pmatrix}. \tag{1,5}$$

Then we see that

$$\overline{M}_P=TM_PT^{-1} \tag{1,6}$$

where

$$T=E_1^{(+)}\dot{+}\dots\dot{+}E_u^{(+)}\dot{+}E_1^{(-)}\dot{+}\dots\dot{+}E_v^{(-)}\dot{+}\begin{pmatrix}0 & E_1^{(c)}\\ E_1^{(c)} & 0\end{pmatrix}\dot{+}\dots\dot{+}\begin{pmatrix}0 & E_w^{(c)}\\ E_w^{(c)} & 0\end{pmatrix}; \tag{1,7}$$

$E_i^{(+)}$, $E_j^{(-)}$ and $E_k^{(c)}$ denote the unit matrices for $M_i^{(+)}$, $M_j^{(-)}$ and $M_k^{(c)}$ respectively[3]. Moreover we have

$$T^2 = E \tag{1,8}$$

Since M is a real matrix i.e., $M=\overline{M}$, by means of (1,1) and (1,6), we

1) $\overline{C}$ denotes the complex conjugate matrix of C.

2) $C\dot{+}D$ denotes $\begin{pmatrix}C & O\\ O & D\end{pmatrix}$.

3) $E_i^{(+)}$, $E_j^{(-)}$ and $E_k^{(c)}$ do not obey to the above notation.

have

$$PM_PP^{-1}=\overline{P}\overline{M}_P\overline{P}^{-1}=\overline{P}TM_PT^{-1}\overline{P}^{-1}=(\overline{P}T)M_P(\overline{P}T)^{-1},$$

if we put $P^{-1}\overline{P}T=R$, it follows

$$RM_P=M_PR, \tag{1,9$_1$}$$

and by (1,8) we obtain

$$\overline{P} = PRT. \tag{1,9$_2$}$$

Now let A be a real matrix belonging to $\mathfrak{A}_{(a)}$ such that $exp\, A=M$ for $M\in\mathfrak{M}_{real}$, then similarly as in the case for M, the set of cheracteristic values of A consist of real numbers and the pairs of complex conjugate numbers, accordingly a must be zero. Furthermore, the imaginary part $I(\mu_i)$ of all characteristic values of A must satisfy the inequality-$\pi<I(\mu_i)<\pi$, therefore $A\in\widetilde{\mathfrak{A}}_{(0)real}$, consequently $\mathfrak{A}_{(0)real}=\widetilde{\mathfrak{A}}_{(0)real}$. Let $M\to L(M)=B$ be the mapping in (1), p. 171,[1] then

$$L(M) = PL(M_P)P^{-1} \tag{1,10}$$

and

$$L(M_P)=L^0(M_1^{(+)})\dotplus\dots\dotplus L^0(M_u^{(+)})\dotplus L^0(M_1^{(-)})\dotplus\dots\dotplus L^0(M_v^{(-)})$$
$$\dotplus L^0(M_1^{(c)})\dotplus L^0(\overline{M_1^{(c)}})\dotplus\dots\dotplus L^0(M_w^{(c)})\dotplus L^0(\overline{M_w^{(c)}}), \tag{1,11}$$

where

$$L^0(M_i^{(+)})=Log\,\sigma_i\cdot E_i^{(+)}+\sum_{s=1}^{s_i^{(+)}-1}(-1)^{s-1}\frac{(N_i^{(+)})^s}{s\,\sigma_i^s}, \quad \begin{pmatrix} N_i^{(+)}=M_i^{(+)}-\sigma_iE_i^{(+)} \\ i=1,\ \dots\dots,\ u; \\ s_i^{(+)}\text{is the order of } M_i^{(+)} \end{pmatrix} \tag{1,12}$$

$$L^0(M_j^{(-)})=Log\,\tau_j\cdot E_j^{(-)}+\sum_{s=1}^{s_j^{(-)}-1}(-1)^{s-1}\frac{(N_j^{(-)})^s}{s\,\tau_j^s}, \quad \begin{pmatrix} N_j^{(-)}=M_j^{(-)}-\tau_jE_j^{(-)} \\ j=1,\ \dots,\ v; \\ s_j^{(-)}\text{ is the order of } M_j^{(-)} \end{pmatrix} \tag{1,13}$$

and

$$L^0(M_k^{(c)})=Log\,\zeta_k\cdot E_k^{(c)}+\sum_{s=1}^{s_k^{(c)}-1}(-1)^{s-1}\frac{(N_k^{(c)})^s}{s\zeta_k^s} \quad \begin{pmatrix} N_j^{(-)}=M_j^{(-)}-\zeta_kE_j^{(-)} \\ k=1,\ \dots,\ w; \\ s_k^{(c)}\text{ is the order of } M_k^{(c)} \end{pmatrix} \tag{1,14}$$

1) K. Morinaga and T. Nôno ibid.

In (1,12), since $\sigma_i > 0$ and $-\pi \quad I(Log\,\sigma_i) < \pi$, it follows that $Log\,\sigma_i$ is real, accordingly $L^0(M_i^{(+)})$ is a real matrix; in (1,13), since $\tau_j < 0$ and $-\pi \leqq I(Log\,\tau_j) < \pi$, it follows that $Log\,\tau_j = Log|\tau_j| - \pi\sqrt{-1}$, accordingly $L^0(M_j^{(-)}) = K^0(M_j^{(-)}) - \pi\sqrt{-1}E_j^{(-)}$, where $K^0(M_j^{(-)})$ is a real matrix; and in (1,14), similarly, since $\zeta_k = \xi_k + \sqrt{-1}\,\eta_k$ $(\eta_k \neq 0)$ and $-\pi \quad I(Log\,\zeta_k) < \pi$, it follows that $Log\,\overline{\zeta_k} = \overline{Log\,\zeta_k}$, accordingly $L^0(\overline{M_k^{(c)}}) = \overline{L^0(M_k^{(c)})}$. Therefore we have

$$L(M_P) = K(M_P) + J(M_P). \tag{1,15}$$

where

$$K(M_P) = L^0(M_1^{(+)}) \dotplus \dots \dotplus L^0(M_u^{(+)}) + K^0(M_1^{(-)}) \dotplus \dots \dotplus K^0(M_v^{(-)})$$
$$\dotplus (L^0(M_1^{(c)}) \dotplus \overline{L^0(M_1^{c})}) \dotplus \dots \dotplus (L^0(M_w^{(c)}) \dotplus \overline{L^0(M_w^{c})}), \tag{1,16}$$

and

$$J(M_P) = 0 \dotplus \dots \dotplus 0 \dotplus (-\pi\sqrt{-1}E_1^{(-)}) \dotplus \dots \dotplus (-\pi\sqrt{-1}E_v^{(-)}) \dotplus 0 \dotplus \dots \dotplus 0. \tag{1,17}$$

From (1,16) we get

$$\overline{K(M_P)} = TK(M_P)T, \tag{1,18}$$

and, since $K(M_P)$ is a polynomial or M_P,[1] denote $PK(M_P)P^{-1}$ by $K(M)$, then we have from (1,9) and (1,18) $\overline{K(M)} = K(M)$, that is, we know that $K(M)$ is a real matrix belonging to $\mathfrak{A}_{(0)}$, i.e., $K(M) \in \mathfrak{A}_{(0)real}$. Similarly from (1,17) we get

$$\overline{J(M_P)} = -J(M_P) \tag{1,19}$$

and, since $J(M_P)$ is a polynomial of M_P[1], denote $PJ(M_P)P^{-1}$ by $J(M)$, then we have from (1,9) and (1,19) $\overline{J(M)} = -J(M)$, that is, we know that $J(M)$ is a pure imaginary matrix belonging to $\mathfrak{A}_{(0)}$. Therefore $L(M)$ is uniquely decomposed into $K(M)$ and $J(M)$. Moreover, since it is obvious from (1,16) and (1,17) that $K(M_P)$ is commutative with $J(M_P)$, we know that $K(M)$ is commutative with $J(M)$.

Thus we have the following theorem:

THEOREM 1. *If B is a matrix belonging to $\mathfrak{A}_{(0)}$ such that $\exp B = M$ for a given matrix $M \in \mathfrak{M}_{real}$, then*

1) Here the coefficient of a polynomial of M may depend on the characteristic values of M. Cf. K. Morinaga and T. Nôno, ibid.

$$B=L(M)=K(M)+J(M)$$

where $K(M)\in \mathfrak{A}_{(0)real}$, $J(M)\in \mathfrak{A}_{(0)}$ *(pure imaginary), and* $K(M)$ *is commutative with* $J(M)$.

Since $J(M)$ is expressed by

$$J(M)=P\{0\dotplus\dots\dotplus 0\dotplus(-\pi\sqrt{-1}E_1^{(-)})\dotplus\dots\dotplus(-\pi\sqrt{-1}E_v^{(-)})\dotplus 0\dotplus\dots\dotplus 0\}P^{-1},$$

B in theorem 1 is real, if and only if all characteristic values of M are not negative. Thus we have the following theorem corresponding to (1) and (2):

THEOREM 2. *A mapping* $A\rightarrow exp\,A=M$ *is the topological mapping from* $\mathfrak{A}_{(0)real}$ *onto* $\tilde{\mathfrak{M}}_{real}$. *And* $\mathfrak{A}_{(0)real}$ *coincides with* $\tilde{\mathfrak{A}}_{(0)real}$.

§ 2. Some properties of local Lie groups.

Let G be an n dimensional local Lie groups,[1] and let $\mathfrak{g}$ be a Lie ring on a real field R, whose basic elements are n linearly independent infinitesimal transformations X_1, ..., X_n in G, in which the multiplication is defined by

$$(X_i,\ X_j)=c_{ij}^{k}X_k,\ (c_{ij}^{k}\in R) \tag{2,1}$$

where c_{ij}^{k} is called the structure tensor of G. We shall introduce a canonical coordinates x^i in G. Let G^* be the domain where the cononical coordinates are defined, then any element of G^* can be uniquely expressed as $exp\,x^iX_i$, and so a mapping $(x^i)\rightarrow exp\,x^iX_i$ is a topological from a neighbourhood U of (0, 0, ..., 0) in n dimensional space R^n onto G^*. Let $exp\,x^iX_i$, $exp\,y^iX_i$ and $exp\,z^iX_i$ be the elements of G such that

$$exp\,z^iX_i=(exp\,x^iX_i)(exp\,y^iX_i)(exp\,x^iX_i)^{-1},$$

then [2]

$$z^i=(exp\,C(x))^i_jy^j, \tag{2,2}$$

where $C(x)$ is a matrix $\|x^ic_{ij}^{k}\|$.

It is well known that the structures of G are completely expressed by the relations between the basic elements is $\mathfrak{g}$; i.e., the relations between the linearly independent infinitesimal transformations in G.

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1) As for the definition of the local Lie groups, see L. Pontrjagin: Topological groups (1939), p. 181.

2) Cf. C. Chevalley: Theory of Lie groups I. 1946, p. 65.

In that case, the relations between the m basic elements mean essentially the relations between the ∞^m elements of G. Contrary to this we shall show that by applying our results in § 1 on the matrix $C(x)$, some properties of G can be represented by the relations between a finite number of elements of G, (i.e., without the conceptions of the infinitesimal transformations.)

REMARKS. (1). Our results hold always for the domain of x such that $C(x)\in\mathfrak{A}^*$ and $x\in G^*$.

(2). We can easily see from (2,2) that for n independent[1] elements $exp\,X_i\,(i=1, 2, \dots, n)$ of G there exists the relation

$$(exp\,X_i)(exp\,X_j)(exp\,X_i)^{-1}=exp\,(\gamma^k_{ij}X_k)$$

where $\gamma^k_{ij}=(exp\,C_i)^k_j$ (C_i is a matrix $\|c^k_{ij}\|$). And we may assume[2] that C_i belong to $\tilde{\mathfrak{A}}_{(0)real}$, then from the results (1) and (2) in § 1 such structure constants of G are uniquely determined by γ^k_{ij}, therefore we may say that the structure of G is represented by γ^k_{ij}.

We shall first prove the theorem:

THEOREM 3. *Denote by $\mathfrak{h}$ an m dimensional linear subspace of Lie ring $\mathfrak{g}$, by H the subset of G corresponding to $\mathfrak{h}$, and by $exp\,x^i_aX_i(a=1, \dots, m)$ any m definite independent elements of H. If $(exp x^iX_i)(exp x^i_aX_i)(exp x^iX_i)^{-1}$ belongs to H for a definite element $exp\,x^iX_i$ of G such that $C(x)\in\mathfrak{A}^*$, then H is invariant by the one parametric local Lie Group: $exp\,tx^iX_i$ where t is a real variable such that $|t|\leqq\alpha$ and $exp\,\alpha x^iX_i\in G$.*[3]

PROOF. We take $x^i_aX_i$ as the parts of a new base of $\mathfrak{g}$ by a linear transformation of basic elements, that is, we may assume $x^i_aX_i$ to be X_a. Then by the assumption in this theorem $(exp\,x^iX_i)\,(exp\,X_a)\,(exp\,x^iX_i)^{-1}$ belongs to H, so it follows that $(exp\,C(x))^i_j\delta^j_a\in\mathfrak{h}$. Hence we have

$$exp\,C(x)=\begin{pmatrix}L_1 & L_3\\ 0 & L_2\end{pmatrix}\quad\text{and}\quad C(x)\in\mathfrak{A}^* \qquad (2,3)$$

where L_1 is a matrix order m. By making use of the result (4) in § 1,

1) The independency of elements in G is defined by the linearly independence of elements in $\mathfrak{g}$.

2) If we take $c'^k_{ij}=\varepsilon c^k_{ij}$ (ε is a sufficiently small real number) for the structure constants c^k_{ij}, then the matrix $\|c'^k_{ij}\|$ belongs to $\tilde{\mathfrak{A}}_{(0)real}$.

3) This condition for the parameter of one parameteric local Lie group is not repeated in the following.

we know that

$$C(x)=\begin{pmatrix} D_1 & D_3 \\ 0 & D_2 \end{pmatrix} \qquad (2,4)$$

where D is a matrix of order m. Therefore we get

$$(x^iX_i,\ X_a)=x^iC_{ia}^kX_k=(C(x))_a^kX_k \quad (a=1,\ \dots,\ m)$$

and by (2,4) the above equation is reduced to

$$(x^iX_i,\ X_a)=(C(x))_a^bX_b \quad (a,\ b=1,\ \dots,\ m),$$

that is, $(x^iX_i,\ X_a)\in\mathfrak{h}$. Thus it is proved that H is invariant by a one parametric local Lie group $exp\ tx^iX_i$.

Moreover by the above result we can easily obtain the following theorem:

THEOREM 4, *Let $\mathfrak{h}$ and $\mathfrak{k}$ be the linear subspaces of dimension l and m in the Lie ring $\mathfrak{g}$ respectively, and let H and K be the subsets of G corresponding to $\mathfrak{h}$ and $\mathfrak{k}$ respectively. If any l definite independent elements $exp\ x_a^iX_i$ $(a=1,\ \dots,\ l)$ of H are transformed into H by any m definite independent elements $exp\ y_b^iX_i$ $(b=1,\ \dots,\ m)$ of K such that $C(y_b)\in\mathfrak{A}^*$, then H is invariant by K; and conversely.*

Furthermore by putting as $H=K$ in the above we have

THEOREM 5. *Let G be a local Lie group, let $\mathfrak{g}$ be the Lie ring corresponding to G, and let H be a subset of G corresponding to an m dimensional linear subspace $\mathfrak{h}$ of $\mathfrak{g}$. In order for H to be a local Lie subgroup of G, it is necessary and sufficient that the following condition is satisfied: $(exp\ x_a^iX_i)\,(exp\ x_b^iX_i)\,(exp\ x_a^iX_i)^{-1}\in H$ for any m definite independent elements $exp\ x_a^iX_i(a=1,\ \dots,\ m)$ of H.*

Next we shall prove the following theorem:

THEOREM 6. *Let G be a local Lie group, and let $\mathfrak{g}$ be the Lie ring corresponding to G. If any definite two elements $exp\ x^iX_i$ and $exp\ y^iX_i$ of G such that $C(x)$ or $C(y)\in\mathfrak{A}^*$ are commutative, then the corresponding elements x^iX_i and y^iX_i of $\mathfrak{g}$ are commutative; consequently, any elements of the one parametric local Lie subgroup $exp\ tx^iX_i$ of G. are commutative with any elements of the one parametric local Lie subgroup $exp\ sy^iX_i$ of G.*

FROOF. Let us assume, first, $C(x)\in\mathfrak{A}^*$. Then

$$exp\ y^iX_i=(exp\ x^iX_i)\,(exp\ y^iX_i)\,(ezp\ x^iX_i)^{-1} \qquad (2,5)$$

so, we have

$$y^i=(exp\ C(x))_j^iy^j \qquad (2,6)$$

Therefore we can take a base of $\mathfrak{g}$ such as $y^i=\delta^i_1$. In this base we have

$$exp\, C(x)=\begin{pmatrix} 1 & \\ 0 & \\ \vdots & ※ \\ 0 & \end{pmatrix}, \quad \begin{matrix}\ulcorner\!\!\longrightarrow j \\ \downarrow \\ i\end{matrix} \quad , \text{ where } C(x)\in\mathfrak{A}^* \tag{2,7}$$

So, by making use of the results (4) in § 1 we get

$$C(x)=\begin{pmatrix} 0 & \\ \vdots & ※ \\ 0 & \end{pmatrix}. \tag{2,8}$$

Therefore it follows that

$$(x^iX_i,\ y^jX_j)=x^iC^k_{ij}y^jX_k=(C(x))^k_1X=0.$$

And for the case $C(y)\in\mathfrak{A}^*$, similarly we have $(x^iX_i,\ y^jX_j)=0$. Thus we have proved this theorem.

Finally we shall prove the following theorem:

THORFM 7. *Let G be a local Lie group, and let $exp\, y^iX_i$ be a element of G such that the matrix $C(y)$ is not nilpotent. If the element $exp\, y^iX_i$ is transformed into $exp\, ry^iX_i$ by a definite element $exp\, x^iX_i$ of G, $C(x)\in\mathfrak{A}^*$, then the one parametric local Lie subgroup $exp\, tx^iX_i$ is commutative with the one parametric local Lie subgroup $exp\, sy^iX_i$.*

PROOF. By the assumption we have

$$(exp\, x^iX_i)\,(exp\, y^iX_i)\,(exp\, x^iX_i)^{-1}=exp\, ry^iX_i \tag{2,9}$$

Making use of an adjoint representation: $x^iX_i\rightarrow C(x)$, (2,9) becomes

$$exp\, C(x)\cdot exp\, C(y)\cdot (exp\, C(x))^{-1}=exp\, rC(y) \tag{2,10}$$

accordingly we have

$$exp\, C(x)\cdot C(y)\cdot exp\, C(x)^{-1}=rC(y) \tag{2,11}$$

Let $\varphi(u)=u^p+a_1u^{p-1}+\cdots+a_{p-1}u+a_p$ be the minimal polynomial of $C(y)$, then we have

$$C(y)^p+a_1C(y)^{p-1}+\cdots+a_{p-1}C(y)+a_pE=0 \tag{2,12}$$

and from (2,11) and (2,12) we get

$$(rC(y))^p+a_1(rC(y))^{p-1}+\cdots+a_{p-1}rC(y)+a_pE=0 \quad (r>0)^{1)}$$

or

1) $r>0$; from theorem 3 and (2,9) we have $(exp\, tx^iX_i)(exp\, y^iX_i)(exp\, tx^iX_i)^{-1}=exy\, r(t)y^iX_i$, where $r(t)$ is a continuous function of t such that $r(0)=1$, $r(1)=r$ and $r(t)\neq 0$.

$$C(y)^p + \frac{a_1}{r} C(y)^{p-1} + \cdots + \frac{a_{p-1}}{r^{p-1}} C(y) + \frac{a_p}{r^p} E = 0. \quad (r > 0) \qquad (2,13)$$

Since $\varphi(u)$ is the minimal polynomial of $C(y)$, from (2,12) and (2,13) we obtain

$$\frac{a_1}{r} = a_1, \quad \frac{a_2}{r^2} = a_2, \ \ldots, \quad \frac{a_p}{r^p} = a_p \qquad (2,14)$$

By the assumption $C(y)$ is not nilpotent, so $a_1, \ldots, a_p$ are not all zero, therefore r, being positive, must be equal to one. Thus from (2,9) we see that $exp\, x^i X_i$ and $exp\, y^i X_i$ are commutative. Consequently, by theorem 6, we can conclude that the one parametric local Lie subgroup $exp\, tx^i X_i$ is commutative with the one parametric local Lie subgroup $exp\, sy^i X_i$.

Moreover, we shall add the following remark:

Let Γ be a set of the element $exp\, x^i X_i$ in G such that x^i satisfies either

$$trace\,(C(x)^2) \equiv C^l_{ik} C^k_{jl} x^i x^j = \gamma_0^2 \qquad (2,15)_1$$

or

$$trace\,(C(x)^2) \equiv C^l_{ik} C^k_{jl} x^i x^j = -\gamma_0^2 \qquad (2,15)_2$$

then Γ is determined independently for the choice of base $X_i (i=1, \ldots, n)$ of $\mathfrak{g}$, since $(2,15)_1$ and $(2,15)_2$ are tensor equations in the Lie ring $\mathfrak{g}$. Furthermore Γ is invariant by any element $exp\, y$ of G. In fact, let $exp\, x$ be any element of Γ, and let us put $exp\, z = exp\, y \cdot exp\, x \cdot (exp\, y)^{-1}$, then, by making use of the adjoint representation it becomes

$$exp\, C(z) = exp\, C(y) \cdot exp\, C(x) \cdot (exp\, C(y))^{-1},$$

accordingly we have

$$C(z) = exp\, C(y) \cdot C(x) \cdot (exp\, C(y))^{-1}$$

therefore we obtain $trace\,(C(z)^2) = trace\,(C(x)^2)$, i. e., $exp\, z$ belongs to Γ.

Specially, in the case where G is a semi-simple and compact Lie group, that is, of which $trace\,(C(x)^2)$ is a negative definite quadratic form[1], Γ represents a non-degenerate quadrics and intersects a straight line through the origin representing a one parametric local Lie subgroup in two points which are symmetric with respect to the origin each other. Therefore the structure of a semi-simple Lie group are represented by the relations between a finite number of points on Γ.[2]

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1) E. Cartan; Selecta, p. 240.

2) We can take a sufficient smal number γ_0 such that Γ is contained in $\mathfrak{A}^*$.