CANONICAL SUBDIRECT FACTORIZATIONS OF LATTICES

By

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G. Birkhoff has proved the following factorization theorem[1]:

Every algebra A with finitary operations can be represented as a subdirect union of subdirectly irreducible algebras. And the subdirect factorizations are closely related to the structure of the lattice[2] $\Theta(A)$ of all congruence relations on A. In particular, for a lattice L $\Theta(L)$ is pseudo-complemented. Using this fact F. Maeda [1] has introduced the canonical subdirect factorization of lattices in which the unique factorization theorem is proved.

In this paper, I show that a lattice L has the canonical subdirect factorization with subdirectly irreducible factors if and only if $\Theta(L)$ is an atomic lattice. And also I give a solution for the Birkhoff's problem 72,[3] that is, a necessary and sufficient condition such that $\Theta(L)$ is a Boolean algebra is that L has a subdirect factorization with simple factors such that the components of arbitrary two elements of L are identical except a finite number of components. I wish to thank Mr. J. Hashimoto for giving me an important suggestion to the last result. I also wish to thank Prof. F. Maeda for his kind guidance.

1. Let L be an arbitrary lattice and let $\Theta(L)$ denote the set of all congruence relations on L, then $\Theta(L)$ forms a complete lattice by defining $\theta \leqq \phi$ if and only if $a \equiv b(\theta)$ implies $a \equiv b(\phi)$. The following properties are well known:

(1) $\Theta(L)$ is an upper continuous, distributive lattice.[4]

(2) In $\Theta(L)$, every element is a meet of completely meet-irreducible elements.[5]

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1) Cf. G. Birkhoff [1] 765, [2] 92. The number in square brackets refer to the list at the end of this paper.

2) G. Birkhoff [1, p. 764] has called it the *structure lattice* of A.

3) Cf. G. Birkhoff [2] 153.

4) Cf. G. Birkhoff [2] 24. A complete lattice L is called *upper continuous* when $a_\delta \uparrow a$ implies $a_\delta \frown b \uparrow a \frown b$. When L is distributive, this is equivalent to the infinite distributive law $\bigvee(a; a \in S) \frown b = \bigvee(a \frown b; a \in S)$ for all $S \subset L$ and $b \in L$. We use 0 and 1 for the zero element and the unit element of $\Theta(L)$ respectively.

5) Cf. G. Birkhoff and O. Frink [1] 304, Theorem 7. In any complete lattice an element a which can not be a meet of elements properly containing a is called *completely meet-irreducible.*

For any element θ of $\Theta(L)$, there exists the greatest element θ^* of the set $\{\phi;\ \phi\frown\theta=0\}$ by (1) and it is called the *pseudo-complement* of θ. The correspondence $\theta\rightarrow\theta^{**}$ is a closure operation in $\Theta(L)$ and the *closed* elements such that $\theta=\theta^{**}$ form a complete Bcolean algebra $\Theta_*(L)$ in which the join is given by the new operation $\theta\vee\phi=(\theta\smile\phi)^{**}$, while the meet operation is the same as in $\Theta(L)$. And the set $\Delta(L)$ of all *dense* elements δ satisfying $\delta^*=0$ is a dual ideal of $\Theta(L)$.[6] Moreover, let $\Theta_z(L)$ be the center of $\Theta(L)$, then it is the set of all elements θ such that $\theta\smile\theta^*=1$ since $\Theta(L)$ is distributive and pseudo-complemented, and we have the following relations:

$$(3)\qquad \Theta(L)\supset\Theta_*(L)\supset\Theta_z(L)$$

where $\Theta_z(L)$ is a sublattice of $\Theta(L)$.[7] Also we have the equality:

$$(4)\qquad (\bigvee(\theta_\alpha;\ \alpha\in I))^*=\bigwedge(\theta_\alpha^*;\ \alpha\in I).^{8)}$$

We use frequently the fact that in any complete lattice the set $\{\theta_\alpha;\ \theta_\alpha>\theta\}$ has the least element θ_0 if and only if θ is completely meet-irreducible.[9] The unit element 1 of $\Theta(L)$ is completely meet-irreducible, but in what follows we consider only the completely meet-irreducible elements which are not 1.

A completely meet-irreducible element is necessarily meet-irreducible. We shall deduce first a property on the meet-irreducible elements of $\Theta(L)$.

LEMMA 1. *The meet-irreducible elements of a Boolean algebra are maximal.*

PROOF. If an element a is not maximal there exists an element b such that $a<b<1$. Since a Boolean algebra is relatively complemented, b has a relative complement b' in the interval $[a, 1]$ and then a is not meet-reducible.

LEMMA 2. *In $\Theta(L)$, a meet-irreducible element is maximal in $\Theta_*(L)$ or dense.*

PROOF. By the distributive law we see that

$$\theta^{**}\frown(\theta\smile\theta^*)=(\theta^{**}\frown\theta)\smile(\theta^{**}\frown\theta^*)=\theta,$$

6) Cf. G. Birkhoff [2] 147-149.

7) Cf. F. Maeda [1] 100.

8) Proof: We have $\bigvee(\theta_\alpha;\alpha\in I)\frown\bigwedge(\theta_\alpha^*;\alpha\in I)=\bigvee(\theta_\alpha\frown\bigwedge(\theta_\alpha^*;\alpha\in I);\alpha\in I)=0$, then $\bigwedge(\theta_\alpha^*;\alpha\in I)\leqq(\bigvee(\theta_\alpha;\alpha\in I))^*$. On the other hand, $\theta_\alpha\leqq\bigvee(\theta_\alpha;\alpha\in I)$ implies $\theta_\alpha^*\geqq(\bigvee(\theta_\alpha;\alpha\in I))^*$ and then $\bigvee(\theta_\alpha^*;\alpha\in I)\geqq(\bigvee(\theta_\alpha;\alpha\in I))^*$.

9) Cf. G. Birkhoff and O. Frink [1] 303, Lemma 6.

and if θ is meet-irreducible we have $\theta^{**}=\theta$ or $\theta\smile\theta^{*}=\theta$, that is, θ is closed or dense since $\theta\smile\theta^{*}=\theta$ implies $\theta^{*}=(\theta\smile\theta^{*})^{*}=\theta^{*}\frown\theta^{**}=0$. And when θ is closed it is meet-irreducible in $\Theta_{*}(L)$, because the meet operation in $\Theta_{*}(L)$ is same as in $\Theta(L)$. Then θ is maximal element of $\Theta_{*}(L)$ by lemma 1.

For the proof of theorem 2 the following lemma is required.

LEMMA 3. *$\Theta(L)$ is a Boolean algebra if and only if $\Delta(L)$ contains only 1.*

PROOF. If $\Delta(L)$ contains only 1, since $\theta\smile\theta^{*}$ is dense by (4) we have $\theta\smile\theta^{*}=1$, then we see that $\Theta(L)=\Theta_{z}(L)$ that is $\Theta(L)$ is a Boolean algebra. Conversely if $\Theta(L)$ is a Boolean algebra, for arbitrary dense element δ, (since $\Theta(L)=\Theta_{z}(L)$) we have $\delta\smile\delta^{*}=1$ which implies $\delta=1$ since $\delta^{*}=0$.

2. By a *subdirect factorization* of a lattice L we mean the system of lattices $\{L_{\alpha};\ \alpha\in I\}$, when L is isomorphic to the subdirect union of $L_{\alpha}(\alpha\in I)$ in Birkhoff's [1, p. 764; 2, p. 91] sense. In this case, if we denote by θ_{α} the congruence relation introduced by the homomorphism $L\to L_{\alpha}$, and we denote by $L_{\theta_{\alpha}}$ the homomorphic image generated by θ_{α}, then $L_{\alpha}\cong L_{\theta_{\alpha}}$ and $\bigwedge(\theta_{\alpha};\ \alpha\in I)=0$. Conversely, when $\{\theta_{\alpha};\ \alpha\in I\}$ is a subset of $\Theta(L)$ such that $\bigwedge(\theta_{\alpha};\ \alpha\in I)=0$, the system of lattices $\{L_{\theta_{\alpha}};\ \alpha\in I\}$ is a subdirect factorization of L.

Let $\{\sigma_{\alpha};\ \alpha\in I\}$ be the set of all completely meet-irreducible elements of $\Theta(L)$, then by the property (2) we have $\bigwedge(\sigma_{\alpha};\ \alpha\in I)=0$ and $\{L_{\sigma_{\alpha}};\ \alpha\in I\}$ is a subdirect factorization of L with subdirectly irreducible factors.[10)]

Let $\{\phi_{\alpha};\ \alpha\in I\}$ is a subset of Θ (L) such that $\phi_{\alpha}=\bigwedge(\phi_{\beta};\ \beta\in I,\ \beta\neq\alpha)$ for all $\alpha\in I$. Then $\bigwedge(\phi_{\alpha};\ \alpha\in I)=\phi_{\alpha}\frown\phi_{\alpha}^{*}=0$ and $\{L_{\phi_{\alpha}};\ \alpha\in I\}$ is a subdirect factorization of L. Such factorization is called *canonical* by F. Maeda [1, p. 100] and in order that $\{L_{\phi_{\alpha}};\ \alpha\in I\}$ is a cononical subdirect factorization of L, it is necessary and sufficient that $\phi_{\alpha}\in\Theta$ (L) for all $\alpha\in I$, and $\bigwedge(\phi_{\alpha};\ \alpha\in I)=0,\ \phi_{\alpha}\vee\phi_{\beta}=1(\alpha\neq\beta)$.[11)]

REMARK 1. Let $\{L_{\phi_{\alpha}};\ \alpha\in I\}$ be a canonical subdirect factorization of L, and if $\phi_{\alpha}^{*}=\bigwedge(\phi_{\beta};\ \beta\in I,\ \beta\neq\alpha)=0$, then $\phi_{\alpha}=\phi_{\alpha}^{**}=1$ and $L_{\phi_{\alpha}}$ is a one element lattice and since we may omit such factor we have $\bigwedge(\phi_{\beta};\ \beta\in I,\ \beta\neq\alpha)\neq0$ and hence $\{L_{\phi_{\beta}};\ \beta\in I,\ \beta\neq\alpha\}$ is not a subdirect factorization of L. In this sense, a canonical subdirect factorization is *irredundant*. Conversely, let $\{L_{\phi_{\alpha}};\ \alpha\in I\}$ be an irredundant subdirect factorization of L with sub-

10) Cf. G. Birkhoff [1] 765, [2] 92, Theorem 10. Also Cf. R. P. Dilworth [1] 352-353.

11) Cf. F. Maeda [1] 100, Theorem 2.1.

directly irreducible factors. Since $\psi_\alpha=\bigwedge(\phi_\beta\,;\,\beta\in I,\ \beta\neq\alpha)\neq 0$ and $\phi_\alpha\frown\psi_\alpha=0$, we have $\phi_\alpha^*\geqq\psi_\alpha>0$ and $\phi_\alpha\notin\Delta(L)$. By lemma 2 ϕ_α is a maximal element of $\Theta_*(L)$ and then $\phi_\alpha\vee\phi_\beta=1(\alpha\neq\beta)$, hence this factorization is canonical.

Also if L has a canonical subdirect factorization with subdirectly irreducible factors, $\Theta_*(L)$ is an atomic Boolean algebra[12] and let $\{\phi_\alpha\,;\,\alpha\in I\}$ be the set of all maximal elements of $\Theta_*(L)$, then $\{L_{\phi_\alpha}\,;\,\alpha\in I\}$ is the unique canonical subdirect factorization with subdirectly irreducible factors and $\{\phi_\alpha^*\,;\,\alpha\in I\}$ is the set of all points of $\Theta_*(L)$.

THEOREM 1. *The following three conditions on a lattice L are equivalent:*

(α) *$\Theta(L)$ is an atomic lattice.*

(β) *The dual ideal $\Delta(L)$ of $\Theta(L)$ is principal.*

(γ) *L has the canonical subdirect factorization with subdirectly irreducible factors.*

PROOF. (i) $(\alpha)\to(\beta)$: Let δ_0 be the join of all points of $\Theta(L)$. For an arbitrary non-zero element θ, there is a point ρ such that $\theta\geqq\rho$, then $\delta_0\frown\theta\geqq\rho>0$. Therefore $\delta_0^*=0$, that is, $\delta_0\in\Delta(L)$. If δ is an arbitrary dense element, then for arbitrary point $\rho(\neq 0)$ $\delta\frown\rho\neq 0$ and this implies $\delta\geqq\rho$. Hence $\delta\geqq\bigvee\rho=\delta_0$, and thus $\Delta(L)$ is a principal dual ideal generated by δ_0.

(ii) $(\beta)\to(\gamma)$: The set of all completely meet-irreducible elements of $\Theta(L)$ are separated by lemma 2 the set $\{\phi_\alpha\,;\,\alpha\in I_1\}$ of maximal elements of $\Theta_*(L)$ and the set $\{\delta_\beta\,;\,\beta\in I_2\}$ of dense elements. From the property (2) we see $\bigwedge(\phi_\alpha\,;\,\alpha\in I_1)\frown\bigwedge(\delta_\beta\,;\,\beta\in I_2)=0$ and $\bigwedge(\delta_\beta\,;\,\beta\in I_2)\geqq\delta_0$ where δ_0 is the element which generates the principal dual ideal $\Delta(L)$, and hence $\bigwedge(\phi_\alpha\,;\,\alpha\in I_1)=0$. Thus $\{L_{\phi_\alpha}\,;\,\alpha\in I_1\}$ is the canonical subdirect factorization of L with subdirectly irreducible factors.

(iii) $(\gamma)\to(\alpha)$: From the assumption and the last statement of remark 1, $\Theta_*(L)$ is atomic and let $\{\phi_\alpha\,;\,\alpha\in I\}$ be the set of all completely meet-irreducible closed elements (that is, maximal elements of $\Theta_*(L)$), then $\bigwedge(\phi_\alpha\,;\,\alpha\in I)=0$.

Let θ be an arbitrary element such that $\theta>\phi_\alpha$, then $\theta^*\leqq\phi_\alpha^*$. Since $\theta^*=\phi_\alpha^*$ implies $\theta\leqq\theta^{**}=\phi_\alpha^{**}=\phi_\alpha$ which is a contradiction, we have $\theta^*<\phi_\alpha^*$. And since ϕ_α^* is a point of $\Theta_*(L)$ and $\theta^*\in\Theta_*(L)$, $\theta^*=0$. Hence $\{\theta\,;\,\theta>\phi_\alpha\}\subset\Delta(L)$ and this set has the least dense element δ_α since ϕ_α is a completely meet-irreducible element of $\Theta(L)$. Since $\Theta(L)$ is distributive, $\delta_\alpha>\phi_\alpha$ implies $\phi_\alpha\smile(\phi_\alpha^*\frown\delta_\alpha)=(\phi_\alpha\smile\phi_\alpha^*)\frown\delta_\alpha$. Since $\phi_\alpha\smile\phi_\alpha^*=\phi_\alpha$ implies $\phi_\alpha\frown\phi_\alpha^*=\phi_\alpha^*>0$

12) Cf. F. Maeda [1] 101, Theorem 2.4. A lattice with zero is called *atomic* if any non-zero element contains at least a point which covers zero.

which is a contradiction, we have $\phi_\alpha \smile \phi_\alpha^* > \phi_\alpha$ and then $\phi_\alpha \smile \phi_\alpha^* \geqq \delta_\alpha$. Hence $\phi_\alpha \smile (\phi_\alpha^* \frown \delta_\alpha) = \delta_\alpha$, and it is clear that $\phi_\alpha \frown (\phi_\alpha^* \frown \delta_\alpha) = 0$. And since δ_α covers ϕ_α we can conclude that $\phi_\alpha^* \frown \delta_\alpha$ is a point of $\Theta(L)$ by the Dedekind's transposition principle.

We put $\delta_0 = \bigvee(\phi_\alpha^* \frown \delta_\alpha;\ \alpha \in I)$, then by the equality (4) we have

$$\delta_0^* = (\bigvee(\phi_\alpha^* \frown \delta_\alpha;\ \alpha \in I))^* = \bigwedge((\phi_\alpha^* \frown \delta_\alpha)^*;\ \alpha \in I) = \bigwedge(\phi_\alpha^{**} \vee \delta_\alpha^*;\ \alpha \in I)^{13)}$$
$$= \bigwedge(\phi_\alpha;\ \alpha \in I) = 0$$

and $\delta_0 \in \Delta(L)$. If δ is an arbitrary dense element, $\delta \frown \delta_\alpha \in \Delta(L)$ since $\Delta(L)$ is a dual ideal of $\Theta(L)$. Hence $\phi_\alpha^* \neq 0$ implies $\phi_\alpha^* \frown \delta_\alpha \geqq \phi_\alpha^* \frown \delta \frown \delta_\alpha > 0$, but $\phi_\alpha^* \frown \delta_\alpha$ is a point and we have $\phi_\alpha^* \frown \delta_\alpha = \phi_\alpha^* \frown \delta \frown \delta_\alpha$ which implies $\delta \geqq \phi_\alpha^* \frown \delta_\alpha$. That is, $\delta \geqq \bigvee(\phi_\alpha^* \frown \delta_\alpha;\ \alpha \in I) = \delta_0$. Then we see that $\Delta(L)$ is a principal dual ideal generated by δ_0. Therefore $\phi_\alpha^* \frown \delta_\alpha \geqq \phi_\alpha^* \frown \delta_0 > 0$ and this implies $\phi_\alpha^* \frown \delta_\alpha = \phi_\alpha^* \frown \delta_0$ since $\phi_\alpha^* \frown \delta_\alpha$ is a point.

The unit 1 contains a point $\phi_\alpha^* \frown \delta_0$. Let θ be a non-zero and non-unit element. By the property (2) θ is a meet of completely meet-irreducible elements and by lemma 2 $\theta = \bigwedge(\phi_\alpha;\ \alpha \in I_1) \frown \bigwedge(\delta_\beta;\ \beta \in I_2)$, where $I_1 \leqq I$ and $\delta_\beta \in \Delta(L)$ for all $\beta \in I_2$. If $I_1 = I$ we have $\theta \leqq \bigwedge(\phi_\alpha;\ \alpha \in I) = 0$ which contradicts our assumption, hence $I_1 < I$. We pick out an index α' from $I - I_1$. Since $\{L_{\phi_\alpha};\ \alpha \in I\}$ is a canonical subdirect factorization of L, we see that $\phi_{\alpha'} = \bigwedge(\phi_\alpha;\ \alpha \in I,\ \alpha \neq \alpha') \leqq \bigwedge(\phi_\alpha;\ \alpha \in I_1)$. But $\bigwedge(\delta_\beta;\ \beta \in I_2) \geqq \delta_0$. Consequently θ contains a point $\phi_{\alpha'}^* \frown \delta_0$ and this completes the proof.

REMARK 2. Let C be a dense-in-itself chain. If θ is a point of $\Theta(C)$, there exist two elements a, b such that $a > b$ in C and $a \equiv b(\theta)$. Since C is dense-in-itself there exists an element c such that $a > c > b$ and let θ' be the congruence relation whose classes consist of the set $\{x;\ c \geqq x \geqq b\}$ and other elements. Then we have $0 < \theta' < \theta$ which contradicts the fact that θ is a point. Thus in $\Theta(C)$ there is not a point, and hence by theorem 1 C has not the canonical subdirect factorization with subdirectly irreducible factors; in other words, any subdirect factorization of C with subdirectly irreducible factors is redundant.

3. R. P. Dilworth[14] has shown that a lattice L is a subdirect union of a finite number of simple[15] lattices if and only if $\Theta(L)$ is a finite Boolean algebra. We attempt the generalization of this result.

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13) Cf. G. Birkhoff [2] 148, (23″).

14) Cf. R. P. Dilworth [1] 353, Theorem 3.3.

15) A lattice L is called *simple* when $\Theta(L)$ contains at most two elements. And it is clear that a simple lattice is subdirectly irreducible.

Now we assume that $\Theta(L)$ is a Boolean algebra, then $\Delta(L)$ contains only 1 by lemma 3. Therefore by theorem 1 and remark 1 $\{L_{\phi_\alpha}; \alpha\in I\}$ is the canonical subdirect factorization with simple factors where $\{\phi_\alpha; \alpha\in I\}$ is the set of all maximal elements of $\Theta(L)=\Theta_*(L)$.[16] Hence every element of $\Theta(L)$ is a meet of maximal elements and since a Boolean algebra is self-dual, every element is a join of points and $\{\phi_\alpha{}^*; \alpha\in I\}$ is the set of all points.

For arbitrary two elements a, b of L, there exists a non-zero congruence relation θ such that $a\equiv b(\theta)$ and $\theta=\bigvee(\phi_\alpha{}^*; \alpha\in I')$ where $I'\leqq I$. Hence[17] there exists a finite subset $\{a_1, a_2, \cdots, a_{n-1}\}$ of L such that

$$a\equiv a_1(\phi^*_{\alpha_1}),\ a_1\equiv a_2(\phi^*_{\alpha_2}), \cdots, a_{n-1}\equiv b(\phi^*_{\alpha_n});\ \alpha_i\in I \text{ for } i=1, 2, \cdots, n.$$

Clearly from this

$$a\equiv b(\bigvee(\phi^*_{\alpha_i};\ i=1, 2, \cdots, n)).$$

However from $\phi^*_{\alpha'}=\bigwedge(\phi_\alpha; \alpha\in I, \alpha\neq\alpha')$ and $\phi_\alpha\smile\phi_\alpha{}^*=1$ by the mathematical induction we obtain the equality;

$$(5)\qquad \bigvee(\phi^*_{\alpha_i};\ i=1, 2, \cdots, n)=\bigwedge(\phi_\alpha;\ \alpha\in I,\ \alpha\neq\alpha_i \ for\ i=1, 2, \cdots, n).$$

In case of $n=1$, (5) is the formula $\phi^*_{\alpha_1}=\bigvee(\phi_\alpha; \alpha\in I, \alpha\neq\alpha_1)$. Suppose (5) is true when $n=r$, then by the distributive law

$$\begin{aligned}\bigvee(\phi^*_{\alpha_i};\ i=1, 2, \cdots, r, r+1) &= \bigvee(\phi^*_{\alpha_i};\ i=1, 2, \cdots, r)\smile\phi^*_{\alpha_{r+1}}\\ &=\bigwedge(\phi_\alpha;\ \alpha\in I,\ \alpha\neq\alpha_i\ for\ i=1, 2, \cdots, r)\smile\bigwedge(\phi_\alpha;\ \alpha\in I,\ \alpha\neq\alpha_{r+1})\\ &=\bigwedge(\phi_\alpha;\ \alpha\in I,\ \alpha\neq\alpha_i\ for\ i=1, 2, \cdots, r, r+1)\\ &\qquad\frown\{\phi_{\alpha_{r+1}}\smile\bigwedge(\phi_{\alpha_i};\ i=1, 2, \cdots, r)\}\end{aligned}$$

and since $\phi_{\alpha_{r+1}}\smile\bigwedge(\phi_{\alpha_i}; i=1, 2, \cdots, r)\geqq\phi_{\alpha_{r+1}}\smile\phi^*_{\alpha_{r+1}}=1$ we see that (5) is true when $n=r+1$, q. e. d. Then

$$a\equiv b(\bigwedge(\phi_\alpha;\ \alpha\in I,\ \alpha\neq\alpha_i\ for\ i=1, 2, \cdots, n)),$$

and we have $a\equiv b(\phi_\alpha)$ for all $\alpha\in I$ except $\alpha_1, \alpha_2, \cdots, \alpha_n$. That is to say that in our factorization the components of a and b are identical except α_i-component for $i=1, 2, \cdots, n$.

Conversely, we have

LEMMA 4. *If a lattice L has a subdirect factorization with simple factors such that the components of arbitrary two elements are identical except a finite*

16) Cf. the relations (3).

17) Cf. G. Birkhoff [2] 23, Theorem 4.

number of components, this factorization is canonical and $\Theta(L)$ is a Boolean algebra.

PROOF. Let $\{L_{\phi_\alpha};\ \alpha\in I\}$ be the given factorization, then the elements ϕ_α are completely meet-irreducible in $\Theta(L)$. For arbitrary two elements a, b of $L(a\neq b)$ we consider $\tau(a, b)=\bigwedge(\theta;\ a\equiv b(\theta))$, that is the least congruence relation by which a, b are equivalent. From the assumption we have $a\equiv b(\phi_\alpha)$ for all $\alpha\in I$ except $\alpha_1, \alpha_2, \cdots, \alpha_n$, that is

$$a\equiv b(\bigwedge(\phi_\alpha;\ \alpha\in I,\ \alpha\neq\alpha_i \text{ for } i=1, 2, \cdots, n))$$

which implies

$$\tau(a, b)\leqq\bigwedge(\phi_\alpha;\ \alpha\in I,\ \alpha\neq\alpha_i \text{ for } i=1, 2, \cdots, n)$$

and since $\bigwedge(\phi_\alpha;\ \alpha\in I)=0$ we have

$$\tau(a, b)\frown\phi_{\alpha_1}\frown\phi_{\alpha_2}\frown\cdots\frown\phi_{\alpha_n}=0\,.$$

By lemma 2, each ϕ_{α_i} is closed or dense element of $\Theta(L)$, however if all $\phi_{\alpha_i}(i=1, 2, \cdots, n)$ are dense we have $\phi_{\alpha_1}\frown\phi_{\alpha_2}\frown\cdots\frown\phi_{\alpha_n}\in\Delta(L)$ since $\Delta(L)$ is a dual ideal of $\Theta(L)$, and this implies $\tau(a, b)=0$ which contradicts the fact that $a\equiv b(\tau(a, b))$. Hence we can assume that

$$\phi_{\alpha_1}, \phi_{\alpha_2}, \cdots, \phi_{\alpha_m}\in\Theta_*(L);\ \phi_{\alpha_{m+1}}, \phi_{\alpha_{m+2}}, \cdots, \phi_{\alpha_n}\in\Delta(L)$$

where $m\geqq 1$. Since $\phi_{\alpha_{m+1}}\frown\phi_{\alpha_{m+2}}\frown\cdots\frown\phi_{\alpha_n}\in\Delta(L)$ we see that

$$\tau(a, b)\frown\phi_{\alpha_1}\frown\phi_{\alpha_2}\frown\cdots\frown\phi_{\alpha_m}=0$$

which implies by the equality (4)

$$\tau(a, b)\leqq(\phi_{\alpha_1}\frown\phi_{\alpha_2}\frown\cdots\frown\phi_{\alpha_m})^*=(\phi_{\alpha_1}^{**}\frown\phi_{\alpha_2}^{**}\frown\cdots\frown\phi_{\alpha_m}^{**})^*=(\phi_{\alpha_1}^{*}\smile\phi_{\alpha_2}^{*}\smile\cdots\smile\phi_{\alpha_m})^{**}.$$

On the other hand, since the factors are simple $\phi_{\alpha_i}(i=1, 2, \cdots, m)$ are maximal in $\Theta(L)$ and hence we have $\phi_{\alpha_i}\smile\phi_{\alpha_i}^*=1$, that is $\phi_{\alpha_i}^*\in\Theta_z(L)$ for $i=1, 2, \cdots, m$. Since $\Theta_z(L)$ is a sublattice of $\Theta(L)$ $\phi_{\alpha_1}^*\smile\phi_{\alpha_2}^*\smile\cdots\smile\phi_{\alpha_m}^*\in\Theta_z(L)\subset\Theta_*(L)$, hence we obtain that

$$\tau(a, b)\leqq\phi_{\alpha_1}^*\smile\phi_{\alpha_2}^*\smile\cdots\smile\phi_{\alpha_m}^*\,.$$

Now we put $\{\phi_\alpha;\ \alpha\in I'\}=\{\phi_\alpha;\ \alpha\in I\}\bigwedge\Theta_*(L)$, then we have $\tau(a, b)\leqq\bigvee(\phi_\alpha^*;\ \alpha\in I')$. But since it is clear that $\bigvee(\tau(a, b);\ a, b\in L)=1$ we have

$$\bigvee(\phi_\alpha^*;\ \alpha\in I')=1 \tag{6}$$

From this we see that

$$\bigwedge(\phi_\alpha;\ \alpha\in I')=\bigwedge(\phi_\alpha^{**};\ \alpha\in I')=(\bigvee(\phi_\alpha^*;\ \alpha\in I'))^*=1^*=0$$

by the equality (4), and since $\phi_\alpha \vee \phi_\beta \geqq \phi_\alpha \smile \phi_\beta = 1$ $(\alpha \neq \beta)$, $\{L_{\phi_\alpha}; \alpha \in I'\}$ is the canonical subdirect factorization with subdirectly irreducible factors.

And then from the third part of the proof of theorem 1 we see that since the element δ_α covering a maximal element ϕ_α is the unit 1, $\phi_\alpha^* \frown \delta_\alpha = \phi_\alpha^*$ is a point of $\Theta(L)$, and $\delta_0 = \bigvee(\phi_\alpha^*; \alpha \in I')$ is the least element of $\Delta(L)$. Thus by the equality (6) and lemma 3 $\Theta(L)$ is a Boolean algebra.

And since $\Theta_*(L) = \Theta(L)$ we have $\{\phi_\alpha; \alpha \in I'\} = \{\phi_\alpha; \alpha \in I\}$, then the given factorization is canonical, completing the proof.

Thus we conclude:

THEOREM 2. *$\Theta(L)$ is a Boolean algebra if and only if the lattice L has a subdirect factorization with simple factors such that the components of arbitrary two elements of L are identical except a finite number of components.*

Finally, we remark that our statements can be established for any algebra with finitary operations whose congruence relations form a distributive lattice, since the structure lattice has the properties (1) and (2).

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