LATTICE THEORETIC CHARACTERIZATION OF GEOMETRIES SATISFYING "AXIOME DER VERKNÜPFUNG"

By

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In my previous paper [1],[1] I have characterized lattice-theoretically an affine geometry of arbitrary dimensions,[2] i.e. a geometry satisfying the Euclidean axiom of parallel lines and "Axiome der Verknüpfung" of D. Hilbert [1],[3] except for the restrictions on the dimensionality.

The purpose of this paper is to characterize lattice-theoretically a geometry satisfying "Axiome der Verknüpfung" alone. The main theorem is as follows:

THEOREM. *An abstract lattice L is isomorphic to the lattice of all subspaces of a space satisfying "Axiome der Verknüpfung" of D. Hilbert* [1], *except for the restrictions on the dimensionality, if and only if L is a strongly plane matroid lattice.*[4]

1. We shall begin by showing several preliminary lemmas.

DEFINITION 1. Let A be a set of points such that for any pair of distinct points p, q there exists a subset $p \vee q$ (called *line*), containing p, q and for any triple of points p, q, r, which are not on a line, there is a subset $p \vee q \vee r$ (called *plane*) containing p, q, r, which satisfy the following conditions:

A. 1. *Two distinct points on a line determine the line.*

A. 2. *Three non-collinear points on a plane determine the plane.*

A. 2′. *The line through two distinct points on a plane is contained in the plane.*

By a *subspace* of A, we mean a subset S such that if p, q are distinct points of S, then $p \vee q \subseteqq S$ and if p, q, r are non-collinear points of S, then

1) The numbers in square brackets refer to the list of the references at the end of the paper.

2) Cf. U. Uasaki [1], Definition 2.

3) Cf. Ibid. 3 and 20.

4) Cf. Definition 3, below.

$p \vee q \vee r \subseteqq S$. If there exist four points p_1, p_2, p_3, p_4 which are not on a plane, the least subspace containing these points is called a *3-space* and is denoted by $p_1 \vee p_2 \vee p_3 \vee p_4$.

A. 4. *If two planes contained in a 3-space have a point in common, then they have at least one more point in common.*

REMARK 1. These conditions A. 1-A. 4 are equivalent to "Axiome der Verknüpfung" I_1-I_8 of D. Hilbert [1], provided that A is especially 3-dimensional, i. e. A is equal to the 3-space $p_1 \vee p_2 \vee p_3 \vee p_4$ for some points p_1, p_2, p_3, p_4 of A. Thus we may consider that our space A is a space satisfying "Axiome der Verknüpfung" of D. Hilbert [1] from which the restrictions on the dimensionality are omitted.

The independence and consistency of these conditions may be easily shown by simple examples.[5]

DEFINITION 2. Let us define:

$$\begin{aligned} p \vee p &= p, \\ p \vee q \vee r &= p \vee q \quad \text{if} \quad r \in p \vee q, \quad \text{and} \\ p \vee q \vee r \vee s &= p \vee q \vee r \quad \text{if} \quad s \in p \vee q \vee r. \end{aligned}$$

S, T be subsets of A. If $S, T \neq 0$, void set, then we shall define $S \oplus T$ to be the set union

$$\bigcup (s_1 \vee s_2 \vee t_1 \vee t_2 ; \ s_1, s_2 \in S, \ t_1, t_2 \in T).$$

When $S=0$, we shall define $S \oplus T = T \oplus S = T$.

REMARK 2. It follows at once from Definition 2:

(1) $S \oplus T = T \oplus S$

(2) $S' \subseteqq S$, $T' \subseteqq T$ imply $S' \oplus T' \subseteqq S \oplus T$.

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5) The following examples i - iv show the independence of A. 1-A. 4, respectively;

(i) Let A be the set of points p, q, r, and let $p \vee q = q \vee r = A$, and $p \vee r = \{p, r\}$.

(ii) Let A be the set of points p, q, r, s, and $x \vee y = \{x, y\}$ for any $x, y \in A$, and let

$$x \vee y \vee z = \begin{cases} \{p, q, r\}, & \text{if } \{x, y, z\} = \{p, q, r\}, \\ A, & \text{otherwise.} \end{cases}$$

(iii) Let A be the set of points p, q, r, s, and

$$x \vee y = \begin{cases} \{q, r, s\}, & \text{if } \{x, y\} \subseteqq \{q, r, s\}, \\ \{x, y\}, & \text{otherwise.} \end{cases}$$

and let $x \vee y \vee z = \{x, y, z\}$, for non-collinear points x, y, z.

(iv) Let A be the set $\{o, p, q, r, s\}$ and $x \vee y = \{x, y\}$ for any $x, y \in A$, and let for any distinct points $x, y, z \ \in A$),

$$x \vee y \vee z = \begin{cases} \{p, q, r, s\}, & \text{if } \{x, y, z\} \subseteqq \{p, q, r, s\}, \\ \{x, y, z\}, & \text{obtherwise.} \end{cases}$$

The consistency of the conditions is secured by the existence of the affine space of three dimensions over the field of real numbers.

LEMMA 1. *Let p_1, p_2, p_3, p_4 be four points of A, then*

$$p_1 \oplus (p_2 \vee p_3 \vee p_4) = p_1 \vee p_2 \vee p_3 \vee p_4 .$$

PROOF. As it follows immediately from Definition 2 that:

$$p_1 \oplus (p_2 \vee p_3 \vee p_4) \subseteqq p_1 \vee p_2 \vee p_3 \vee p_4 ,$$

it is sufficient to prove the converse inequality.

For this purpose, we shall show:

(*) $p \in p_1 \vee p_2 \vee p_3 \vee p_4$ implies $p \in p_1 \vee p_2 \vee q$ for some point $q \in p_2 \vee p_3 \vee p_4$.

We can assume without loss of generality that p_1, p_2, p_3, p_4 are not on a plane and $p \notin p_1 \vee p_2$. Since the two planes $p \vee p_1 \vee p_2$, $p_2 \vee p_3 \vee p_4$ have the point p_2 in common, there exists a point $q\,(\neq p_2)$ such that

$$q \in p \vee p_1 \vee p_2 , \quad \text{and} \quad q \in p_2 \vee p_3 \vee p_4 .$$

It follows from A. 2, $p \in q \vee p_1 \vee p_2$, completing the proof of (*). While (*) shows $p_1 \vee p_2 \vee p_3 \vee p_4 \subseteqq p_1 \oplus (p_2 \vee p_3 \vee p_4)$, which is desired.

This lemma shows that $p_{i_1} \oplus (p_{i_2} \vee p_{i_3} \vee p_{i_4}) = p_1 \oplus (p_2 \vee p_3 \vee p_4)$, for any permutation i_1, i_2, i_3, i_4 of 1, 2, 3, 4.

LEMMA 2. *If S is a subspace containing a point s_0, and if p is a point of A, then $p \oplus S = \bigvee (p \vee s_0 \vee s\,;\ s \in S)$.*

PROOF. Let q be a point of $p \oplus S$, then $q \in p \vee r_1 \vee r_2$, for some points $r_1, r_2 \in S$, whence clearly $q \in p \vee r_1 \vee r_2 \vee s_0$. It follows from the proposition (*) that $q \in p \vee s_0 \vee s$ for some $s \in r_1 \vee r_2 \vee s_0$, whence $s \in S$, in view of the fact that S is a subspace of A. Consequently we have $q \oplus S \subseteqq \bigvee (p \vee s_0 \vee s\,;\ s \in S)$. The converse inequality is obvious from Definition 2.

LEMMA 3. *Let p_1, p_2, p_3, p_4, p_5 be points of A, then*

$$p_1 \oplus (p_2 \vee p_3 \vee p_4 \vee p_5) = p_2 \oplus (p_1 \vee p_3 \vee p_4 \vee p_5) .$$

PROOF. Let q be any point of $p_1 \oplus (p_2 \vee p_3 \vee p_4 \vee p_5)$, then we have by Lemma 2, $q \in p_1 \vee p_2 \vee r$, for some point $r \in p_2 \vee p_3 \vee p_4 \vee p_5$, whence by (*) $r \in p_2 \vee p_3 \vee s$, for some point $s \in p_3 \vee p_4 \vee p_5$.

It follows from Definition 2:

$$\begin{aligned} q \in\ & p_1 \oplus (p_2 \vee p_3 \vee s) \\ =\ & p_2 \oplus (p_1 \vee p_3 \vee s) \qquad \text{(by Lemma 1)} \\ \subseteqq\ & p_2 \oplus (p_1 \vee p_3 \vee p_4 \vee p_5) \quad \text{(by Remark 2)} \end{aligned}$$

Thus it holds $p_1 \oplus (p_2 \vee p_3 \vee p_4 \vee p_5) \subseteqq p_2 \oplus (p_1 \vee p_3 \vee p_4 \vee p_5)$.

By a similar way, we have the converse inequality, completing the proof.

This result shows that $p_{i_1}\oplus(p_{i_2}\vee p_{i_3}\vee p_{i_4}\vee p_{i_5})=p_1\oplus(p_2\vee p_3\vee p_4\vee p_5)$, for any permutation i_1, i_2, i_3, i_4, i_5 of 1, 2, 3, 4, 5.

LEMMA 4. *Let p_1, p_2, p_3, p_4, p_5 be points of A. Then*

$$p_1\oplus(p_2\vee p_3\vee p_4\vee p_5)=(p_1\vee p_2)\oplus(p_3\vee p_4\vee p_5).$$

PROOF. Let q be a point of $p_1\oplus(p_2\vee p_3\vee p_4\vee p_5)$, then by a similar way as Lemma 3, there exists a point s with $q\in p_1\vee p_2\vee p_3\vee s$, and $s\in p_3\vee p_4\vee p_5$, whence $q\in(p_1\vee p_2)\oplus(p_3\vee p_4\vee p_5)$, by Definition 2. It follows:

$$p_1\oplus(p_2\vee p_3\vee p_4\vee p_5)\subseteqq(p_1\vee p_2)\oplus(p_3\vee p_4\vee p_5).$$

By a similar way, we obtain the converse inequality, completing the proof.

LEMMA 5. *Let S be a subspace containing a point s_0, and let T be any subset of A. Then*

$$S\oplus T=\bigcup(s_0\vee s\vee t_1\vee t_2;\ s\in S,\ t_1, t_2\in T).$$

PROOF. Let p be any point of $S\oplus T$, then we have:

$$\begin{aligned} p &\in s_1\vee s_2\vee t_1\vee t_2, \quad \text{where } s_1, s_2\in S;\ t_1, t_2\in T, \\ &\subseteqq(t_1\vee t_2)\oplus(s_0\vee s_1\vee s_2) \quad \text{(by Definition 2)} \\ &= t_1\oplus(t_2\vee s_0\vee s_1\vee s_2) \quad \text{(by Lemma 4)}. \end{aligned}$$

It follows from Lemma 2 that $p\in t_1\vee t_2\vee r$, for some point $r\in t_2\vee s_0\vee s_1\vee s_2$, whence $r\in t_2\vee s_0\vee s$, for some point $s\in s_0\vee s_1\vee s_2$. Consequently $p\in t_1\vee t_2\vee s_0\vee s$, where $s\in S$, since S is a subspace. Thus we have:

$$S\oplus T\subseteqq\bigcup(s_0\vee s\vee t_1\vee t_2;\ s\in S,\ t_1, t_2\in T).$$

This completes the proof, since the converse inequality is trivial.

LEMMA 6. *Let S be a subspace, and T be any subset of A, and let p be a point. Then $(p\oplus S)\oplus T\subseteqq p\oplus(S\oplus T)$.*

PROOF. Let q be any point of $(p\oplus S)\oplus T$, then we have: $q\in r_1\vee r_2\vee t_1\vee t_2$, where $t_1, t_2\in T$, and $r_1, r_2\in p\oplus S$, whence $r_i\in p\vee s_0\vee s_i$, for some points $s_i\in S(i=1, 2)$ by Lemma 2, s_0 being a point of S.[6] It follows: $q\in(p\vee s_0\vee s_1\vee s_2)\oplus(t_1\vee t_2)$, whence by Lemma 5, $q\in p\vee r\vee t_1\vee t_2$, for some point $r\in p\vee s_0\vee s_1\vee s_2$. Hence by Lemma 1, there exists a point s with $r\in p\vee s_0\vee s$, and $s\in s_0\vee s_1\vee s_2\subseteqq S$. It follows:

6) We may assume the existence of the point s_0, since otherwise the result is trivial.

$$q \in (p \vee s_0 \vee s) \oplus (t_1 \vee t_2)$$
$$= p \oplus (s_0 \vee s \vee t_1 \vee t_2) \quad \text{(by Lemma 3 and 4)}$$
$$\subseteqq p \oplus (S \oplus T).$$

Hence we have the result.

LEMMA 7. *A subset S of A is a subspace if and only if $S \oplus S = S$.*

PROOF. By Definition 1, S is a subspace if and only if $p, q, r \in S$ imply $p \vee q \vee r \subseteqq S$. This is equivalent to $S \oplus S \subseteqq S$ by Definition 2 and Lemma 5. And it is also equivalent to $S \oplus S = S$, since, $S \subseteqq S \oplus S$ in general.

LEMMA 8. *Let S be a subspace and p be a point of A, then $p \oplus S$ is the least subspace containing p and S.*

PROOF. Repeated applications of Lemma 6 and 7 yield:

$$(p \oplus S) \oplus (p \oplus S) \subseteqq p \oplus (p \oplus S).$$

Since it is easily seen from Definition 2 and Lemma 2 that $p \oplus (p \oplus S) = p \oplus S$, we obtain $(p \oplus S) \oplus (p \oplus S) \subseteqq p \oplus S$, whence $(p \oplus S) \oplus (p \oplus S) = p \oplus S$, because the converse inequality is obvious. Hence $p \oplus S$ is a subspace of A by Lemma 7.

Clearly p and S are contained in $p \oplus S$, while any subspace containing p and S contains $p \oplus S$, whence it is the least subspace containing p and S.

2. In this section, we shall characterize the lattice of all subspaces of the space A in Definition 1.

DEFINITION 3. If in a lattice L with 0, $a < b$ implies $a < a \smile p \leqq b$ for some point p, then L is called *relatively atomic.*

Let $\{a_\delta ; \delta \in D\}$ be a directed set of elements in a complete lattice L. If $a_\delta \uparrow a$ implies $a_\delta \frown b \uparrow a \frown b$ for any element b, then L is called *upper continuous.*

A lattice is called *semi-modular* if it satisfies:

(ξ') *If a and b cover c, and $a \neq b$, then $a \smile b$ covers a and b.*

By a *matroid lattice* it is meant a lattice which is relatively atomic, upper continuous, and semi-modular.

A lattice L with 0 called *strongly plane* if for any points p, q, r, and any element a such that $p \leqq q \smile a$, $r \leqq a$, there exists a point s with $p \leqq q \smile r \smile s$ and $s \leqq a$.

REMARK 3. A relatively atomic, upper continuous lattice is semi-modular if and only if it satisfies:

(η') *If p, q are points, and $a < a \smile p \leqq a \smile q$, then $a \smile p = a \smile q$.*[7]

THEOREM 1. *An abstract lattice $\mathfrak{A}$ is isomorphic to the lattice of all subspaces of a space in Difinition 1, if and only if $\mathfrak{A}$ is a strongly plane matroid lattice.*

PROOF. (i) Let $\mathfrak{A}$ be the set of all subspaces of A. Then it is easily shown that $\mathfrak{A}$ is a relatively atomic, upper continuous lattice ordered by set-inclusion.[8] It follows immediately from Lemma 8 and Lemma 2 that $\mathfrak{A}$ is strongly plane. Next let $a < a \smile p \leqq a \smile q$, p, q being points. Then $p \leqq q \smile a$ and $p \frown a = 0$. We may assume without loss of generality that there is a point r with $r \leqq a$. Since $\mathfrak{A}$ is strongly plane, there exists a point $s \leqq a$ such that $p \leqq q \smile r \smile s$, whence $q \leqq p \smile r \smile s$ by A. 2.[9] It follows $q \leqq p \smile a$, and so $q \smile a \leqq p \smile a$. Hence we have $a \smile p = a \smile q$. Consequently $\mathfrak{A}$ is semi-modular in view of Remark 3. Thus $\mathfrak{A}$ is a strongly plane matroid lattice.

(ii) Conversely let $\mathfrak{A}$ be a strongly plane matroid lattice, and let $A(\mathfrak{A})$ be the set of points of $\mathfrak{A}$. We shall define lines and planes in $A(\mathfrak{A})$ as follows:

$$p \vee q = \{s;\ s \leqq p \smile q,\ s \in A(\mathfrak{A})\},$$
$$p \vee q \vee r = \{s;\ s \leqq p \smile q \smile r,\ s \in A(\mathfrak{A})\},$$

Then it is easily shown that $A(\mathfrak{A})$ satisfies A. 1, 2, 2$'$, and 4 in Definition 1, and that $\mathfrak{A}$ is isomorphic to the lattice of all subspaces of the space $A(\mathfrak{A})$.[10]

3. Now we shall give a remark to the characterization of an affine geometry.

In a space A in Definition 1, two lines $p \vee q$, $r \vee s$ are called *parallel* to each other and denoted by $p \vee q \parallel r \vee s$ provided that they are contained in the same plane and have no point in common.

If especially A satisfies the following condition:

A. 3. *If p, q, r are non-collinear points, then there exists one and only one line $p \vee s$ with $p \vee s \parallel q \vee r$,*

then A. 2$'$ is redundant,[11] and A is an affine space. Cf. U. Sasaki, [1] Definition 2.

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7) Cf. F. Maeda [2] 180 Theorem 2.
8) Cf. F. Maeda [1] 93 Theorem 2. 1.
9) Obviously $p \smile r \smile s = p \vee r \vee s$, since $p \vee r \vee s$ is the least subspace containing p, r and s.
10) Cf. U. Sasaki [1] Theorem 2. 2.
11) Cf. U. Sasaki [1] Lemma 1. 1.

Hence we obtain the following theorem in view of Theorem 1.

THEOREM 2. *An abstract lattice L is isomorphic to the lattice of all subspaces of an affine space of arbitrary dimensions, if and only if L is a strongly plane matroid lattice satisfying the following*:

(α) *Let p, q, r be independent points of L, then there exists one and only one element l such that*

$$p < l < p \smile q \smile r \quad \text{and} \quad l \frown (q \smile r) = 0 .$$

While, in view of U. Sasaki [2] Theorem 2, a relatively atomic, upper continuous lattice is semi-modular and strongly plane if and only if it is *semi-modular in the sense of Wilcox*, i.e.

(A) $(b, c)M$, $b \frown c = 0$ *imply* $(c, b)M$, *and*

(B) $b \frown c \neq 0$ *implies* $(b, c)M$.[12]

Thus Theorem 2 may be stated as follows:

An abstract lattice L is isomorphic to the lattice of all subspaces of an affine space of arbitrary dimensions, if and only if L is a relatively atomic, upper continuous lattice which is semi-modular in the sense of Wilcox and satisfies the condition (α) cited above.

It was the main theorem of U. Sasaki [1], obtained by making use of the result of Wilcox [1].

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12) By (b, c) M, we mean that $a \leq c$ implies $(a \smile b) \frown c = a \smile (b \frown c)$.