# LATTICE THEORETIC CHARACTERIZATION OF AN AFFINE GEOMETRY OF ARBITRARY DIMENSIONS

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G. Birkhoff [1][1)] and K. Menger [1] have characterized lattice theoretically projective and affine geometries of finite dimensions, while a projective geometry of arbitrary dimensions, finite or infinite, has been characterized by O. Frink [1] and W. Prenowitz [1].

The purpose of this paper is to characterize the lattice of all subspaces of an affine space of arbitrary dimensions. The main theorem is as follows:

THEOREM. *An abstract lattice $L$ is isomorphic to the lattice of all subspaces of an affine space if and only if $L$ is relatively atomic, upper continuous lattice which is semi-modular in the sense of Wilcox and satisfies the following condition:*

*Let $p$, $q$, $r$ be independent atomic elements of $L$, then there exists one and only one element $l$ such that $p < l < p \cup q \cup r$, and $l \cap (q \cup r) = 0$.*

In the appendix, we shall give a proof that the axiom $I_7$ of Hilbert [1] p. 4 is equivalent to the transitivity of parallel lines in a 3-space.

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## § 1. A Projective Space and an Affine Space.

DEFINITION 1.1. Let $G$ be a set of points. If for any pair of distinct points $p$, $q$ of $G$, there exists a subset $p+q$ (called *line* of $G$) containing $p$ and $q$, which satisfies the following conditions, then $G$ is called a *projective space*.[2)]

P.1. *Two distinct points on a line determine the line.*

P.2. *If $p, q, r$ are points not all on the same line, and $u$ and $v (u \neq v)$ are points such that $p, q, u$ are on a line and $p, r, v$ are on a line, then there is a point $w$ such that $q, r, w$ are on a line, and also $u, v, w$ are on a line.*

1) The numbers in square brackets refer to the list of references at the end of the paper.

2) Cf. Birkhoff [2] 116.

P. 3. *Every line contains at least three points.*

By a *linear subspace* of $G$, it is meant the subset $S$ such that $p, q \in S$ imply $p+q \subseteqq S$.

Let $S$, $T$ be subsets of $G$. If $S, T \neq 0$, void set, we define $S+T$ to be the set union $\bigvee(p+q\,;\ p \in S,\ q \in T)$, where by $p+p$ we mean the set $\{p\}$ containing $p$ alone. If $T=0$ we define $S+T=T+S=S$.[3)]

REMARK 1.1. It follows immediately from P. 2 that if $p, q, r$ are points of $G$, then $p+(q+r)=(p+q)+r$,[4)] whence it may be denoted by $p+q+r$.

DEFINITION 1.2. Let $A$ be a set of points. If for any pair of distinct points $p, q$, there exists a subset $p \vee q$ (called *line* of $A$) containing $p, q$, and if for any triple $p, q, r$ of non-collinear points,[5)] there exists a subset $p \vee q \vee r$ (called *plane* of $A$) containing $p, q, r$, which satisfy the following conditions A. 1-A. 4, then $A$ is called an *affine space*.[6)]

A. 1. *Two distinct points on a line determine the line.*

A. 2. *Three non-collinear points on a plane determine the plane.*

Two lines $p \vee q$, $r \vee s$ are called to be *parallel* to each other and denoted by $p \vee q \parallel r \vee s$ provided that they are contained in the same plane and have no point in common.

A. 3. *If $p, q, r$ are non-collinear points, then there exists one and only one line $r \vee s$ such that $r \vee s$ is parallel to $p \vee q$.*

By a *subspace* of $A$, we mean the subset $S$ such that $p, q \in S$ implies $p \vee q \subseteqq S$, and $p, q, r \in S$ implies $p \vee q \vee r \subseteqq S$.

Let $T$ be any subset of $A$. The smallest subspace containing $T$ is called the subspace *generated* by $T$.[7)] If four points $p, q, r, s$ are not on a plane, the subspace generated by the set of these points is called a 3-*space* and is denoted by $p \vee q \vee r \vee s$.

A. 4. *If two planes in a 3-space have a common point, then they have at least one more point in common.*

REMARK 1.2. Let us suppose that $A$ is especially an affine space of three dimensions in the sense that $A$ is generated by four points which are not on a plane, then the conditions A. 1-A. 4 together with this assumption are equivalent to "Axiome der Verknüpfung" I together with

3) Cf. Prenowitz [1] 664.

4) Cf. ibid. 665 Theorem 3, J 5.

5) If three points $p, q, r$ are on some line, then they are called to be collinear, and otherwise non-collinear.

6) As to the definition of an affine plane, cf. Birkhoff [2] 110.

7) The intersection of all subspaces containing $T$ is the subspace generated by $T$.

"Axiom der Parallelen" IV of Hilbert [1] 3, 20.[8] Hence we may consider the Definition 1.2 is obtained by omitting the axiom which restricts the dimensionality from the system of axioms I, IV. The situation is similar as a projective space of arbitrary dimensions, finite or infinite, can be defined as in Definition 1.1, which is obtained by omitting the postulates which restrict the dimensionality from postulates for 3-dimensional projective space.

Now we shall construct an affine space of arbitrary dimensions, finite or infinite, over an arbitrary fieid $F$.[9] Let $\alpha$ be any cardinal number, let $T$ be a set of elements of cardinal number $\alpha$, and let $t$ be a variable ranging over the set $T$. Let $F$ be any field. By a *coordinate function*, we shall mean the function $f(t)$ defined over $T$ such as $f(t)\in F$, and $f(t)=0$ except for a finite set of value of $t$. Let $AS(\alpha; F)$ be the set of all coordinate functions. For any three elements $p_i=f_i(t)$, $(i=1, 2, 3)$, we shall define as follows:

$$p_1 \vee p_2 = \{ \textstyle\sum_{i=1}^2 \lambda_i f_i(t);\ \lambda_1, \lambda_2 \in F,\ \lambda_1+\lambda_2=1\},$$
$$p_1 \vee p_2 \vee p_3 = \{ \textstyle\sum_{i=1}^3 \lambda_i f_i(t);\ \lambda_1, \lambda_2, \lambda_3 \in F,\ \lambda_1+\lambda_2+\lambda_3=1\}.$$

Then it is easily shown that $AS(\alpha; F)$ satisfies A.1-A.4.[10]

Conversely we can introduce a coordinate system into an affine space if it contains four points which are not on a plane, and $A$ is isomorphic to an $AS(\alpha; F)$ for some cardinal number $\alpha$ and for some field $F$.[11]

REMARK 1.3. We find in remark 1.2 a proof of the consistency of the conditions A.1-A.4. The independence of these conditions can be shown as follows:

(1) Let $A=\{1, 2, 3\}$, $1\vee 2=2\vee 3=A$ and $1\vee 3=\{1, 3\}$, then $A$ satisfies A.2-A.4 vacuously but does not satisfy A.1. This proves that A.1 is independent of the others.

(2) Let $A=\{1, 2, 3, 4\}$, $i\vee j=\{i, j\}$ $(i, j=1, 2, 3, 4)$ and

$$i\vee j\vee k=\begin{cases}\{1, 2, 3\}, & \text{if } \{i, j, k\}=\{1, 2, 3\}\\ A, & \text{otherwise.}\end{cases}$$

From this example it is shown that A.2 is independent of the others.

8) As to the axiom $I_6$ of Hilbert [1] 4, cf. Lemma 1.1 below.

9) As regards the case of finite dimensions, cf. Birkhoff [2], 102-3, and Iyanaga [1] 58, 71.

10) The proof is similar to the case of finite dimensions, since from the definition, $f(t)\neq 0$ merely on the finite subset of $T$.

11) Cf. Frink [1] § 6, and Theorem 1.1 below. Cf. also Iyanaga [1] 90 Theorem 10, 95 Theorem 10.

(3) Let $A$ be a projective space of 3 dimensions over an arbitrary field, then $A$ satisfies A. 1, A. 2 and A. 4 but does not satisfy A. 3. It follows that A. 3 is independent of the others.

(4) Let $A$ be an affine space of 4 dimensions over the prime field $F$ of characteristic 2, that is, $A=\{(x, y, z, t);\ x, y, z, t=0, 1 \in F\}$. By $p\oplus q$ and $p\oplus q\oplus r$, we shall denote respectively the line and plane defined in Remark 1.2. Then $p\oplus q$ consists of $p$ and $q$, if $p, q$ are distinct. And for three distinct points $p_i=(x_i, y_i, z_i, t_i)$ $(i=1, 2, 3)$, $p_1\oplus p_2\oplus p_3$ contains one and only one more point $p_4=(x_4, y_4, z_4, t_4)$ with

$$\sum_{i=1}^4 x_i = 0, \quad \sum_{i=1}^4 y_i = 0, \quad \sum_{i=1}^4 z_i = 0, \quad \sum_{i=1}^4 t_i = 0.$$

Now let us consider a transformation $T$ of $A$ onto itself such as

$$Tp = \begin{cases} (0, 1, 0, 0), & \text{if } p=(1, 0, 0, 0) \\ (1, 0, 0, 0), & \text{if } p=(0, 1, 0, 0) \\ p, & \text{otherwise.} \end{cases}$$

and let us define $p\vee q$ and $p\vee q\vee r$ as follows:

$$p\vee q = p\oplus q$$

$$p\vee q\vee r = \begin{cases} T(Tp\oplus Tq\oplus Tr), & \text{if } p, q, r \in S,\ S \text{ being the 3-space determined by the equation } t=0. \\ p\oplus q\oplus r, & \text{otherwise.} \end{cases}$$

Suppose that $A'$ is the set $A$ and $p\vee q$ and $p\vee q\vee r$ are respectively the line and the plane of $A'$. Then it is clear that $A'$ satisfies A. 1.-A. 3. But it does not satisfy A. 4. For, put

$$p=(0, 0, 0, 0), \quad q=(0, 1, 0, 0), \quad q'=(1, 0, 0, 0),$$
$$a=(0, 0, 1, 0), \quad b=(1, 0, 1, 0), \quad r=(0, 0, 1, 1),$$
$$s=(1, 0, 1, 1),$$

then we have

$$p\vee q \parallel a\vee b, \quad a\vee b \parallel r\vee s, \quad \text{and} \quad r\vee s \parallel p\vee q',$$

it follows that

$$p\vee r\vee s, \quad p\vee a\vee b \subseteqq a\vee p\vee r\vee s,$$

while the two planes $p\vee r\vee s$ and $p\vee a\vee b$ have no more point in common. This shows that A. 4 is independent of the others.

REMARK 1.4. In Definition 1.2, we adopted A. 4 according to the conventional form, but it might be replaced by

A. 4′: $p\vee q \parallel x\vee y$, $x\vee y \parallel r\vee s$ *and* $r \notin p\vee x\vee y$ *imply* $p\vee q \parallel r\vee s$.

In fact, A. 4′ is used instead of A. 4 in the sequel except in Lemma 1. 2, which shows A. 4 implies A. 4′. As to the relation of these conditions, cf. the appendix and Remark 2. 4.

LEMMA 1.1 *In an affine space, if two distinct points* $q$ *and* $r$ *are contained in a plane* $p_1\vee p_2\vee p_3$, *then the line* $q\vee r$ *is contained in the plane.*

PROOF. At least one of the points $p_1$, $p_2$, $p_3$ (say $p_1$) is not on the line $q\vee r$, whence by A. 2 $p_1\vee q\vee r=p_1\vee p_2\vee p_3$ and by A. 3 there exists a point $s$ with $p_1\vee s \parallel q\vee r$. By the definition of parallel lines, $q\vee r$ and $p_1\vee s$ are contained in a plane, which is equal to $p_1\vee q\vee r=p_1\vee p_2\vee p_3$, by A. 2.

LEMMA 1. 2. *In an affine space, the following condition is satisfied:*

A. 4′: $p\vee q \parallel x\vee y$, $x\vee y \parallel r\vee s$, *and* $r \notin p\vee x\vee y$ *imply* $p\vee q \parallel r\vee s$.

PROOF. Since $p\vee q \parallel x\vee y$, it holds $q\in p\vee x\vee y$. It follows that $r\vee p\vee q \subseteqq r\vee p\vee x\vee y$, and clearly $r\vee x\vee y \subseteqq r\vee p\vee x\vee y$. Hence $r\vee p\vee q$, and $r\vee x\vee y$ have a common point $s'(\neq r)$ by A. 4. Two lines $p\vee q$, $r\vee s'$ have no common point, since otherwise it would be contained in $x\vee y$, contrary to $p\vee q \parallel x\vee y$. Furthermore, they are contained in a plane $r\vee x\vee y$, whence $r\vee s' \parallel p\vee q$. Similarly $r\vee s' \parallel x\vee y$. Since $r\vee s \parallel x\vee y$, it follows from A. 3 that $r\vee s'=r\vee s$, whence $p\vee q \parallel r\vee s$, completing the proof.

We note here that the remaining lemmas of this section shall be proved by making use of A. 4′, instead of A. 4.

LEMMA 1. 3. *In an affine spce,* $p\vee a \parallel p'\vee a'$, $p\vee b \parallel p'\vee b'$, $p\vee p' \parallel a\vee a'$, $p\vee p' \parallel b\vee b'$ *and* $p' \notin p\vee a\vee b$ *imply* $a\vee b \parallel a'\vee b'$.

PROOF. Since $a\vee a' \parallel p\vee p'$, $p\vee p' \parallel b\vee b'$, it follows that $a\vee a' \parallel b\vee b'$ by A. 4′. So the points $a$, $a'$, $b$ and $b'$ are on a plane. Suppose that $a\vee b$ and $a'\vee b'$ were not parallel. Let $c$ be their common point, then there would exist points $d, e$ with $c\vee d \parallel p\vee a$, $c\vee e \parallel p\vee b$ by A. 3, and $c\in p\vee a\vee b$ by Lemma 1.1 and $d, e\in p\vee a\vee b$. It would follow from A. 2 that $p\vee a\vee b=c\vee d\vee e$. Similarly $p'\vee a'\vee b'=c\vee d\vee e$, since $c\vee d \parallel p'\vee a'$, and $c\vee e \parallel p'\vee b'$ by A. 4′. This contradicts the assumption $p' \notin p\vee a\vee b$. Hence we have

$$a\vee b \parallel a'\vee b'.$$

LEMMA 1. 4. *Let* $p, a, b$ *be non-collinear points of an affine space. If* $p\vee a \parallel p'\vee a'$, $p\vee b \parallel p'\vee b'$, *and* $c\in p\vee a\vee b$, *then there exists a point* $c'$ *with* $p\vee c \parallel p'\vee c'$ *and* $c'\in p'\vee a'\vee b'$.

PROOF. If $p' \in p \vee a \vee b$, then the lemma is immediate from A. 3, so let us assume the contrary. Furthermore by A. 3 we can assume without loss of generality that $p \vee p' \parallel a \vee a'$ and $p \vee p' \parallel b \vee b'$. It follows from Lemma 1. 3 that $a \vee b \parallel a' \vee b'$. First consider the case $p \vee c \parallel a \vee b$. Then there exists a point $c'$ with $p' \vee c' \parallel p \vee c$ by A. 3, and we have $p' \vee c' \parallel a' \vee b'$ by A. 4′, whence $c' \in p' \vee a' \vee b'$. And so $c'$ is the desired point. Next suppose that $p \vee c$ and $a \vee b$ are not parallel, then we can assume that $c \in a \vee b$. It follows that there exists a point $c'$ with $c \vee c' \parallel a \vee a'$, $c' \in a' \vee b'$ by A. 3, whence $p \vee c \parallel p' \vee c'$ by Lemma 1. 3 and $c' \in p' \vee a' \vee b'$ by Lemma 1. 1, completing the proof.

DEFINITION 1. 3. Let $A(\neq 0)$ be an affine space. For any two lines $p \vee q$, $r \vee s$ of $A$, we define $(p \vee q)\rho(r \vee s)$ to be either $p \vee q \parallel r \vee s$ or $p \vee q = r \vee s$. Then the relation $\rho$ is an equivalence relation in view of A. 3 and A. 4′. Hence $\rho$ classifies the set of lines of $A$. Each class shall be called "*point at infinity*" on the lines which belong to the class, and be denoted by $\alpha, \beta, \cdots$ or sometimes by $x, y, \cdots$. If it is necessary, the point at infinity on the line $p \vee q$ is denoted by $\alpha_{pq}$. By $\Gamma$ we denote the set of all points at infinity.

Let $G(A)$ be the set-union of $A$ and $\Gamma$, and let us define the *lines* of $G(A)$ as follows:

(1) If $p, q \in A$, $p \neq q$, then the line $p+q$ is defined to be the set-union of $p \vee q$ and $\alpha_{pq}$.

(2) If $p \in A$, $\alpha \in \Gamma$, then the line $p+\alpha=\alpha+p$ is defined to be the set-union of $p \vee q$ and $\alpha$, where $q$ is a point of $A$ such that $p \vee q$ belongs to the class $\alpha$.

(3) If $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$, then the line $\alpha+\beta$ is defined to be the set of all points at infinity $\gamma_{pc}$, where $p$ is a point of $A$ and $c$ is any point on the plane $p \vee a \vee b$, $a, b$ being points of $A$ with $p \vee a \in \alpha$, and $p \vee b \in \beta$.

If especially $p=q$ or $\alpha=\beta$, then we shall define respectively $p+q=\{p\}$, or $\alpha+\beta=\{\alpha\}$.

REMARK 1. 5. It follows at once from Definition 1. 3 that a line of $G(A)$ containing two distinct points at infinity is contained in $\Gamma$.

REMARK 1. 6. In Definition 1. 3 (2), the existence of the point $q$, and the independence of $p+\alpha$ from $q$, are shown by A. 3.

In the case (3), it follows at once from Lemma 1. 4 that $\alpha+\beta$ is not dependent on the points $p, a, b$.

The remainder of this section is devoted to showing that $G(A)$ is a projective space and that $\Gamma$ is a maximal proper linear subspace of $G(A)$.

In order to prove P. 1, it is sufficient to show the following:

LEMMA 1. 5. *Let $x$, $y$ and $z$ be points of $G(A)$. Then*

$$z \in x+y \text{ and } y \neq z \text{ imply } x+y=y+z.$$

PROOF. First let us assume that $x, y, z \in \Gamma$. In this case, let $p$ be a point of $A$ and let $a, b$ be points of $A$ with $p \vee a \in x$, $p \vee b \in y$, then by Definition 1. 3 (3), there exists a point $c(\in A)$ with $p \vee c \in z$, $c \in p \vee a \vee b$ and $p \vee b \neq p \vee c$. Since $p, b, c$ and $p, a, b$ are non-collinear, it follows from A. 2 that $p \vee a \vee b = p \vee b \vee c$, whence $x+y=y+z$ by Definition 1. 3.

In the other cases, the lemma follows at once from Definition 1. 3 and A. 1.

LEMMA 1. 6. *Any line of $G(A)$ contains at least three points.*

PROOF. We need only to consider the case $\alpha, \beta \in \Gamma$ and $\alpha \neq \beta$, since otherwise the lemma follows immediately from Definition 1. 3 (1), (2).

Let $p$ be a point of $A$, and $a, b(\in A)$ be the points with $p \vee a \in \alpha$, $p \vee b \in \beta$, then we have $p \vee a \neq p \vee b$, whence $p, a, b$ are non-collinear. Let $\gamma$ be the point at infinity on the line $p \vee c$ such as $p \vee c \parallel a \vee b$, then clearly $\gamma \in \alpha+\beta$ and $\gamma \neq \alpha, \beta$, completing the proof.

In order to prove P. 2, it is sufficient to show the following:

LEMMA. 1. 7. *Let $x, y, z$ be non-collinear points of $G(A)$. If $u \in x+y$, $v \in x+z$ and $u \neq v$, then $u+v$, $y+z$ have a common point.*

PROOF. We can assume that $u \neq x, y$ and $v \neq x, z$ without loss of generality, since otherwise the result is obvious.

First let us suppose that $y+z$, $u+v \subseteqq \Gamma$. Then it follows that $x \in \Gamma$, from Remark 1. 5. Let $p$ be a point of $A$ and let $p \vee a \in x$, $p \vee b \in y$, $p \vee c \in z$ $p \vee d \in u$, and $p \vee e \in v$, then $a, b, c, d, e \in A$ and $a \notin p \vee b \vee c$ since $x, y, z$ are non-collinear. Since $u \in x+y$, it follows that $d \in p \vee a \vee b$, whence we can assume that $p \vee a \parallel b \vee d$, and $a \vee d \parallel p \vee b$. Similarly we can assume that $a \vee e \parallel p \vee c$ and $p \vee a \parallel c \vee e$. It follows from Lemma 1. 3 that $d \vee e \parallel b \vee c$. There exists by A. 3 a point $p'$ with $p \vee p' \parallel b \vee c$ and $p \vee p' \parallel d \vee e$ by A. 4'. Hence the point at infinity $\alpha_{pp'}$ is contained in $y+z$ and $u+v$ by Definition 1. 3 (3).

Next suppose that one of $y+z$ and $u+v$ (say $u+v$) is contained in $\Gamma$. Then the point at infinity $\alpha_{yz}$ is the common point of $y+z$ and $u+v$.

When none of $y+z$ and $u+v$ is contained in $\Gamma$, then the applications of A. 3 show immediately the lemma.

Now we obtain the final result of this section.

THEOREM 1.1. *Let $A$ be an affine space. Then the extended space $G(A)=A\cup\Gamma$ is a projective space and $\Gamma$ is a maximal linear subspace of $G(A)$.*[12]

PROOF. In view of Lemma 1.5, 1.6, 1.7 and Remark 1.5 it remains to prove the maximality of $\Gamma$. Let $Q$ be a linear subspace with $\Gamma\subset Q$, then there exists a point $q\in Q-\Gamma$. Let $q+\Gamma$ be the linear subspace generated by $q$ and $\Gamma$, then we have

$$\Gamma\subset q+\Gamma\subseteqq Q\subseteqq G(A).$$

Now let $p\in G(A)$, then either $p\in\Gamma$ or $p\in A$. In the former case, we have clearly $p\in q+\Gamma$. In the latter case, we may assume that $p\neq q$, whence $\alpha_{pq}\in\Gamma$. It follows that $p\in q+\alpha_{pq}\subseteqq q+\Gamma$. whence $G(A)\subseteqq q+\Gamma$. Thus we have $G(A)=Q$, completing the proof.

REMARK 1.6. It is not difficult to show that conversely if $G$ is a projective space and $\Gamma$ is a maximal linear subspace of $G$, then $A=G-\Gamma$ is an affine space provided that we defind $p\vee q=(p+q)\cap A$, and $p\vee q\vee r=(p+q+r)\cap A$.

## § 2. Lattice Theoretic Characterization of an Affine Geometry.

DEFINITION 2.1. Let $b, c$ be elements of a lattice. By $(b, c)M$, it is meant that $(a\cup b)\cap c=a\cup(b\cap c)$ for any element $a$ with $a\leqq c$.

Let $L$ be a lattice with 0. If the elements of $L$ satisfy the following conditions:

(A) $(b, c)M$, $b\cap c=0$ imply $(c, b)M$, and

(B) $b\cap c\neq 0$ implies $(b, c)M$,

then $L$ is called to be *semi-modular in the sense of Wilcox.*[13]

REMARK 2.1. If $L$ is semi-modular in the sense of Wilcox, then $L$ satisfies the following condition:

($\xi'$) *If $x, y$ cover $a$ and $x\neq y$, then $x\cup y$ covers $x$ and $y$.* Cf. G. Birkhoff [2] 101, Ex. 1.

LEMMA 2.1. *Let $A$ be an affine space, and let $\Gamma$ and $G(A)$ be the sets defined in Definition 1.3. By $\mathfrak{G}_0$, we shall denote the family which consists of the void set 0 and all linear suspaces of the projective space $G(A)$ which*

12) For the plane affine geometry, cf. Birkhoff [2] 110, Theorem 10. And for the finite dimensional affine geometry over a field, cf. ibid. 103.

13) Cf. Wilcox [1] 496.

*are not contained in* $\Gamma$. *Then* $\mathfrak{S}_0$ *is a semi-modular lattice in the sense of Wilcox.*

PROOF. It is well known that the set $\mathfrak{S}$ of all linear subspaces of the projective space $G(A)$ is a complemented modular lattice[14] partially ordered by set-inclusion, and the join of $S^*$, $T^*$ is $S^*+T^*$,[15] while the meet is the intersection $S^*\frown T^*$. It follows at once that $\mathfrak{S}_0$ is a lattice, and the joint of $S^*$, $T^*$ is $S^*+T^*$ as above, while the meet $S\times T^*$ is as follows:

$$S^*\times T^* = \begin{cases} 0, & \text{if } S^*\frown T^*\subseteqq\Gamma, \\ S^*\frown T^*, & \text{otherwise.} \end{cases}$$

Hence the lemma follows immediately from the result of Wilcox [1] 498.

It is noted that $\mathfrak{S}_0$ is not a sublattice of $\mathfrak{S}$.

THEOREM 2.1 *The set* $\mathfrak{A}$ *of all subspaces of an affine space* $A$ *is a relatively atomic, upper continuous*[16] *lattice whice is semi-modular in the sense of Wilcox and satisfies the following condition:*

($\alpha$) *Let* $p$, $q$, $r$ *be independent atomic elements of* $\mathfrak{A}$, *then there exists one and only one element* $l$ *such that* $p<l<p\smile q\smile r$ *and* $l\frown(q\smile r)=0$.[17]

PROOF. (i) We shall omit the proof that $\mathfrak{A}$ is a relatively atomic, upper continuous lattice, since it is analogous to that of F. Maeda [1] 93, Theorem 2.1.

(ii) In order to prove that $\mathfrak{A}$ is semi-modular in the sense of Wilcox, it is sufficient from Lemma 2.2 to show that $\mathfrak{A}\cong\mathfrak{S}_0$.

Let $S\in\mathfrak{A}$, then $S+S$ is the union of $S$ and the set of points at infinity on the lines of $A$ contained in $S$, whence we can prove $S+S\in\mathfrak{S}_0$. For we need only to prove that $\alpha, \beta\in S+S$, ($\alpha, \beta\in\Gamma$) imply $\alpha+\beta\subseteqq S+S$, since otherwise the result is obvious. Let $p\vee q\in\alpha$, $r\vee s\in\beta$ where $p, q, r, s\in S$, and let $p\vee t \parallel r\vee s$, then $t\in p\vee r\vee s$, whence $t\in S$. It follows that $p\vee q\vee t \subseteqq S$, whence $\alpha+\beta\subseteqq S+S$ by Definition 1.3 (3).

While for any $S^*\in\mathfrak{S}_0$, $S^*\frown A$ is an element of $\mathfrak{A}$, and $(S^*\frown A)+(S^*\frown A) = S^*$ and $(S+S)\frown A=S$. Hence $S\rightarrow S+S$, and $S^*\rightarrow S^*\frown A$ are one-to-one correspondences between $\mathfrak{A}$ and $\mathfrak{S}_0$, which are obviously order-preserving, whence we have $\mathfrak{A}\cong\mathfrak{S}_0$.[18]

14) Cf. Frink [1] 455 Theorem 5. Prenowitz [1] 679 Corollary 3.
15) Cf. Frink [1] 454 Theorem 3. Prenowitz [1] 666 Theorem 7.
16) Cf. F. Maeda [1] 88 Definition 1.1, and 89 Definition 1.4.
17) Cf. Menger [1] 477 Law +8.
18) Cf. Birkhoff [2] 103 Theorem 4.

(iii) ($\alpha$) follows immediately from A. 3.

DEFINITION 2.2. Let $L$ be a relatively atomic, upper continuous lattice satisfying the condition ($\xi'$) stated in Remark 2.1, then $L$ is called a *matroid* lattice.[19]

REMARK 2.2. It follows from Remark 2.1 that the lattice $\mathfrak{A}$ in Theorem 2.1 is a matroid lattice.

REMARK 2.3. We note that in a relatively atomic, upper continuous lattice, ($\xi'$) is equivalent to the exchange axiom:

($\eta'$) $a < p \cup a \leqq q \cup a$ *imply* $p \cup a = q \cup a$.

Cf. F. Maeda [2] 180 Theorem 2.

LEMMA 2.2. *Let $L$ be a matroid lattice satisfying $(B)$ of Definition 2.1, then $L$ satisfies the following condition.*

(C) $s \leqq p \cup a$, $q \leqq a$ *imply* $s \leqq p \cup q \cup r$ *for some atomic elements* $r (\leqq a)$, *where* $p, q, s$ *are atomic elements and* $a$ *is any element of* $L$.

By a *strongly planer* lattice, we shall mean a lattice with 0 satisfying the condition (C).

PROOF. If $p=q$ or $s \leqq p \cup q$, then the lemma is obvious, so let us assume that $(p, q, s) \perp$[20]. Since $a \cap (s \cup p \cup q) \geqq q > 0$, it follows from the hypothesis that $(a, s \cup p \cup q) M$, whence we have

$$(p \cup a) \cap (s \cup p \cup q) = p \cup \{a \cap (s \cup p \cup q)\}.$$

The assumption $s \leqq p \cup a$ shows that

$$s \leqq p \cup \{a \cap (s \cup p \cup q)\},$$

so we have by upper continuity of $L$,[21]

$$s \leqq p \cup q \cup r_1 \cup \cdots \cup r_n \text{ for some } r_i \leqq a \cap (s \cup p \cup q) \ (i = 1, 2, \cdots, n).$$

Since $(p, q, s) \perp$, there exists at least one atomic element $r_k$ with $r_k \not\leqq p \cup q$, and so $(p, q, r_k) \perp$. Since $p, q, r_k \leqq s \cup p \cup q$, it follows that

$$p \cup q \cup r_k = p \cup q \cup s.\text{[22]}$$

Thus the $r_k$ is the desired one.

COROLLARY. *The lattice of all subspaces of an affine space $A$ is strongly planer.*

19) Cf. F. Maeda [2] 181 Definition 4.

20) $p \cap q = 0$, $(p \cup q) \cap s = 0$ imply $(p, q, s) \perp$. Cf. U Sasaki and S. Fujiwara [1] 184 Lemma 2 (1).

21) Cf. F. Maeda [1] 90 Lemma 1.3.

22) Cf. U. Sasaki and S. Fujiwara [1] 184 Lemma 2 (2).

This follows at once from Theorem 2.1, Remark 2.2, and Lemma 2.2.

THEOREM 2.2. *Let $\mathfrak{A}$ be a relatively atomic, upper continuous lattice which is semi-modular in the sense of Wilcox and satisfies the condition* ($\alpha$) *in Theorem* 2.1. *Then the set $A(\mathfrak{A})$ of all atomic elements of $\mathfrak{A}$, is an affine space, provided that we define as follows:*

$$p \vee q = \{r \in A(\mathfrak{A});\ r \leqq p \cup q\}$$
$$p \vee q \vee r = \{s \in A(\mathfrak{A});\ s \leqq p \cup q \cup r\}$$

*And $\mathfrak{A}$ is isomorphic to the lattice $\mathfrak{A}(A(\mathfrak{A}))$ of all subspaces of $A(\mathfrak{A})$.*

PROOF. (i) A.1, A.2 follows immediately from Remark 2.2 and 2.3 and A.3 follows from ($\alpha$).

(ii) In order to show A.4, it is sufficient to prove that if $p, q, r$ ; $p, s, t$ and $p_1, p_2, p_3, p_4$ are respectively independent sets and if

$$p \cup q \cup r,\ p \cup s \cup t \leqq p_1 \cup p_2 \cup p_3 \cup p_4,$$

then there exists an atomic element $s'$ with $s' \leqq (p \cup q \cup r) \frown (p \cup s \cup t)$ and $s' \neq p$. We can assume without loss of generality that $p, q, r, s$ are independent, whence we have

$$p \cup q \cup r \cup s = p_1 \cup p_2 \cup p_3 \cup p_4 .$$

Since $t \leqq s \cup p \cup q \cup r$, it follows from Lemma 2.2 that there exists an atomic element $s'$ with $t \leqq s \cup p \cup s'$, and $s' \leqq p \cup q \cup r$. Since $t \not\leqq s \cup p$, it follows that $s' \neq p$ and $s' \leqq p \cup s \cup t$, which is to be proved.

(iii) Next we shall show $\mathfrak{A} \cong \mathfrak{A}(A(\mathfrak{A}))$. Put

$$S(a) = \{p \in A(\mathfrak{A});\ p \leqq a\} \quad \text{for any} \quad a \in \mathfrak{A}, \quad \text{and}$$
$$a(S) = \bigvee(p;\ p \in S) \quad \text{for any} \quad S \in \mathfrak{A}(A(\mathfrak{A})).$$

Clearly $S(a) \in \mathfrak{A}(A(\mathfrak{A})$ and $a(S) \in \mathfrak{A}$. Since $\mathfrak{A}$ is relatively atomic, it follows at once that $a(S(a))=a$. While the upper continuity of $\mathfrak{A}$ and Lemma 2.2 show that $S(a(S))=S$. Therefore $a \to S(a)$, and $S \to a(S)$ are one-to-one correspondences between $\mathfrak{A}$ and $\mathfrak{A}(A(\mathfrak{A}))$ which preserve the order relation. Hence we have

$$\mathfrak{A} \cong \mathfrak{A}(A(\mathfrak{A})).$$

REMARK 2.4. (ii) of the proof of Theorem 2.2 shows that A.4′ implies A.4 under the conditions A.1-A.3. It follows from this result and Lemma 1.2 that the conditions A.1-A.4 are equivalent to A.1-A.3 together with A.4′. As to the direct proof of the equivalence, cf. Appendix.

THEOREM 2.3.[23] *An abstract lattice $L$ is isomorphic to the lattice of all subspaces of an affine space if and only if $L$ is relatively atomic, upper continuous lattice which is semi-modular in the sense of Wilcox and satisfies the following condition:*

($\alpha$) *Let $p, q, r$ be independent atomic elements of $L$, then there exists exactly one element $l$ such that*

$$p < l < p \cup q \cup r \quad \text{and} \quad l \cap (q \cup r) = 0 .$$

This is an immediate consequence of Theorem 2.1 and 2.2.

## APPENDIX

In this appendix we shall give a direct proof that the conditions A.1-A.4 are equivalent to A.1-A.3 and A.4'.

In Lemma 1.2, we have shown that A.1-A.4 imply A.4'. Hence it is sufficient to prove that conversely A.1-A.3 and A.4' imply A.4. We begin by listing the conditions we shall employ.

DEFINITION 1. Let $A$ be a set of points. Suppose that for any pair of distinct points $p, q$, there exists a subset $p \vee q$ (called *line* of $A$) containing $p, q$, and that for any triple $p, q, r$ of non-collinear points, there exists a subset $p \vee q \vee r$ (called *plane* of $A$) containing $p, q, r$, which satisfy the following conditions.

A.1. *Two distinct points on a line determine the line.*

A.2. *Three non-collinear points on a plane determine the plane.*

Two lines $p \vee q$, $r \vee s$ are called to be *parallel* to each other and are denoted by $p \vee q \parallel r \vee s$, provided that they are contained in the same plane and have no point in common.

A.3. *If $p, q, r$ are non-collinear points then there exists one and only one line $r \vee s$ such that $r \vee s$ is parallel to $p \vee q$.*

A.4'. $p \vee q \parallel x \vee y$, $x \vee y \parallel r \vee s$, *and* $r \notin p \vee x \vee y$ *imply* $p \vee q \parallel r \vee s$.

DEFINITION 2. By a *subspace* of $A$, we mean the subset $S$ such that $p, q \in S$ implies $p \vee q \subseteqq S$, and $p, q, r \in S$ implies $p \vee q \vee r \subseteqq S$.

Let $T$ be any subset of $A$. The smallest subspace containing $T$ is called the subspace *generated* by $T$. If four points $p, q, r, s$ are not on a

23) This may be formulated as follows:

*An abstract lattice $L$ is isomorphic to the lattice of all subspaces of an affine space if and only if $L$ is strongly planer matroid lattice satisfying the condition* ($\alpha$). Cf. the corollary to Lemma 2.2 and the proof of Theorem 2.2.

plane, the subspace generated by the set of these four points is called a *3-space* and is denoted by $p \vee q \vee r \vee s$.

It is our purpose to prove that $A$ satisfies the following condition.

A. 4. *If two planes contained in a 3-space have a common point, then they have at least one more point in common.*

We remark that Lemma 1. 1 and 1. 3, used in the subsequent arguments, have been proved by A. 1–A. 3 and A. 4′, without making use of A. 4, and so they are satisfied in $A$. We shall list them again.

LEMMA 1. 1. *If two distinct points $q, r$ are contained in a plane, then the line $q \vee r$ is contained in the plane.*

LEMMA 1. 3. $p \vee a \parallel p' \vee a'$, $p \vee b \parallel p' \vee b'$, $p \vee p' \parallel a \vee a'$, $p \vee p' \parallel b \vee b'$, *and* $p' \notin p \vee a \vee b$ *imply* $a \vee b \parallel a' \vee b'$.

Now let $p_1, p_2, p_3, p_4$ be four fixed points which are not on a plane, and let us denote by $p_1 \oplus (p_2, p_3, p_4)$ the set

$$\{p\,;\ p \in p_1 \vee p_2 \vee t,\quad t \in p_2 \vee p_3 \vee p_4\}.$$

LEMMA I. *Both the following two conditions are necessary and sufficient in order to* $p \in p_1 \oplus (p_2, p_3, p_4)$:

($\alpha$) $p \in p_1 \vee q \vee r$, *for some points* $q, r\ (\in p_2 \vee p_3 \vee p_4)$,

($\beta$) $p \in p_1 \vee s$, *for some point* $s\ (\in p_2 \vee p_3 \vee p_4)$, or
$p \vee p_1 \parallel p_2 \vee t$, *for some point* $t\ (\in p_2 \vee p_3 \vee p_4)$.

PROOF. First we shall show that ($\alpha$) implies ($\beta$).

Let us suppose that there exist two distinct points $q, r\ (\in p_2 \vee p_3 \vee p_4)$ such that $p \in p_1 \vee q \vee r$. Since the lines $p \vee p_1$ and $q \vee r$ are on a plane, they are either parallel to each other or intersect in a point $s$. In the former case, let $p_2 \vee t \parallel q \vee r$ (A. 3). Then it is clear that $t \in p_2 \vee p_3 \vee p_4$. While we have in view of A. 4′, $p \vee p_1 \parallel p_2 \vee t$, In the latter case, it holds $s \in p_2 \vee p_3 \vee p_4$ from Lemma 1. 1, and $p \in p_1 \vee s$, since $p, p_1, s$ are collinear. Thus it is proved that ($\alpha$) implies ($\beta$).

We can assert immediately from the definition that $p \in p_1 \oplus (p_2, p_3, p_4)$ implies ($\alpha$), and that ($\beta$) implies $p \in p_1 \oplus (p_2, p_3, p_4)$. Thus the conditions ($\alpha$), ($\beta$) and $p \in p_1 \oplus (p_2, p_3, p_4)$ are equivalent.

LEMMA II. $q \in p_1 \oplus (p_2, p_3, p_4)$, *and* $q \notin p_2 \vee p_3 \vee p_4$ *imply* $p_1 \oplus (p_2, p_3, p_4) = q \oplus (p_2, p_3, p_4)$.

PROOF. Let us suppose $q \in p_1 \oplus (p_2, p_3, p_4)$, and $q \notin p_2 \vee p_3 \vee p_4$. We shall show first

$$p_1 \oplus (p_2, p_3, p_4) \subseteqq q \oplus (p_2, p_3, p_4).$$

Let $r \in p_1 \oplus (p_2, p_3, p_4)$. Then, since also $q \in p_1 \oplus (p_2, p_3, p_4)$, the following cases occur in view of Lemma I ($\beta$):

*Case* 1. $q \in p_1 \vee q'$, $r \in p_1 \vee r'$, where $q', r' \in p_2 \vee p_3 \vee p_4$.

We can assume that $q, q', r$ are non-collinear. Since $p_1 \in q \vee q'$ by A. 1, it holds $p_1 \in q \vee q' \vee r'$ by Lemma 1. 1, and it follows from $r \in p_1 \vee r'$ that $r \in q \vee q' \vee r'$, again by Lemma 1. 1. This proves in view of Lemma 1 ($\alpha$), $r \in q \oplus (p_2, p_3, p_4)$.

*Case* 2. $q \in p_1 \vee q'$, and $p_1 \vee r \parallel p_2 \vee r'$ where $q', r' \in p_2 \vee p_3 \vee p_4$, or
$r \in p_1 \vee r'$, and $p_1 \vee q \parallel p_2 \vee q'$ where $q', r' \in p_2 \vee p_3 \vee p_4$.

It is sufficient to prove the former, since the proof is similar.

Let us assume $q' \notin p_2 \vee r'$, since the proof is analogous even if $q' \in p_2 \vee r'$. Then there exists a point $s$ such that $q' \vee s \parallel p_2 \vee r'$ by A. 3, and clearly $s \in p_2 \vee p_3 \vee p_4$. While in view of A. 4′, we have $p_1 \vee r \parallel q' \vee s$, whence $r \in p_1 \vee q' \vee s$. Since $q, q', s$ are non-collinear points on $p_1 \vee q \vee s$, it holds by A. 2, that $p_1 \vee q' \vee s = q \vee q' \vee s$, whence $r \in q \vee q' \vee s$, which proves $r \in q \oplus (p_2, p_3, p_4)$ by Lemma 1 ($\alpha$).

*Case* 3. $p_1 \vee q \parallel p_2 \vee q'$, $p_1 \vee r \parallel p_2 \vee r'$, where $q', r' \in p_2 \vee p_3 \vee p_4$.

We can assume without loss of generality that $p_1 \vee p_2 \parallel q \vee q'$, and $p_1 \vee p_2 \parallel r \vee r'$. It follows at once from Lemma 1. 3 that $q \vee r \parallel q' \vee r'$, whence $r \in q \vee q' \vee r'$, which proves $r \in q \oplus (p_2, p_3, p_4)$ by Lemma I ($\alpha$).

Thus it has been proved that $p_1 \oplus (p_2, p_3, p_4) \subseteq q \oplus (p_2, p_3, p_4)$. The converse inequality may be proved in similar arguments.

LEMMA III. *The set* $p_1 \oplus (p_2, p_3, p_4)$ *is the 3-space* $p_1 \vee p_2 \vee p_3 \vee p_4$.

PROOF. First we shall show that $p_1 \oplus (p_2, p_3, p_4)$ is a subspace. For the purpose, it is sufficient to prove the following two propositions (a) and (b) by the definition of the subspace.

(a) $q_1, q_2 \in p_1 \oplus (p_2, p_3, p_4)$, *and* $q \in q_1 \vee q_2$ *imply* $q \in p_1 \oplus (p_2, p_3, p_4)$.

We may assume that either $q_1$ or $q_2$ (say $q_1$) is not on the plane $p_2 \vee p_3 \vee p_4$, since otherwise the result is obvious from Lemma 1. 1. It follows from Lemma II that $p_1 \oplus (p_2, p_3, p_4) = q_1 \oplus (p_2, p_3, p_4)$, whence it holds $q_2 \in q_1 \oplus (p_2, p_3, p_4)$. So we have in view of Lemma I ($\alpha$), $q_2 \in q_1 \vee r \vee s$ for some $r, s \in p_2 \vee p_3 \vee p_4$. Since $q \in q_1 \vee q_2$, it holds $q \in q_1 \vee r \vee s$, which proves the result in view of Lemma I ($\alpha$).

(b) $q_1, q_2, q_3 \in p_1 \oplus (p_2, p_3, p_4)$, *and* $q \vee q_1 \parallel q_2 \vee q_3$ *imply* $q \in p_1 \oplus (p_2, p_3, p_4)$.

We may assume without loss of generality that either $q_2$ or $q_3$ (say $q_3$)

are not on $p_2 \vee p_3 \vee p_4$. It follows from Lemma II,

$$p_1 \oplus (p_2, p_3, p_4) = q_3 \oplus (p_2, p_3, p_4),$$

whence $q_1, q_2 \in q_3 \oplus (p_2, p_3, p_4)$. So the following cases occur:

*Case* 1 a). $q_2 \in q_3 \vee r_2$, and $q_1 \in q_3 \vee r_1$, where $r_1, r_2 \in p_2 \vee p_3 \vee p_4$.

Since $q_1 \in q_3 \vee r_1$, it holds $q_1 \in q_3 \vee r_1 \vee r_2$. It follows from $q \vee q_1 \parallel q_3 \wedge r_2$, $q \in q_3 \vee r_1 \vee r_2$, which proves the result in view of Lemma I ($\alpha$).

*Case* 1 b). $q_2 \in q_3 \vee r_2$, and $q_1 \vee q_3 \parallel p_2 \vee r_2$, where $r_1, r_2 \in p_2 \vee p_3 \vee p_4$.

Let $s$ be a point with $s \vee r_2 \parallel p_2 \vee r_1$ (A. 3), then clearly $s \in p_2 \vee p_3 \vee p_4$. While we have in view of A. 4′, $q_1 \vee q_3 \parallel r_2 \vee s$, whence it holds $q_1 \in q_3 \vee r_2 \vee s$. Since $q_1 \vee q \parallel q_3 \vee r_2$, it is evident $q \in r_2 \vee r_2 \vee s$, which shows the result.

*Case* 2. $q_3 \vee q_2 \parallel p_2 \vee r_2$, where $r_2 \in p_2 \vee p_3 \vee p_4$.

We may assume $q_1 \notin p_2 \vee p_3 \vee p_4$, since otherwise the result is obvious. It follows from Lemma II, $q_1 \oplus (p_2, p_3, p_4) = p_1 \oplus (p_2, p_3, p_4)$.

Since $q \vee q_1 \parallel q_2 \vee q_3$, and $q_2 \vee q_3 \parallel p_2 \vee r_2$, it holds $q \vee q_1 \parallel p_2 \vee r_2$, whence $q \in q_1 \vee p_2 \vee r_2$, which proves the result.

Thus it has been proved that $p_1 \oplus (p_2, p_3, p_4)$ is a subspace.

While it is clear that $p_1 \oplus (p_2, p_3, p_4)$ contains the points $p_1, p_2, p_3$, and $p_4$, and that it is contained in any subspace containing these four points. So we have $p_1 \oplus (p_2, p_3, p_4) = p_1 \vee p_2 \vee p_3 \vee p_4$.

COROLLARY. *Four non-coplaner points on a 3-space determine the 3-space.*

PROOF. Let $p_1, p_2, p_3$, and $p_4$ be non-coplaner points. Let us suppose that the points $p, q, r$, and $s$ are non-coplaner and are contained in the 3-space $p_1 \vee p_2 \vee p_3 \vee p_4$. At least one of the points $p, q, r, s$, (say $p$) is not on the plane $p_2 \vee p_3 \vee p_4$, it follows from Lemma II, and III, $p_1 \vee p_2 \vee p_3 \vee p_4 = p \vee p_2 \vee p_3 \vee p_4$. Repeated applications of this argument show that $p_1 \vee p_2 \vee p_3 \vee p_4 = p \vee q \vee r \vee s$.

*Now we shall show* A. 4.

Let $p \vee q \vee r$ and $p \vee s \vee t$ be two planes contained in a 3-space $p_1 \vee p_2 \vee p_3 \vee p_4$. We can assume $s, t \notin p \vee q \vee r$, since otherwise the result is obvious. It follows from the above corollary,

$$p_1 \vee p_2 \vee p_3 \vee p_4 = s \vee p \vee q \vee r.$$

Hence it holds $t \in s \vee p \vee q \vee r$. It follows from Lemma III and the definition that there exists a point $u$ such that $t \in s \vee p \vee u$, and $u \in p \vee q \vee r$. While $u$ and $p$ are distinct, since otherwise the point $t$ would be on the line $p \vee s$.

Thus the two planes $p \vee q \vee r$ and $p \vee s \vee t$ have a common point distinct from $p$.

In conclusion, we obtain the following

THEOREM. *Under the conditions* A. 1-A. 3, *the two conditions* A. 4 *and* A. 4′ *are equivalent to each other.*

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