ITERATION OF CERTAIN FINITE TRANSFORMATION.

By

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Preface.

Given the finite transformation of the form as follows:

$$(0.1) \qquad T: \quad 'x^{\nu} = \varphi^{\nu}(x) = a^{\nu}_{\mu} x^{\mu} + a^{\nu}_{\mu_1 \mu_2} x^{\mu_1} x^{\mu_2} + \cdots\cdots .^{(1)}$$

We consider the case where the absoulte values of all the eigen values λ_{ν} of $\| a^{\nu}_{\mu} \|$ are unity. In the previous paper[2], it was shown that, when T is majorized, the equations of Schröder for T can be solved and, that, by means of their solutions, T is reduced to the linear transformation such that $'x^{\nu} = \lambda_{\nu} x^{\nu}$ (not summed by ν). When the arguments of the eigen values are all commensurable with 2π, by iteration of certain times, T can be reduced to the transformation where the eigen values are all unity. In this paper, we deal with this case. When T is majorized, the reduced transformation becomes an identical transformation[3]. In this paper, we do not assume the condition of majorizedness. By effecting a suitable linear transformation of the variables x^{ν}, without loss of generality, we may assume that $\| a^{\nu}_{\mu} \|$ is of Jordan's form. Thus the transformation which becomes a subject in this paper, may be written as follows:

$$(0.2) \qquad T: \quad 'x^{\nu} = \varphi^{\nu}(x) = x^{\nu} + \delta_{\nu-1} x^{\nu-1} + a^{\nu}_{\mu_1 \mu_2} x^{\mu_1} x^{\mu_2} + \cdots\cdots ,$$

where $\delta_{\nu-1} = 0$ or 1.

In the case of one variable, the characters of the transformation of the form (0.2) were studied by means of iteration[4], and, in the neighborhood of the origin, there was found a domain, of which all the points converge to the origin always remaining in it when the transformation is

1) $a^{\nu}_{\mu} x^{\mu}$, $a^{\nu}_{\mu_1 \mu_2} x^{\mu_1} x^{\mu_2}$, ... mean that $\sum_{\mu} a^{\nu}_{\mu} x^{\mu}$, $\sum_{\mu_1, \mu_2} a^{\nu}_{\mu_1 \mu_2} x^{\mu_1} x^{\mu_2}$, ... In the following, we use this convention of tensor calculus.

2) M. Urabe, ***Application of majorized group of transformations to functional equations***, this Journal Vol. 16, No. 2 (1952), pp. 267-283.

3) do.

4) M. P. Fatou, Bull. Soc. Math. (1919).
G. Julia, Jour. Math. Pures Appl. (1918).
J. Malmquist, 1er mémoire, Ark. Mat. Ast. Fys. (1921).

is infinitely iterated on these points.

In this paper, extending Julia's method[1] to the case of n variables, we seek for analogous domains in the neighborhood of the origin, and we shall find some conditions for existence of such domains. In the last, we remark on an application to the solutions of the differential equations in the neighborhood of the singularity.

In the case of two variables, J. Malmquist[2] has studied iteration of the transformation of the form (0.2) of the special kind and he has applied his results to the differential equations. The transformations which he has discussed, are those which have invariant variety of two dimensions. For such transformations, their iteration is reduced to that of the transformation of one variable. In this paper, we consider the general transformation, consequently our discussions aie entirely diffeient from his.

In this paper, we classify the transformation T as follows:

$$\begin{cases} \text{type A} \begin{cases} \mathrm{A}_1: & T \text{ where } \| a^\nu_\mu \| \text{ is of diagonal form and at least one of} \\ & a^\nu_{\mu_1\mu_2}\text{'s does not vanish;} \\ \mathrm{A}_2: & T \text{ where } \| a^\nu_\mu \| \text{ is not of diagonal form;} \end{cases} \\ \text{type B} \quad : \; T \text{ where } \| a^\nu_\mu \| \text{ is of diagonal form and all } a^\nu_{\mu_1\mu_2}\text{'s vanish.} \end{cases}$$

Chapter I. Transformation of type A.

§ 1. Condition I for the transformation of type A_1.

We consider the transformation of type A_1, which is written as follows:

$$T: \quad 'x^\nu = \varphi^\nu(x) = x^\nu + a^\nu_{\mu_1\mu_2} x^{\mu_1} x^{\mu_2} + \cdots\cdots . \tag{1.1}$$

In the space E_{2n} of the complex numbers $(x^1, x^2, \ldots, x^n)$, we consider the hypersphere S with center α^ν and with radius r, of which the hypersurface passes through the origin.

Put

$$\alpha^\nu = r r_\nu e^{i\omega_\nu}, \tag{1.2}$$

where $r_\nu \geqq 0$ and $\sum_\nu r_\nu^2 = 1$. We assume that, *for any r such that $r \leqq r_0$, all the points of the hypersurface of any hypersphere with fixed r_ν and ω_ν passing through the origin are transformed by T to the inner points of that hypersphere*

1) G. Julia, ibid.

2) J. Malmqnist, 2^e mémoire, Ark. Mat. Ast. Fys. (1921).

except for the origin. We denote the family of these hyperspheres by $\mathfrak{S}$. Take any hypersphere $S \in \mathfrak{S}$ and any point P on its hypersurface. Then it is readily seen that P is transformed to the inner point of S or to the origin by iteration of T on P. Therefore the set of poits $\{T^k(P)\}$ $(k=0, 1, 2, \ldots\ldots)$ has at least one point of accumulation. When P is the origin, $T^k(P)$ is always the origin. We exclude this case. Let any one of the points of accumulation be P_0. We shall show that this P_0 must be the origin. If P_0 is not the origin, there exists a hypersphere $S_0 \in \mathfrak{S}$ passing through P_0. By our assumption, $P_1 = T(P_0)$ is an inner point of S_0 and not the origin. We consider the hypersphere V with the center P_1 in the interior of S_0. Then there exists a hypersphere $S' \in \mathfrak{S}$ which contains V and lies in the interior of S_0. Then, by the continuity of T, there exists a hypersphere V_0 with the center P_0 in the exterior of S' such that $T(V_0) \subset V$. Since P_0 is a point of accumulation of $\{T^k(P)\}$, there exists a sequence $\{k_i\}$ such that $\{T^{k_i}(P)\}$ converges to P_0. Consequently, if we take G sufficiently large, then, for $k_i > G$, $T^{k_i}(P) \in V_0$. Then, from $T(V_0) \subset V$, $T^{k_i+1}(P) \in V$. From our assumption, for any k, $T^k(V) \subset S'$, namely $T^{k_i+k}(P) \in S'$. Then, for any $k > k_i$, $T^k(P) \in S'$. This is inconsistent with the assumption that P_0 is a point of accumulation of $\{T^k(P)\}$. Thus P_0 must be the origin. Namely there exists only one point of accumulation of $\{T^k(P)\}$. Now evidently $T^k(P) \in S$ for any k. Consequently any subsequence of $\{T^k(P)\}$ has a point of accumulation, which is the origin. Thus we see that $\{T^k(P)\}$, always remaining in S, converges to the origin. Now S is any hypersphere belonging to the family $\mathfrak{S}$, consequently we see that *any point of the hypersphere S converges to the origin always remaining in it when T is infinitely iterated on that point.*

Thus we seek for the condition that there exists such a hypersphere. Consequently our condition gives a sufficient condition that there exists a domain, of which all the points converge to the origin remaining in it when T is infinitely iterated on these points.

The coordinates x^ν of any point P on the hypersurface of the hypersphere S are expressed as follows:

$$x^\nu - \alpha^\nu = r\rho_\nu e^{i'\theta_\nu}, \tag{1.3}$$

where $\rho_\nu \geqq 0$ and $\sum_\nu \rho_\nu^2 = 1$. Then the square of the distance d from the center α^ν to $P'('x) = T(P)$ becomes as follows:

$$(1.4) \qquad d^2 = \sum_\nu ('x^\nu - \alpha^\nu)('\bar{x}^\nu - \bar{\alpha}^\nu) \text{ (1)}$$

$$= \sum_\nu \{(x^\nu - \alpha^\nu) + a^\nu_{\lambda\mu} x^\lambda x^\mu + \cdots\}\{(\bar{x}^\nu - \bar{\alpha}^\nu) + \bar{a}^\nu_{\lambda\mu}\bar{x}^\lambda \bar{x}^\mu + \cdots\}$$

$$= \sum_\nu (x^\nu - \alpha^\nu)(\bar{x}^\nu - \bar{\alpha}^\nu) + \sum_\nu a^\nu_{\lambda\mu} x^\lambda x^\mu (\bar{x}^\nu - \bar{\alpha}^\nu) + \sum_\nu \bar{a}^\nu_{\lambda\mu}\bar{x}^\lambda \bar{x}^\mu (x^\nu - \alpha^\nu) + \cdots\cdots$$

$$= r^2 + 2r^3 R + [r]_4,$$

where $[\cdots]_4$ denotes the sum of the terms of the fourth and higher orders of the argument. We calculate R, namely the real part of $\frac{1}{r^3}\sum_\nu a^\nu_{\lambda\mu} x^\lambda x^\mu (\bar{x}^\nu - \bar{\alpha}^\nu)$. For this purpose, we put as follows:

$$(1.5) \qquad a^\nu_{\lambda\mu} = R^\nu_{\lambda\mu} e^{i'\Omega^\nu_{\lambda\mu}} \quad \text{and} \quad x^\nu = r X_\nu e^{i\tau_\nu}.$$

Then, from (1.2) and (1.3), we have

$$(1.6) \qquad X_\nu \cos\tau_\nu = r_\nu \cos\omega_\nu + \rho_\nu \cos{}'\theta_\nu, \; X_\nu \sin\tau_\nu = r_\nu \sin\omega_\nu + \rho_\nu \sin{}'\theta_\nu.$$

Substituting (1.5) and (1.6) into R, we calculate R as follows:

$$R = \sum_{\lambda,\mu,\nu} R^\nu_{\lambda\mu} X_\lambda X_\mu \rho_\nu \cos('\Omega^\nu_{\lambda\mu} + \tau_\lambda + \tau_\mu - '\theta_\nu)$$

$$= \sum_{\lambda,\mu,\nu} R^\nu_{\lambda\mu}\rho_\nu[\cos{}'\Omega^\nu_{\lambda\mu} \cdot X_\lambda X_\mu(\cos\tau_\lambda \cos\tau_\mu \cos{}'\theta_\nu + \cos\tau_\lambda \sin\tau_\mu \sin{}'\theta_\nu$$
$$- \sin\tau_\lambda \sin\tau_\mu \cos{}'\theta_\nu + \sin\tau_\lambda \cos\tau_\mu \sin{}'\theta_\nu)$$
$$- \sin{}'\Omega^\nu_{\lambda\mu} \cdot X_\lambda X_\mu(\sin\tau_\lambda \cos\tau_\mu \cos{}'\theta_\nu + \sin\tau_\lambda \sin\tau_\mu \sin{}'\theta_\nu$$
$$+ \cos\tau_\lambda \sin\tau_\mu \cos{}'\theta_\nu - \cos\tau_\lambda \cos\tau_\mu \sin{}'\theta_\nu)]$$

$$= \sum_{\lambda,\mu,\nu} R^\nu_{\lambda\mu}\rho_\nu[\cos{}'\Omega^\nu_{\lambda\mu}\{(r_\lambda \cos\omega_\lambda + \rho_\lambda \cos{}'\theta_\lambda)(r_\mu \cos\omega_\mu + \rho_\mu \cos{}'\theta_\mu)\cos{}'\theta_\nu + \cdots\cdots\}$$
$$- \sin{}'\Omega^\nu_{\lambda\mu}\{(r_\lambda \sin\omega_\lambda + \rho_\lambda \sin{}'\theta_\lambda)(r_\mu \cos\omega_\mu + \rho_\mu \cos{}'\theta_\mu)\cos{}'\theta_\nu + \cdots\cdots\}]$$

$$= \sum_{\lambda,\mu,\nu} R^\nu_{\lambda\mu}\rho_\nu[r_\lambda r_\mu \cos('\Omega^\nu_{\lambda\mu} + \omega_\lambda + \omega_\mu - '\theta_\nu) + r_\lambda\rho_\mu \cos('\Omega^\nu_{\lambda\mu} + \omega_\lambda + '\theta_\mu - '\theta_\nu)$$
$$+ r_\mu\rho_\lambda \cos('\Omega^\nu_{\lambda\mu} + \omega_\mu + '\theta_\lambda - '\theta_\nu) + \rho_\lambda\rho_\mu \cos('\Omega^\nu_{\lambda\mu} + '\theta_\lambda + '\theta_\mu - '\theta_\nu)].$$

Put

$$(1.7) \qquad '\Omega^\nu_{\lambda\mu} + \omega_\lambda + \omega_\mu - \omega_\nu = \Omega^\nu_{\lambda\mu} \quad \text{and} \quad '\theta_\nu = \omega_\nu + \theta_\nu.$$

Then we have:

$$(1.8) \qquad R = R(r, \omega; \rho, \theta)$$
$$= \sum_{\lambda,\mu,\nu} R^\nu_{\lambda\mu}\rho_\nu[r_\lambda r_\mu \cos(\Omega^\nu_{\lambda\mu} - \theta_\nu) + r_\lambda\rho_\mu(\Omega^\nu_{\lambda\mu} + \theta_\mu - \theta_\nu)$$
$$+ r_\mu\rho_\lambda \cos(\Omega^\nu_{\lambda\mu} + \theta_\lambda - \theta_\nu) + \rho_\lambda\rho_\mu \cos(\Omega^\nu_{\lambda\mu} + \theta_\lambda + \theta_\mu - \theta_\nu)].$$

For the origin, $\theta_\nu \equiv \pi \pmod{2\pi}$ and $\rho_\nu = r_\nu$, consequently $R=0$. We assume that, except foı the origin, $R<0$ for all ρ_ν and θ_ν. Then, if we take arbitrary small positive number δ, then, for ρ_ν and θ_ν satisfying at least one of the relations as follows:

1) The bars on the letters mean the conjugate imaginaries.

(1.9) $$|\rho_\nu - r_\nu| \geqq \delta \quad \text{and} \quad \pi - \delta \geqq \theta_\nu \geqq -\pi + \delta\,,$$

there exists R_0 such that $R \leqq R_0 < 0$. Then, from (1.4), there exists a small number r_0 such that, for $r \leqq r_0$, $d < r$, namely any point P on the hypersurface of S which satisfies at least one of the relations (1.9) is transformed by T to an inner point of S.

Next, we consider the point P which does not satisfy any of the relations (1.9), or in other words, does satisfy the following relations:

(1.10) $$|\rho_\nu - r_\nu| < \delta \quad \text{and} \quad \pi - \delta < \theta_\nu < \pi + \delta\,.$$

Put

(1.11) $$\rho_\nu = r_\nu + \varepsilon\eta_\nu, \quad \theta_\nu = \pi + \varepsilon\varepsilon_\nu\,,$$

where η_ν and ε_ν are not all zero and $|\eta_\nu|$, $|\varepsilon_\nu| \leqq 1$, and we assume that

(1.12) $$|\varepsilon| < \delta\,.$$

For such $P(x^\nu)$, from (1.6), we have

(1.13) $$\begin{aligned} X_\nu \cos\tau_\nu &\doteqdot (-\eta_\nu \cos\omega_\nu + \varepsilon_\nu r_\nu \sin\omega_\nu)\varepsilon\,, \\ X_\nu \sin\tau_\nu &\doteqdot (-\eta_\nu \sin\omega_\nu - \varepsilon_\nu r_\nu \cos\omega_\nu)\varepsilon\,. \end{aligned}$$ [1]

Consequently it follows that

(1.14) $$X_\nu \doteqdot |\varepsilon| \sqrt{\eta_\nu^2 + \varepsilon_\nu^2 r_\nu^2}\,.$$

We put as follows:

(1.15) $$x^\nu = \sigma\sigma_\nu e^{i\tau_\nu}\,,$$

where $\sum_\nu \sigma_\nu^2 = 1$ and $\sigma_\nu \geqq 0$. Then, comparing these with (1.5), we have

(1.16) $$\sigma\sigma_\nu = rX_\nu\,,$$

consequently it follows that

(1.17) $$\sigma = r\sqrt{\sum_\nu X_\nu^2} \doteqdot |\varepsilon|\, r\sqrt{\sum_\nu \eta_\nu^2 + \sum_\nu \varepsilon_\nu^2 r_\nu^2}\,.$$

For ρ_ν and θ_ν of (1.11), calculating R, we have:

(1.18) $$R \doteqdot \varepsilon^2 H_2\,,$$

where

(1.19) $$H_2 = \sum_{\lambda,\mu,\nu} R^\nu_{\lambda\mu} r_\nu [(r_\lambda \varepsilon_\lambda \eta_\mu + r_\mu \varepsilon_\mu \eta_\lambda) \sin\Omega^\nu_{\lambda\mu} + (r_\lambda r_\mu \varepsilon_\lambda \varepsilon_\mu - \eta_\lambda \eta_\mu) \cos\Omega^\nu_{\lambda\mu}]\,.$$

When $H_2 \neq 0$, from our assumption that $R < 0$ except for the origin,

1) $\doteqdot$ means equality except for the infinitesimals of higher order than those written explicitly.

it follows that $H_2<0$. Now, for ρ_ν and θ_ν of (1.11), from (1.15) and (1.17), $x^\nu=|\varepsilon| r\sigma_\nu\sqrt{\sum_\nu \eta_\nu^2+\sum_\nu \varepsilon_\nu^2 r_\nu^2}\, e^{i\tau_\nu}$, consequently, in (1.4), all the terms of $[r]_4$ are of the order at least three with regard to ε. Then (1.4) is written as follows:

$$d^2 \doteqdot r^2+r^3\varepsilon^2[2H_2+r\varepsilon(\cdots\cdots)]\,.$$

Then, for any sufficiently small number ε, it follows that $d<r$ except for $\varepsilon=0$. Thus the point P which satisfies (1.10), is also transformed to an inner point of S. Thus we have

Theorem 1. *For the transformation*

$$T:\quad {}'x^\nu=\varphi^\nu(x)=x^\nu+a^\nu_{\lambda\mu}x^\lambda x^\mu+\cdots\cdots,$$

we put as follows:

$$a^\nu_{\lambda\mu}=R^\nu_{\lambda\mu}e^{i'\Omega^\nu_{\lambda\mu}},\quad {}'\Omega^\nu_{\lambda\mu}+\omega_\lambda+\omega_\mu-\omega_\nu=\Omega^\nu_{\lambda\mu}\,.$$

For the following function

$$\begin{aligned} R&=R(r,\,\omega\,;\,\rho,\,\theta)\\ &=\sum_{\lambda,\mu,\nu}R^\nu_{\lambda\mu}\rho_\nu[r_\lambda r_\mu\cos(\Omega^\nu_{\lambda\mu}-\theta_\nu)+r_\lambda\rho_\mu\cos(\Omega^\nu_{\lambda\mu}+\theta_\mu-\theta_\nu)\\ &\qquad +r_\mu\rho_\lambda\cos(\Omega^\nu_{\lambda\mu}+\theta_\lambda-\theta_\nu)+\rho_\lambda\rho_\mu\cos(\Omega^\nu_{\lambda\mu}+\theta_\lambda+\theta_\mu-\theta_\nu)]\,,\end{aligned}$$

where $r_\nu,\,\rho_\nu\geqq 0$ and $\sum r_\nu^2=\sum\rho_\nu^2=1$, we assume that there exists a set of $(r_\nu,\,\omega_\nu)$ such that $R<0$ for all $(\rho_\nu,\,\theta_\nu)$ except for $(\rho_\nu=r_\nu,\,\theta_\nu\equiv\pi(\mathrm{mod}\ 2\pi))$, and that $\frac{1}{2}\left[\frac{d^2}{d\varepsilon^2}R(r,\omega\,;\,r_\nu+\varepsilon\eta_\nu,\,\pi+\varepsilon\varepsilon_\nu)\right]_{\varepsilon=0}=H_2\neq 0$ for any $(\eta_\nu,\,\varepsilon_\nu)$ such that $|\eta_\nu|,\,|\varepsilon_\nu|\leqq 1$ except for $\eta_\nu=\varepsilon_\nu=0$. Then, in the space E_{2n} of the complex numbers x^ν, there exists a small hypersphere passing through the origin with the center $\alpha^\nu=rr_\nu e^{i\omega_\nu}$ and with the radius r, such that all the points of that hypersphere converge to the origin remaining in it when T is infinitely iterated on these points.

When $H_2=0$ for certain $(\eta_\nu,\,\varepsilon_\nu)$, R is expanded with regard to ε as follows: $R=\varepsilon^3H_3+\varepsilon^4H_4+\cdots\cdots$. However, from our assumption that $R<0$ except for the origin, it must be $H_3=0$. When $H_4\neq 0$, from (1.4), we have:

$$d^2=r^2+r^3\varepsilon^3\{2\varepsilon H_4+rK+r\varepsilon(\cdots\cdots)\}\,,\tag{1.20}$$

where K is the coefficient produced from the sum $\sum_{\lambda_1,\lambda_2,\lambda_3,\nu}a^\nu_{\lambda_1\lambda_2\lambda_3}x^{\lambda_1}x^{\lambda_2}x^{\lambda_3}\times$ $\times(\bar{x}^\nu-\bar{\alpha}^\nu)$ and their conjugates. If $K=0$, we can deduce $d<r$ as in the case where $H_2\neq 0$, consequently in this case there exists a hypersphere of

Theorem 1. If $K\neq 0$, according as $K>0$ or <0, for positive ε, $d>r$ or $<r$ and, for negative ε, $d<r$ or $>r$, consequently in general there does not exist the hypersphere of Theorem 1. When $H_4=0$, similar reasonings are continually applied.

When there exist many hyperspheres of Theorem 1, it is evident that the domain constructed by the sum of such hyperspheres has the same character as each hypersphere, namely all the points of that domain converge to the origin by infinite iteration of the transformation always remaining in it.

Example 1. *The case where* $n=1$.

We assume that $R^1_{11}=c\neq 0$. Julia discussed this case, but his discussions seem insufficient for lack of investigation on the neighborhood of the origin.[1] However, correctness of his conclusion is easily seen as follows:

In this case, since $\eta_1=0$, we have:

$$R = c[\cos(\Omega-\theta)+2\cos\Omega+\cos(\Omega+\theta)] = 4c\cos\Omega\cos^2\frac{\theta}{2}\,;$$
$$H_2 = c\varepsilon_1^2\cos\Omega\,.$$

Consequently there exists the hypersphere with ω such that $\cos('\Omega+\omega)<0$.

Example 2. $a^1_{11}=a^2_{12}=\cdots\cdots=a^n_{1n}<0$ *and other* $a^\nu_{\lambda\mu}$*'s and all* $a^\nu_{\lambda_1\lambda_2\lambda_3}$*'s vanish.*

Put $R^\nu_{1\nu}=c>0$. We adopt r_ν and ω_ν such that $\omega_\nu=0$ and $r_1=1$, $r_2=r_3=\cdots=r_n=0$. Then $\Omega^\nu_{1\nu}=\pi$, and R is calculated as follows:

$$\begin{aligned}
R &= c\rho_1\{\cos(\pi-\theta_1)+2\rho_1\cos\pi+\rho_1^2\cos(\pi+\theta_1)\}\\
&\qquad +2c\rho_2\{\rho_2\cos\pi+\rho_1\rho_2\cos(\pi+\theta_1)\}\\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\qquad +2c\rho_n\{\rho_n\cos\pi+\rho_1\rho_n\cos(\pi+\theta_1)\}\\
&= -c\rho_1(\cos\theta_1+2\rho_1+\rho_1^2\cos\theta_1)-2c(\rho_2^2+\cdots+\rho_n^2)(1+\rho_1\cos\theta_1)\\
&= -c\rho_1\cos\theta_1(1-\rho_1^2)-2c(1+\rho_1\cos\theta_1)\quad (\because\ \rho_1^2+\cdots+\rho_n^2=1)\\
&= -c\{2+\rho_1(3-\rho_1^2)\cos\theta_1\}\,.
\end{aligned}$$

Now it is readily seen that $0<\rho(3-\rho_1^2)<2$ for $0<\rho_1<1$. Consequently $R<0$ except for $\rho_1=1$ and $\theta_1=\pi$, namely except for the origin.

Next, put $\rho_1=1+\varepsilon\eta_1$, $\rho_2=\varepsilon\eta_2$, $\ldots\ldots$, $\rho_n=\varepsilon\eta_n$. Then, from $\sum_\nu\rho_\nu^2=1$, $2\varepsilon\eta_1+\varepsilon^2\sum_\nu\eta_\nu^2=0$, consequently $\eta_1=-\frac{1}{2}\varepsilon\sum_\nu\eta_\nu^2$. Therefore, instead of the above substitution, we put as follows:

1) G. Julia, ibid.

$$\rho_1 = 1+\varepsilon^2\eta_1, \quad \rho_2 = \varepsilon\eta_2, \quad \ldots\ldots\ldots\ldots, \quad \rho_n = \varepsilon\eta_n .$$

Putting $\theta_\nu = \pi + \varepsilon\varepsilon_\nu$, we have:

$$H_2 \doteqdot -c\varepsilon_1^2 .$$

Therefore, for $\varepsilon_1 \neq 0$, $H_2 \neq 0$. For $\varepsilon_1 = 0$, $\theta_1 = \pi$. For this value, R becomes as follows:

$$\begin{aligned} R &= -c(\rho_1^3 - 3\rho_1 + 2) \\ &= -c(\rho_1 - 1)^2(\rho_1 + 2) \\ &= -c\varepsilon^4\eta_1^2(3+\varepsilon^2\eta_1) . \end{aligned}$$

If $\eta_1 = 0$, then $\rho_1 = 1$ and $\theta_1 = \pi$, namely this point is the origin. Therefore, except for the origin, $\eta_1 \neq 0$, consequently $H_4 \neq 0$. Now, by our assumption that $a^\nu_{\lambda_1\lambda_2\lambda_3} = 0$, in (1.20), $K = 0$. Therefore, by the remark after Theorem 1, it follows that $d < r$ except for the origin.

Summarizing the results, we see that *there exists the hypersphere of Theorem 1 with the center* $(r, 0, \ldots, 0)$ *for sufficiently small* r.

§ 2. The necessary conditions.

We seek for the necessary conditions that there may exist hyperspheres of Theorem 1. For this purpose, at first, we seek for the maximum of R under the condition that one of ρ_ν's is unity and the other ρ_ν's vanish. Under this condition, R is written as follows:

$$R = R_\nu = \sum_{\lambda,\mu} R^\nu_{\lambda\mu} r_\lambda r_\mu \cos(\Omega^\nu_{\lambda\mu} - \theta_\nu) + 2\sum_\lambda R^\nu_{\lambda\nu} r_\lambda \cos\Omega^\nu_{\lambda\nu} + R^\nu_{\nu\nu}\cos(\Omega^\nu_{\nu\nu} + \theta_\nu). \tag{2.1}$$

Putting $\dfrac{dR}{d\theta_\nu} = 0$, we have:

$$\sum_{\lambda,\mu} R^\nu_{\lambda\mu} r_\lambda r_\mu \sin(\Omega^\nu_{\lambda\mu} - \theta_\nu) - R^\nu_{\nu\nu}\sin(\Omega^\nu_{\nu\nu} + \theta_\nu) = 0 ,$$

therefore

$$\begin{aligned} &\Big(\sum_{\lambda,\mu} R^\nu_{\lambda\mu} r_\lambda r_\mu \sin\Omega^\nu_{\lambda\mu} - R^\nu_{\nu\nu}\sin\Omega^\nu_{\nu\nu}\Big)\cos\theta_\nu \\ &\qquad = \Big(\sum_{\lambda,\mu} R^\nu_{\lambda\mu} r_\lambda r_\mu \cos\Omega^\nu_{\lambda\mu} + R^\nu_{\nu\nu}\cos\Omega^\nu_{\nu\nu}\Big)\sin\theta_\nu . \end{aligned} \tag{2.2}$$

Put

(2.3)

$$\begin{aligned} D_\nu &= \sqrt{\Big(\sum_{\lambda,\mu} R^\nu_{\lambda\mu} r_\lambda r_\mu \sin\Omega^\nu_{\lambda\mu} - R^\nu_{\nu\nu}\sin\Omega^\nu_{\nu\nu}\Big)^2 + \Big(\sum_{\lambda,\mu} R^\nu_{\lambda\mu} r_\lambda r_\mu \cos\Omega^\nu_{\lambda\mu} + R^\nu_{\nu\nu}\cos\Omega^\nu_{\nu\nu}\Big)^2} \\ &= \sqrt{\sum_{\lambda,\mu,\lambda_1,\mu_1} R^\nu_{\lambda\mu} R^\nu_{\lambda_1\mu_1} r_\lambda r_\mu r_{\lambda_1} r_{\mu_1} \cos(\Omega^\nu_{\lambda\mu} - \Omega^\nu_{\lambda_1\mu_1}) + 2\sum_{\lambda,\mu} R^\nu_{\lambda\mu} R^\nu_{\nu\nu} r_\lambda r_\mu \cos(\Omega^\nu_{\lambda\mu} + \Omega^\nu_{\nu\nu}) + (R^\nu_{\nu\nu})^2} . \end{aligned}$$

Substituting θ_ν determined by (2.2) into $\dfrac{d^2R}{d\theta_\nu^2}$, we have:

$$\begin{aligned}\frac{d\,R}{d\theta_\nu^2} &= -[\sum_{\lambda,\mu} R^\nu_{\lambda\mu} r_\lambda r_\mu \cos(\Omega^\nu_{\lambda\mu} - \theta_\nu) + R^\nu_{\nu\nu}\cos(\Omega^\nu_{\nu\nu}+\theta_\nu)] \\ &= -\frac{s}{D_\nu}[\sum_{\lambda,\mu} R^\nu_{\lambda\mu} r_\lambda r_\mu \cos\Omega^\nu_{\lambda\mu}(\sum_{\lambda_1,\mu_1} R^\nu_{\lambda_1\mu_1} r_{\lambda_1} r_{\mu_1}\cos\Omega^\nu_{\lambda_1\mu_1} + R^\nu_{\nu\nu}\cos\Omega^\nu_{\nu\nu}) \\ &\quad + \sum_{\lambda,\mu} R^\nu_{\lambda\mu} r_\lambda r_\mu \sin\Omega^\nu_{\lambda\mu}(\sum_{\lambda_1,\mu_1} R^\nu_{\lambda_1\mu_1} r_{\lambda_1} r_{\mu_1}\sin\Omega^\nu_{\lambda_1\mu_1} - R^\nu_{\nu\nu}\sin\Omega^\nu_{\nu\nu}) \\ &\quad + R^\nu_{\nu\nu}\cos\Omega^\nu_{\nu\nu}(\cdots) - R^\nu_{\nu\nu}\sin\Omega^\nu_{\nu\nu}(\cdots)] \\ &= -sD_\nu\,,\end{aligned}$$

where $s=+1$ or -1. When $D_\nu \neq 0$, for θ_ν such that $s=+1$, $\frac{d\,R}{d\theta_\nu^2}<0$, namely R_ν becomes maximum for such θ_ν. The maximum values of R_ν are easily found as follows:

$$R_\nu = D_\nu + 2\sum_\lambda R^\nu_{\lambda\nu} r_\lambda \cos\Omega^\nu_{\lambda\nu}\,. \tag{2.4}$$

When $D_\nu=0$, from (2.3), it follows that

$$\sum_{\lambda,\mu} R^\nu_{\lambda\mu} r_\lambda r_\mu \cos\Omega^\nu_{\lambda\mu} + R^\nu_{\nu\nu}\cos\Omega^\nu_{\nu\nu} = 0\,,$$
$$\sum_{\lambda,\mu} R^\nu_{\lambda\mu} r_\lambda r_\mu \sin\Omega^\nu_{\lambda\mu} - R^\nu_{\nu\nu}\sin\Omega^\nu_{\nu\nu} = 0\,.$$

Therefore, in this case, we have: $R=2\sum_\lambda R^\nu_{\lambda\nu} r_\lambda \cos\Omega^\nu_{\lambda\nu}$. Then (2.4) is valid whether D_ν vanishes or not. Then, from (1.4), we have

Theorem 2. *In order that there may exist hyperspheres of Theorem 1, it is necessary that, for any ν,*

$$D_\nu + 2\sum_\lambda R^\nu_{\lambda\nu} r_\lambda \cos\Omega^\nu_{\lambda\nu} \leqq 0\,.$$

For example, we consider the case where $a^\nu_{\lambda\mu}$ vanish except for $a^\nu_{\nu\nu}$. Put $R^\nu_{\nu\nu}=c_\nu$ and $\Omega^\nu_{\nu\nu}=\Omega_\nu$, then the necessary conditions become as follows:

$$c_\nu[\sqrt{r_\nu^4+2r_\nu^2\cos 2\Omega_\nu+1} + 2r_\nu\cos\Omega_\nu] \leqq 0\,.$$

However $\sqrt{r_\nu^4+2r_\nu^2\cos 2\Omega_\nu+1} > |2r_\nu\cos\Omega_\nu|$ except for the case $r_\nu=1$. Therefore, when at least two of $a^\nu_{\nu\nu}$'s do not vanish, the necessary conditions are not valid, namely, in this case, there does not exist the hypersphere.

§ 3. Condition I for the transformation of type A_2.

We consider the transformation of type A_2, which is written as follows:

$$T:\quad 'x^\nu = \varphi^\nu(x) = x^\nu + \delta_{\nu-1}x^{\nu-1} + a^\nu_{\lambda\mu}x^\lambda x^\mu + \cdots\cdots\,, \tag{3.1}$$

where $\delta_{\nu-1}=0$ or 1. In the space E_{2n} of the complex numbers $(x^1, x^2, \ldots, x^n)$, as in § 1, we seek for the hypersphere of Theorem 1. Then, the square of the distance d from the center α^ν to the transform P' of the point P on the hypersuface becomes as follows:

$$
\begin{aligned}
(3.2) \qquad d^2 &= \sum_\nu ('x^\nu - \alpha^\nu)('\bar{x}^\nu - \bar{\alpha}^\nu) \\
&= \sum_\nu \{(x^\nu - \alpha^\nu) + \delta_{\nu-1} x^{\nu-1} + a^\nu_{\lambda\mu} x^\lambda x^\mu + \cdots\cdots\} \\
&\qquad \times \{(\bar{x}^\nu - \bar{\alpha}^\nu) + \delta_{\nu-1}\bar{x}^{\nu-1} + \bar{a}^\nu_{\lambda\mu}\bar{x}^\lambda \bar{x}^\mu + \cdots\cdots\} \\
&= \sum_\nu (x^\nu - \alpha^\nu)(\bar{x}^\nu - \bar{\alpha}^\nu) + \sum \delta_{\nu-1} x^{\nu-1}(\bar{x}^\nu - \bar{\alpha}^\nu) + \sum \delta_{\nu-1}\bar{x}^{\nu-1}(x^\nu - \alpha^\nu) \\
&\qquad + \sum_\nu \delta^2_{\nu-1} x^{\nu-1}\bar{x}^{\nu-1} + \cdots\cdots \\
&= r^2 + r^2 R + [r]_3 .
\end{aligned}
$$

As in §1, by means of (1.6), we calculate R as follows:

$$
\begin{aligned}
R &= 2\sum_\nu \delta_{\nu-1} X_{\nu-1}\rho_\nu \cos(\tau_{\nu-1} - '\theta_\nu) + \sum_\nu \delta^2_{\nu-1} X^2_{\nu-1} \\
&= 2\sum_\nu \delta_{\nu-1}\rho_\nu [(r_{\nu-1}\cos\omega_{\nu-1} + \rho_{\nu-1}\cos '\theta_{\nu-1})\cos '\theta_\nu + (r_{\nu-1}\sin\omega_{\nu-1} \\
&\qquad + \rho_{\nu-1}\sin '\theta_{\nu-1})\sin '\theta_\nu] + \sum_\nu \delta^2_{\nu-1}[r^2_{\nu-1} + \rho^2_{\nu-1} + 2r_{\nu-1}\rho_{\nu-1}\cos('\theta_{\nu-1} - \omega_{\nu-1})] \\
&= 2\sum_\nu \delta_{\nu-1}\rho_\nu [r_{\nu-1}\cos('\theta_\nu - \omega_{\nu-1}) + \rho_{\nu-1}\cos('\theta_\nu - '\theta_{\nu-1})] \\
&\qquad + \sum_\nu \delta^2_{\nu-1}[r^2_{\nu-1} + \rho^2_{\nu-1} + 2r_{\nu-1}\rho_{\nu-1}\cos('\theta_{\nu-1} - \omega_{\nu-1})] .
\end{aligned}
$$

From (1.7), we have:

$$
\begin{aligned}
(3.3) \quad R &= 2\sum_\nu \delta_{\nu-1}\rho_\nu [r_{\nu-1}\cos(\theta_\nu + \omega_\nu - \omega_{\nu-1}) + \rho_{\nu-1}\cos(\theta_\nu - \theta_{\nu-1} + \omega_\nu - \omega_{\nu-1})] \\
&\qquad + \sum_\nu \delta^2_{\nu-1}(r^2_{\nu-1} + \rho^2_{\nu-1} + 2r_{\nu-1}\rho_{\nu-1}\cos\theta_{\nu-1}) .
\end{aligned}
$$

We put unity one of such ρ_ν's that $\delta_{\nu-1}=0$ and $\delta_\nu=1$. Then the first sum of the right-hand side vanishes, consequently it follows that

$$
R = r_1^2 + \cdots + (r_\nu^2 + 1 + 2r_\nu \cos\theta_\nu) + \cdots\cdots .
$$

Now $r_\nu^2 + 1 + 2r_\nu\cos\theta_\nu$ becomes $(r_\nu+1)^2 > 0$ for $\theta_\nu = 0$. Namely R becomes positive for a certain point which is not the origin.

Thus we see that, *for the transformation of type* A_2, *there does not exist the hypersphere of Theorem 1.*

§ 4. Condition II for the transformation of type A_1.

In each x^ν-plane, we consider the circle $\mathring{C}_\nu$ passing through the origin O with the center $re^{i\omega_\nu}$ and with the radius r. Then, in the space E_{2n} of the complex numbers $(x^1, x^2, \dots, x^n)$, the cylindrical domain D is constructed by these $\mathring{C}_\nu$'s. We consider the condition that any point $P(x^\nu)$ on the boundary hypersurface of D is transformed to the inner point of D except for the origin.

In the x^ν-plane, we construct a circle C_ν which touches $\overset{\circ}{C}_\nu$ at the origin O and passes through the point x^ν. When the point $P(x^\nu)$ lies on the boundary of D, the point x^ν in the x^ν-plane lies on the boundary or in the interior of $\overset{\circ}{C}_\nu$. Let the center of C_ν be α^ν, then it follows readily that

$$\alpha^\nu = r r_\nu e^{i\omega_\nu}, \quad \text{where} \quad 0 \leqq r_\nu \leqq 1. \tag{4.1}$$

Let the angle between the radius vector passing through α^ν and that passing through x^ν be θ_ν. Then it follows that

$$x^\nu = 2 r r_\nu \cos\theta_\nu e^{i(\omega_\nu+\theta_\nu)}, \quad x_\nu - \alpha_\nu = r r_\nu e^{i(\omega_\nu+2\theta_\nu)}, \quad \text{and} \quad -\frac{\pi}{2} \leqq \theta_\nu \leqq \frac{\pi}{2}. \tag{4.2}$$

Now we consider the condition. If $P(x^\nu)$ on the boundary of D is transformed to an inner point of D, then x^ν on the boundary of $\overset{\circ}{C}_\nu$ must be transformed to an inner point of $\overset{\circ}{C}_\nu$. At first, we seek for the condition for this. Let the given transformation of type A_1 be

$$T: \quad {}'x^\nu = \varphi^\nu(x) = x^\nu + a^\nu_{\lambda\mu} x^\lambda x^\mu + \cdots\cdots, \tag{4.3}$$

and put $a^\nu_{\lambda\mu} = R^\nu_{\lambda\mu} e^{i\,'\Omega^\nu_{\lambda\mu}}$. The square of the distance d_ν from the center $\alpha^\nu = re^{i\omega_\nu}$ of $\overset{\circ}{C}_\nu$ to the transform ${}'x^\nu$ of x^ν on $\overset{\circ}{C}_\nu$ becomes as follows[1]:

$$\begin{aligned} d_\nu^2 &= ({}'x^\nu - \alpha^\nu)({}'\bar{x}^\nu - \bar{\alpha}^\nu) \\ &= \{(x^\nu - \alpha^\nu) + a^\nu_{\lambda\mu} x^\lambda x^\mu + \cdots\}\{(\bar{x}^\nu - \bar{\alpha}^\nu) + \bar{a}^\nu_{\lambda\mu}\bar{x}^\lambda \bar{x}^\mu + \cdots\} \\ &= (x^\nu - \alpha^\nu)(\bar{x}^\nu - \bar{\alpha}^\nu) + a^\nu_{\lambda\mu} x^\lambda x^\mu (\bar{x}^\nu - \bar{\alpha}^\nu) + \bar{a}^\nu_{\lambda\mu}\bar{x}^\lambda \bar{x}^\mu (x^\nu - \alpha^\nu) + \cdots \\ &= r^2 + 8r^3 R_\nu + [r]_4, \end{aligned} \tag{4.4}$$

where R_ν is the real part of $\frac{1}{4r^3} a^\nu_{\lambda\mu} x^\lambda x^\mu (\bar{x}^\nu - \bar{\alpha}^\nu)$. Then, by our condition, it must be $R_\nu \leqq 0$. From (4.2), R_ν is calculated as follows:

$$\begin{aligned} R_\nu &= \sum_{\lambda,\mu} R^\nu_{\lambda\mu} r_\lambda r_\mu \cos\theta_\lambda \cos\theta_\mu \cos({}'\Omega^\nu_{\lambda\mu} + \omega_\lambda + \omega_\mu - \omega_\nu + \theta_\lambda + \theta_\mu - 2\theta_\nu) \\ &= R^\nu_{\nu\nu}\cos^2\theta_\nu \cos\Omega^\nu_{\nu\nu} + 2\sum_{\lambda\neq\nu} R^\nu_{\lambda\nu} r_\lambda \cos\theta_\lambda \cos\theta_\nu \cos(\Omega^\nu_{\lambda\nu} + \theta_\lambda - \theta_\nu) \\ &\quad + \sum_{\lambda,\mu\neq\nu} R^\nu_{\lambda\mu} r_\lambda r_\mu \cos\theta_\lambda \cos\theta_\mu \cos(\Omega^\nu_{\lambda\mu} + \theta_\lambda + \theta_\mu - 2\theta_\nu), \end{aligned} \tag{4.5}$$

where

$$\Omega^\nu_{\lambda\mu} = {}'\Omega^\nu_{\lambda\mu} + \omega_\lambda + \omega_\mu - \omega_\nu. \tag{4.6}$$

For the sake of the subsequent discussions, we prove

Lemma. *If Ω is given arbitrarily and p, q are such that $p \geqq q \geqq 0$, and $p+q>1$, then, for θ, φ both lying in $[-\pi/2, \pi/2]$, there exists a value θ_0 of θ such that $-\pi/2 < \theta_0 < \pi/2$ and*

---

1) When x^ν is on $\overset{\circ}{C}_\nu$, $r_\nu = 1$.

(L) $$\cos(\Omega+p\theta_0-q\varphi)\geqq\eta>0$$

for all φ sufficiently near to $\pi/2$ or to $-\pi/2$, where η is a suitable small number.

Proof. We consider the intervals $I_1\left[-(p+q)\frac{\pi}{2},\ (p-q)\frac{\pi}{2}\right]$ and $I_2\left[-(p-q)\frac{\pi}{2},\ (p+q)\frac{\pi}{2}\right]$. Then both intervals have the common interval $I_c\left[-(p-q)\frac{\pi}{2},\ (p-q)\frac{\pi}{2}\right]$ and the sum interval $I_s\left[-(p+q)\frac{\pi}{2},\ (p+q)\frac{\pi}{2}\right]$. The length of I_s is evidently $(p+q)\pi>\pi$. Then there exist ξ_0 and sufficiently small numbers δ, η such that, for ξ such that $|\xi-\xi_0|\leqq\delta$, there hold $\cos\xi\geqq\eta>0$ and also at least one of the following relations:

$$\Omega-(p+q)\frac{\pi}{2}<\xi<\Omega+(p-q)\frac{\pi}{2}\quad\text{or}\quad\Omega-(p-q)\frac{\pi}{2}<\xi<\Omega+(p+q)\frac{\pi}{2}\,.$$

For the former, we put $\xi=\Omega+p\theta_0-q\frac{\pi}{2}$, then, from the inequality, it follows: $-\frac{\pi}{2}<\theta_0<\frac{\pi}{2}$. For $\varphi=\frac{\pi}{2}-\varepsilon$ where ε is an arbitrary positive number such that $\varepsilon q\leqq\delta$, we put $\xi'=\Omega+p\theta_0-q\varphi$, then $\xi'=\xi_0+\varepsilon q$, consequently $|\xi'-\xi_0|\leqq\delta$. Then we have: $\cos\xi'=\cos(\Omega+p\theta_0-q\varphi)\geqq\eta>0$. For the latter of the inequalities, in the same manner, we have the same conclusion for $\varphi=-\frac{\pi}{2}+\varepsilon$. Thus, in either case, the lemma is valid.

Now we return to (4.5).

For $\theta_\nu=\pi/2$, it follows that $R_\nu=-\sum_{\lambda,\,\mu\neq\nu}R^\nu_{\lambda\mu}r_\lambda r_\mu\cos\theta_\lambda\cos\theta_\mu\cos(\Omega^\nu_{\lambda\mu}+\theta_\lambda+\theta_\mu)$. we take any one of r_λ's and put the other r_λ's zero. Then R_ν becomes as follows: $R_\nu=-R^\nu_{\lambda\lambda}r^2_\lambda\cos^2\theta_\lambda\cos(\Omega^\nu_{\lambda\lambda}+2\theta_\lambda)$. Now it is evident that the above lemma is valid also when the inequality (L) is replaced by $\cos(\Omega+p\theta_0-q\varphi)\leqq\eta<0$. Consequently there exists $\mathring{\theta}_\lambda$ such that $\cos\mathring{\theta}_\lambda>0$ and $\cos(\Omega^\nu_{\lambda\lambda}+2\mathring{\theta}_\lambda)<0$. Then, if $R^\nu_{\lambda\lambda}\neq0$, it becomes that $R_\nu>0$ for $\theta_\lambda=\mathring{\theta}_\lambda$. This is inconsistent with our assumption. Therefore it must be $R^\nu_{\lambda\lambda}=0$ for $\lambda\neq\nu$. Next we take any two of r_λ's and put the other r_λ's zero. Then R_ν becomes as follows: $R_\nu=-2R^\nu_{\lambda\mu}r_\lambda r_\mu\cos\theta_\lambda\cos\theta_\mu\cos(\Omega^\nu_{\lambda\mu}+\theta_\lambda+\theta_\mu)$ for chosen r_λ and r_μ. Put $\theta_\lambda=\theta_\mu=\theta$. Then, by the same reasonings as above, we see that $R^\nu_{\lambda\mu}=0$. Thus it must be $R^\nu_{\lambda\mu}=0$, for λ, $\mu\neq\nu$ whether $\lambda=\mu$ or not.

Then R_ν becomes as follows:

$$R_\nu=\cos\theta_\nu[R^\nu_{\nu\nu}\cos\theta_\nu\cos\Omega^\nu_{\nu\nu}+2\sum_{\lambda\neq\nu}R^\nu_{\lambda\nu}r_\lambda\cos\theta_\lambda\cos(\Omega^\nu_{\lambda\nu}+\theta_\lambda-\theta_\nu)]\,.$$

Therefore, for θ_ν such that $-\pi/2<\theta_\nu<\pi/2$, from our condition, it must be

$$R^\nu_{\nu\nu}\cos\theta_\nu\cos\Omega^\nu_{\nu\nu}+2\sum_{\lambda\neq\nu}R^\nu_{\lambda\nu}r_\lambda\cos\theta_\lambda\cos(\Omega^\nu_{\lambda\nu}+\theta_\lambda-\theta_\nu)\leqq0\,.$$

If $R^\nu_{\lambda\nu}\neq0$ for certain λ, then we take r_λ corresponding to that λ, and put the other r_λ's zero. Then the above relation becomes as follows:

$$R^\nu_{\nu\nu}\cos\theta_\nu\cos\Omega^\nu_{\nu\nu}+2R^\nu_{\lambda\nu}r_\lambda\cos\theta_\lambda\cos(\Omega^\nu_{\lambda\nu}+\theta_\lambda-\theta_\nu)\leqq 0\,. \tag{4.7}$$

Now, by the lemma, for θ_ν sufficiently near to $\pi/2$ or to $-\pi/2$, there exists $\overset{\circ}{\theta}_\lambda$ such that $\cos\overset{\circ}{\theta}_\lambda>0$ and $\cos(\Omega^\nu_{\lambda\nu}+\overset{\circ}{\theta}_\lambda-\theta_\nu)\geqq\eta>0$. Put $\theta_\nu=\frac{\pi}{2}-\varepsilon$ or $\theta_\nu=-\frac{\pi}{2}+\varepsilon$, and let ε tend to zero. Then, for sufficiently small ε, the left-hand side of (4.7) becomes positive. This is a contradiction. Thus it must be $R^\nu_{\lambda\nu}=0$ for $\lambda\neq\nu$.

Thus, from the above results, it follows that

$$R_\nu=R^\nu_{\nu\nu}\cos\Omega^\nu_{\nu\nu}\cos^2\theta_\nu\,. \tag{4.8}$$

Next, we consider θ_ν sufficiently near to $\pi/2$ or to $-\pi/2$. In this case, we must investigate the terms of higher order. Writing (4.4) minutely, we have:

$$\begin{aligned}
(4.9)\quad d_\nu^2 &= ('x^\nu-\alpha^\nu)('\bar{x}^\nu-\bar{\alpha}^\nu)\\
&=\{(x^\nu-\alpha^\nu)+a^\nu_{\nu\nu}x^\nu x^\nu+a^\nu_{\lambda_1\lambda_2\lambda_3}x^{\lambda_1}x^{\lambda_2}x^{\lambda_3}+\cdots\cdots\\
&\quad +a^\nu_{\lambda_1\ldots\lambda_{N-1}}x^{\lambda_1}\cdots x^{\lambda_{N-1}}+a^\nu_{\lambda_1\ldots\lambda_N}x^{\lambda_1}\cdots x^{\lambda_N}+\cdots\cdots\}\times\\
&\quad\times\{(\bar{x}^\nu-\bar{\alpha}^\nu)+\bar{a}^\nu_{\nu\nu}\bar{x}^\nu\bar{x}^\nu+\bar{a}^\nu_{\lambda_1\lambda_2\lambda_3}\bar{x}^{\lambda_1}\bar{x}^{\lambda_2}\bar{x}^{\lambda_3}+\cdots\cdots\\
&\quad +\bar{a}^\nu_{\lambda_1\ldots\lambda_{N-1}}\bar{x}^{\lambda_1}\cdots\bar{x}^{\lambda_{N-1}}+\bar{a}^\nu_{\lambda_1\ldots\lambda_N}\bar{x}^{\lambda_1}\cdots\bar{x}^{\lambda_N}+\cdots\cdots\}\\
&=(x^\nu-\alpha^\nu)(\bar{x}^\nu-\bar{\alpha}^\nu)+\{a^\nu_{\nu\nu}x^\nu x^\nu(\bar{x}^\nu-\bar{\alpha}^\nu)+\bar{a}^\nu_{\nu\nu}\bar{x}^\nu\bar{x}^\nu(x^\nu-\alpha^\nu)\}\\
&\quad+\{a^\nu_{\lambda_1\lambda_2\lambda_3}x^{\lambda_1}x^{\lambda_2}x^{\lambda_3}(\bar{x}^\nu-\bar{\alpha}^\nu)+a^\nu_{\nu\nu}x^\nu x^\nu\bar{a}^\nu_{\nu\nu}\bar{x}^\nu\bar{x}^\nu\\
&\quad+\bar{a}^\nu_{\lambda_1\lambda_2\lambda_3}\bar{x}^{\lambda_1}\bar{x}^{\lambda_2}\bar{x}^{\lambda_3}(x^\nu-\alpha^\nu)\}+\cdots\cdots\\
&\quad+\{a^\nu_{\lambda_1\ldots\lambda_{N-1}}x^{\lambda_1}\cdots x^{\lambda_{N-1}}(\bar{x}^\nu-\bar{\alpha}^\nu)+\cdots+\bar{a}^\nu_{\lambda_1\ldots\lambda_{N-1}}\bar{x}^{\lambda_1}\cdots\bar{x}^{\lambda_{N-1}}(x^\nu-\alpha^\nu)\}\\
&\quad+\{a^\nu_{\lambda_1\ldots\lambda_N}x^{\lambda_1}\cdots x^{\lambda_N}(\bar{x}^\nu-\bar{\alpha}^\nu)+a^\nu_{\lambda_1\ldots\lambda_{N-1}}\bar{a}^\nu_{\nu\nu}x^{\lambda_1}\cdots x^{\lambda_{N-1}}\bar{x}^\nu\bar{x}^\nu+\cdots\cdots\\
&\quad+\bar{a}^\nu_{\lambda_1\ldots\lambda_{N-1}}a^\nu_{\nu\nu}\bar{x}^{\lambda_1}\cdots\bar{x}^{\lambda_{N-1}}x^\nu x^\nu+\bar{a}^\nu_{\lambda_1\ldots\lambda_N}\bar{x}^{\lambda_1}\cdots\bar{x}^{\lambda_N}(x^\nu-\alpha^\nu)\}+\cdots\cdots.
\end{aligned}$$

Let K_N be the real part of $\frac{1}{2^N r^{N+1}}a^\nu_{\lambda_1\ldots\lambda_N}x^{\lambda_1}\cdots x^{\lambda_N}(\bar{x}^\nu-\bar{\alpha}^\nu)$, then, putting $a^\nu_{\lambda_1\ldots\lambda_N}=R^\nu_{\lambda_1\ldots\lambda_N}e^{i'\overset{\circ}{\Omega}{}^\nu_{\lambda_1\ldots\lambda_N}}$ and $\Omega^\nu_{\lambda_1\ldots\lambda_N}={}'\Omega^\nu_{\lambda_1\ldots\lambda_N}+\omega_{\lambda_1}+\cdots+\omega_{\lambda_N}-\omega_\nu$, we have:

(4.10)

$$\begin{aligned}
K_N&=\sum_{\lambda_1,\cdots,\lambda_N}R^\nu_{\lambda_1\ldots\lambda_N}r_{\lambda_1}\cdots r_{\lambda_N}\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_N}\cos(\Omega^\nu_{\lambda_1\ldots\lambda_N}+\theta_{\lambda_1}+\cdots+\theta_{\lambda_N}-2\theta_\nu)\\
&=\sum_{\lambda_1,\cdots,\lambda_N\neq\nu}R^\nu_{\lambda_1\ldots\lambda_N}r_{\lambda_1}\cdots r_{\lambda_N}\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_N}\cos(\Omega^\nu_{\lambda_1\ldots\lambda_N}+\theta_{\lambda_1}+\cdots+\theta_{\lambda_N}-2\theta_\nu)\\
&\quad+N\cos\theta_\nu\sum_{\lambda_1,\cdots,\lambda_{N-1}\neq\nu}R^\nu_{\nu\lambda_1\ldots\lambda_{N-1}}r_{\lambda_1}\cdots r_{\lambda_{N-1}}\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_{N-1}}\cos(\Omega^\nu_{\nu\lambda_1\ldots\lambda_{N-1}}\\
&\quad+\theta_{\lambda_1}+\cdots+\theta_{\lambda_{N-1}}-\theta_\nu)+\binom{N}{2}\cos^2\theta_\nu\sum_{\lambda_1,\cdots,\lambda_{N-2}}R^\nu_{\nu\nu\lambda_1\ldots\lambda_{N-2}}r_{\lambda_1}\cdots r_{\lambda_{N-2}}\cos\theta_{\lambda_1}\cdots\\
&\quad\cdots\cos\theta_{\lambda_{N-2}}\cos(\Omega^\nu_{\nu\nu\lambda_1\ldots\lambda_{N-2}}+\theta_{\lambda_1}+\cdots\cdots+\theta_{\lambda_{N-2}})\,.
\end{aligned}$$

We denote the general term in the first and second sum of the right-hand side of the above expression by S_1 and S_2 respectively. Put $\theta_{\lambda_1}=\theta_{\lambda_2}=\cdots=\theta_{\lambda_N}=\theta$, then, since $N\geqq 3$, by the lemma, we see that there exists θ_0 such that $\cos\theta>0$ and $\cos(\Omega^{\nu}_{\lambda_1\ldots\lambda_N}+N\theta_0-2\theta_\nu)>0$, $\cos(\Omega^{\nu}_{\nu\lambda_1\ldots\lambda_{N-1}}+(N-1)\theta_0-\theta_\nu)>0$, for θ_ν sufficiently near to $\frac{\pi}{2}$ or to $-\frac{\pi}{2}$. Thus it is seen that, for θ_ν sufficiently near to $\frac{\pi}{2}$ or to $-\frac{\pi}{2}$, there exists a set of $(\theta_{\lambda_1}, \theta_{\lambda_2}, \ldots, \theta_{\lambda_N})$ such that $\cos\theta_{\lambda_1}, \ldots, \cos\theta_{\lambda_N}>0$ and $S>0$ and a set of $(\theta_{\lambda_1}, \theta_{\lambda_2}\ldots, \theta_{\lambda_{N-1}})$ such that $\cos\theta_{\lambda_1}, \ldots, \cos\theta_{\lambda_{N-1}}>0$ and $S_2>0$.

We assume that $R^{\nu}_{\lambda_1\lambda_2\lambda_3}, \ldots, R^{\nu}_{\lambda_1\lambda_2\ldots\lambda_{N-1}}=0$ for $\lambda_1\neq\nu, \ldots, \lambda_{N-1}\neq\nu$. Then from (4.9), for $\theta_\nu=\frac{\pi}{2}$ or $-\frac{\pi}{2}$, we have:

$$d_\nu^2=r^{\cdot}+2^{N+1}r^{N+1}K_N+[r]_{N+2},$$

where $K_N=\sum S_1$. We take arbitrarily $r_{\lambda_1}, \ldots, r_{\lambda_N}$ and put the other r_λ's zero. Let the distinct ones of $r_{\lambda_1}, \ldots, r_{\lambda_N}$ be $\rho_1, \ldots, \rho_p$. Put

$$\begin{cases} \rho_1=\sigma, \\ \rho_2=\sigma^{N+1}, \\ \cdots\cdots\cdots \\ \rho_p=\sigma^{(N+1)^{p-1}}, \end{cases}$$

where $0\leqq\sigma\leqq 1$. If we write $K_N=\sum S_1$ as follows:

$$\sum_{m_1+\cdots+m_p=N} R_{m_1\cdots m_p}(\theta)\rho_1^{m_1}\cdots\rho_p^{m_p},$$

then it follows that

$$K_N=\sum S_1=\sum_{m_1+\cdots+m_p=N} R_{m_1,\cdots m_p}(\theta)\sigma^{m_1+m_2(N+1)+\cdots+m_p(N+1)^{p-1}},$$

and, here the terms of the left-hand side correspond in one-to-one to those of the right-hand side. Now, from our assumption, $K_N\leqq 0$. Then, taking sifficiently small σ, by means of the preceding results on S_1, we see that it must be $R^{\nu}_{\lambda_1\ldots\lambda_N}=0$ for $\lambda\neq\nu, \ldots, \lambda_N\neq\nu$. Thus, by induction, we see that, foı any $N\geqq 3$, $a^{\nu}_{\lambda_1\ldots\lambda_N}=0$ where $\lambda_1\neq\nu, \ldots, \lambda_N\neq\nu$.

Next, we assume that $R^{\nu}_{\nu\lambda_1\lambda_2}, \ldots, R^{\nu}_{\nu\lambda_1\ldots\lambda_{N-2}}=0$ for $\lambda_1\neq\nu, \ldots, \lambda_{N-2}\neq\nu$. Then, from (4.9), we have:

$$\begin{aligned} d_\nu^2 &= r^2+r^3\cos^2\theta_\nu[8R^{\nu}_{\nu\nu}\cos\Omega^{\nu}_{\nu\nu}+r(\cdots)+\cdots+r^{N-3}(\cdots)] \\ &\quad +r^{N+1}\cos\theta_\nu[2^{N+1}K'_N+\cos\theta_\nu(\cdots)] \\ &\quad +\cos\theta_\nu\cdot[r]_{N+2}, \end{aligned}$$

where $K'_N=K_N/\cos\theta_\nu=N\sum S_2+\cos\theta_\nu(\cdots)$. Put $\theta_\nu=-\frac{\pi}{2}+\varepsilon$ or $\frac{\pi}{2}-\varepsilon$, where

ε is an arbitrary small positive number, then, for sufficiently small r and ε, it follows that

$$d_\nu^2 \doteqdot r^2 + r^3\varepsilon^2 \cdot 8R^\nu_{\nu\nu} \cos \Omega^\nu_{\nu\nu} + r^{N+1}\varepsilon \cdot 2^{N+1} N \sum S_2 .$$

Then, by our assumption, $\sum S_2 \leqq 0$, therefore, by similar reasonings as above, it must be $R^\nu_{\nu\lambda_1\ldots\lambda_{N-1}}=0$ for $\lambda_1 \neq \nu, \ldots, \lambda_{N-1}\neq\nu$. Then, by induction, we see that, for any $N\geqq 3$, $a^\nu_{\nu\lambda_1\ldots\lambda_{N-1}}=0$ for $\lambda_1\neq\nu, \ldots, \lambda_{N-1}\neq\nu$.

Summarizing the results, it is seen that, *in order that there may exist a circular cylindrical domain, of which all the points of the boundary are transformed to the inner points except for the origin, it is necessary that the transformation T is of the form as follows:*

$$T: \quad 'x^\nu = \varphi^\nu(x) = x^\nu + (x^\nu)^2[a^\nu + a^\nu_\lambda x^\lambda + a^\nu_{\lambda_1\lambda_2} x^{\lambda_1} x^{\lambda_2} + \cdots] . \tag{4.11}$$

Next we consider the converse. In the x^ν-plane, we take a point x^ν arbitrarily on the boundary or in the interior of $\overset{\circ}{C}_\nu$. Then there exists a circle C_ν passing through the point x^ν. For such x^ν, from (4.9), the distance d_ν from the center $\alpha^\nu = rr_\nu e^{i\omega_\nu}$ to the transform $'x^\nu$ of x^ν becomes as follows:

$$d_\nu^2 = r^2 r_\nu^2 + r^3 r_\nu^3 \cos^3 \theta_\nu (8R^\nu_{\nu\nu} \cos \Omega^\nu_{\nu\nu} + [r]_1) . \tag{4.12}$$

If $R^\nu_{\nu\nu}\neq 0$, then, for $'\Omega^\nu_{\nu\nu}$ such that $\cos('\Omega^\nu_{\nu\nu}+\omega_\nu)<0$, there exists r_0 such that, for $r\leqq r_0$, $d_\nu < rr_\nu$ except for $\theta_\nu = \frac{\pi}{2}$ or $-\frac{\pi}{2}$, namely, in the x^ν-plane, there exists a circle C_ν, of which all the points of the periphery are transformed by T to the inner point of that circle except for the origin.

Thus, summarizing the results, by means of the same reasonings as in the out-set of §1, we have

Theorem 3. *In order that there may exist a circular cylindrical domain D, of which all the points of the boundary are transformed by T to its inner points except for the origin, it is necessary that the transformation T is of the form as follows:*

$$T: \quad 'x^\nu = \varphi^\nu(x) = x^\nu + (x^\nu)^2[a^\nu + a^\nu_\lambda x^\lambda + a^\nu_{\lambda_1\lambda_2} x^{\lambda_1} x^{\lambda_2} + \cdots\cdots] .$$

Conversely, when the transformation T is of the above form and $a^\nu\neq 0$ for all ν, then there exists a cylindrical domain of the above characters, and moreover, in this case, in each x^ν-plane, there exists a circle such that all the points of that circle converge to the origin always remaining in it when T is infinitely iterated on these points.

The transformation in the above theorem resembles closely the transformation constructed by direct summation of those of one variable. From this aspect, it is readily seen that, for this transformation, Fatou's method[1] is applicable and then the same conclusion is deduced.

§5. Condition II for the transformation of type A_2.

In this paragraph, we consider the transformation of type A_2, which is written as follows:

$$(5.1)\qquad T:\quad 'x^\nu = \varphi^\nu(x) = x^\nu + \delta_{\nu-1}x^{\nu-1} + a^\nu_{\lambda\mu}x^\nu x^\mu + \cdots\cdots,$$

where $\delta_{\nu-1}=0$ or 1. For this transformation, we consider the distance d_ν from the center $\alpha^\nu = re^{i\omega_\nu}$ of $\mathring{C}_\nu$ to the transform $'x^\nu$ of x^ν on $\mathring{C}_\nu$. Then d^2_ν is calculated as follows:

$$\begin{aligned}(5.2)\qquad d^2_\nu &= ('x^\nu - \alpha^\nu)('\bar{x}^\nu - \bar{\alpha}^\nu)\\ &= \{(x^\nu-\alpha^\nu)+\delta_{\nu-1}x^{\nu-1}+a^\nu_{\lambda\mu}x^\lambda x^\mu+\}\{(\bar{x}^\nu-\bar{\alpha}^\nu)+\delta_{\nu-1}\bar{x}^{\nu-1}+\bar{a}^\nu_{\lambda\mu}\bar{x}^\lambda\bar{x}^\mu+\cdots\cdots\}\\ &= r^2+\{\delta_{\nu-1}x^{\nu-1}(\bar{x}^\nu-\bar{\alpha}^\nu)+\delta_{\nu-1}\bar{x}^{\nu-1}(x^\nu-\alpha^\nu)+\delta^2_{\nu-1}x^{\nu-1}\bar{x}^{\nu-1}\}+\cdots\cdots\\ &= r^2+4r^2R_\nu+[r]_3\,.\end{aligned}$$

Now, by means of (4.2), we have:

$$(5.3)\qquad R_\nu = \delta_{\nu-1}r_{\nu-1}\cos\theta_{\nu-1}\cos\{(\omega_{\nu-1}-\omega_\nu)+\theta_{\nu-1}-2\theta_\nu\} + \delta^2_{\nu-1}r^2_{\nu-1}\cos^2\theta_{\nu-1}\,.$$

For ν such that $\delta_{\nu-1}=1$, we vary θ_ν from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Then the first term of the right-hand side of (5.3) attains the maximum value $r_{\nu-1}\cos\theta_{\nu-1}\geqq 0$. Then, from (5.3), for this value of θ_ν, $R_\nu>0$ for $r_{\nu-1}\neq 0$, $\cos\theta_{\nu-1}\neq 0$.

Thus we see that, *for the transformation of type A_2, there does not exist the circular cylindrical domain such that all the points of the boundary of the domain are transformed to its inner points except for the origin.*

(To be continued in our next)

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1) M. P. Fatou, ibid.