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Generalization of Poincaré-Bendixson Theorem

By

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Introduction.

Given the system of the differential equations

$$(E) \qquad \frac{dx}{dt}=X(x,y), \qquad \frac{dy}{dt}=Y(x,y).$$

To this system corresponds a vector field $F=(X, Y)$ in the phase plane of the variables x and y. Then, on the existence of limit-cycles of (E), Poincaré-Bendixson theorem asserts that, if two concentric Bendixson curves C_1 and C_2 which bound a closed region R free of critical points are crossed in opposite senses by the field vectors, then there exists at least one limit-cycle lying in R. In this paper, generalizing this theorem, we shall show that, *even if the boundary curves contact in some points with the field vectors, the same conclusion is also valid.*

We assume that the boundary simply closed curves C_1 (outer) and C_2 (inner) are continuously differentiable up to the second order(1) and that the functions $X(x,y)$ and $Y(x,y)$ are continuous with their derivatives in the open region G which contains the closed region R. Then our theorem will assert that, *if R does not contain critical points and C_1 and C_2 are crossed in opposite senses or touched by the field vectors, then there exists at least one limit-cycle in R.*

§ 1. Variation of the inclinations.

In this paragraph, we denote any one of C_1 and C_2 by C. Let $x=x(s)$, $y=y(s)$ be the equations of C, where s is the length of the arc. Let the direction cosines of the tangent and the normal of C in any point $P(x,y)$ be (α, β) and (l, m) respectively. We make the convention that the positive sense of the normal is such that the tangent and the normal have the positive orientation. Then, for these direction cosines, it is evidently valid that

$$(1.1) \qquad \begin{vmatrix} \alpha & \beta \\ l & m \end{vmatrix} = +1.$$

1) When the equations of the curve are given in the form $x=x(s)$, $y=y(s)$, where s is the length of the arc, we say that the curve is continuously differentiable up to the n-th order when the functions of the right-hand sides of the equations are continuously differentiable with regard to s up to the n-th order.

We take a point $P_1(x_1, y_1)$ on the normal in P so that $PP_1=\varepsilon\rho$, where ε is a small positive number, and we denote the locus of the point P_1 by C'. Then, for any point of C', it is valid that

$$\begin{cases} x_1=x+\varepsilon\rho l\,, \\ y_1=y+\varepsilon\rho m\,. \end{cases}$$

Differentiating both sides of these equations with regard to s, we have:

$$\begin{cases} \dot{x}_1=\alpha+\varepsilon(\dot{\rho}l+\rho\kappa\alpha)\,, \\ \dot{y}_1=\beta+\varepsilon(\dot{\rho}m+\rho\kappa\beta)\,,^{(1)} \end{cases}$$

where κ is a curvature with the suitable sign. Hence the direction cosines (α_1, β_1) of the tangent of C' become

$$\begin{cases} \alpha_1=\alpha+\delta\alpha=\alpha+\varepsilon(\dot{\rho}l+\rho\kappa\alpha)/1+\varepsilon\rho\kappa=\alpha+\varepsilon\dot{\rho}l\,, \\ \beta_1=\beta+\delta\beta=\beta+e(\dot{\rho}m+\rho\kappa\beta)/1+\varepsilon\rho\kappa=\beta+\varepsilon\dot{\rho}m\,.^{(2)} \end{cases}$$

Thus the variations $(\delta\alpha, \delta\beta)$ of the direction cosines of the tangent become

$$\text{(1.2)} \qquad \begin{cases} \delta\alpha=\varepsilon\dot{\rho}l\,, \\ \delta\beta=\varepsilon\dot{\rho}m\,. \end{cases}$$

For the field vector in $P_1(x_1, y_1)$, we have:

$$\begin{cases} X(x+\varepsilon\rho l, y+\varepsilon\rho m)=X(x,y)+\varepsilon\rho\bar{X}\,, \\ Y(x+\varepsilon\rho l, y+\varepsilon\rho m)=Y(x,y)+\varepsilon\rho\bar{Y}\,, \end{cases}$$

where

$$\text{(1.3)} \qquad \begin{cases} \bar{X}=lX_x+mX_y\,, \\ \bar{Y}=lY_x+mY_y\,. \end{cases}$$

Consequently the variation of the field vector becomes

$$\text{(1.4)} \qquad \begin{cases} \delta X=\varepsilon\rho\bar{X}\,, \\ \delta Y=\varepsilon\rho\bar{Y}\,. \end{cases}$$

Now let the inclinations of the field vectors from C and C' measured in positive sense be θ and $\theta_1=\theta+\delta\theta$ respectively. Then

$$\tan\theta=\frac{\dfrac{Y}{X}-\dfrac{\beta}{\alpha}}{1+\dfrac{Y}{X}\cdot\dfrac{\beta}{\alpha}}=\frac{\alpha Y-\beta X}{\alpha X+\beta Y}\,.$$

Consequently, differentiating both sides, we have:

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1) The dots over the letters mean differentiation with regard to s.

2) We have neglected the terms of the second and higher orders of ε. In the following we make use of this convention.

$$\delta\theta = \frac{X\cdot\delta Y - \delta X\cdot Y}{X^2+Y^2} - (\alpha\cdot\delta\beta - \delta\alpha\cdot\beta).$$

Substituting (1.2) and (1.4) in the right hand side, we have:

$$\delta\theta = -(\alpha m - \beta l)\varepsilon\dot{\rho} + \frac{X\bar{Y} - \bar{X}Y}{X^2+Y^2}\varepsilon\rho.$$

Thus, by means of (1.1), we obtain:

(1.5)
$$\delta\theta = -\varepsilon\left[\dot{\rho} + \rho\frac{\bar{X}Y - X\bar{Y}}{X^2+Y^2}\right].$$

§ 2. Deformation of the boundary curves.

For any given functions $\Theta_1(s)$ and $\Theta_2(s)$, we consider the linear differential equations as follows:

(2.1)
$$\dot{\rho}_i + \rho_i\frac{\bar{X}Y - X\bar{Y}}{X^2+Y^2} = \Theta_i, \quad (i=1, 2).$$

Then the solutions of these equations such that $\rho(0)=0$ are given as follows:

(2.2)
$$\rho_i(s) = e^{-\phi(s)}\int_0^s \Theta_i e^{\phi(s)} ds,$$

where

(2.3)
$$\phi(s) = \int_0^s \frac{\bar{X}Y - X\bar{Y}}{X^2+Y^2} ds.$$

Now, by our assumption, for C_1, it is valid that $0 \leqq \theta \leqq \pi$. Corresponding to C_1, we construct the curve C_1' so that $\rho(s)$ becomes $\rho_1(s)$ of (2.2) with $\Theta_1 = \theta - \frac{\pi}{2}$. Then, for such C_1', from (1.5), it follows that

(2.4)
$$\delta\theta = -\varepsilon\Theta_1 = \varepsilon\left(\frac{\pi}{2} - \theta\right).$$

Consequently, for sufficiently small ε, it follows that

$$0 < \varepsilon\frac{\pi}{2} \leqq \theta + \delta\theta \leqq \pi - \varepsilon\frac{\pi}{2} < \pi.$$

Namely, for the inclination θ_1 of the field vector from C_1', it is valid that

(2.5)
$$0 < \theta_1 < \pi.$$

Similarly, corresponding to C_2, the curve C_2' is constructed so that $\rho(s)$ becomes $\rho_2(s)$ of (2.2) with $\Theta_2 = \theta + \frac{\pi}{2}$. Then, as in the former case, for C_2', it holds that

(2.6) $$\delta\theta=\varepsilon\left(-\frac{\pi}{2}-\theta\right),$$

and, for the inclination θ_1 of the field vector from C_2', it is valid that

(2.7) $$-\pi<\theta_1<0.$$

Now, let the periphery of C_i be L_i. Then, from our assumption, for $0\leqq s\leqq L_i$, $\phi(s)$ and Θ_i's are continuous, consequently $\rho_i(s)$'s determined by (2.2) are bounded there. Hence, for both C_i''s, there exists a fixed positive number K such that

(2.8) $$|\rho_i(s)|\leqq K \quad \text{for} \quad 0\leqq s\leqq L_i.$$

Then, for sufficiently small ε, the curve C_i' lies in the εK-neighborhood of the curve C_i, consequently C_i' lies in G and moreover is free of critical points. Thus we see that there exists really the curve C_i' for which (2.5) or (2.7) holds. From (2.8), it is evident that $C_i'\to C_i$ as $\varepsilon\to 0$.

§ 3. Existence of the limit-cycles.

When both the curves C_i's do not contact in any point with the field vectors, our theorem is evidently valid by Poincaré-Bendixson theorem.

In the following, we consider the case where at least one of curves C_i's contacts in some points with the field vectors. When the curve C_i contacts in some points with the field vectors, by the method of the preceding paragraph we make the curve C_i' taking one of the contact points with the field vectors as the starting point for measuring the length of the arc. Let this starting point be A_i. Then, when the point P moves along the curve C_i in the positive sense starting from A_i, the corresponding point P_1 moves along the curve C_i' starting from A_i, and when P returns to A_i after one round, the corresponding point P_1 reaches the certain point A_i' on the curve C_i'. Then it is evident that A_i' lies on the normal of C_i in A_i and moreover $A_iA_i'=\varepsilon\rho_iL_i$.

Let the curve composed of the segment A_iA_i' and of the arc A_iA_i' of C_i' be $\bar{C}_i'$. Then it is proved that, for sufficiently small ε, the curve $\bar{C}_i'$ is simply closed. For, if there exists a multiple point on $\bar{C}_i'$, then it must be an intersection of the normals of C_i in at least two different points. We consider the sequence $\{\varepsilon_n\}$ of ε such that $\varepsilon_n\to 0$ as $n\to\infty$, and we assume that, for each n, there exist two points Q_n and Q_n' on C_i such that the normals in Q_n and Q_n' intersect in a point S_n which lies from Q_n and Q_n' in the distance smaller than ε_nK. Then each set of $\{Q_n\}$ and $\{Q_n'\}$ has at least one point of accumulation. Then, for a suitable subsequence $\{m\}$ of $\{n\}$, there exist Q and Q' such that $Q_mQ_m'<2\varepsilon_mK$ and $Q_m\to Q$, $Q_m'\to Q'$ as $m\to\infty$. Consequently $QQ'\to 0$ as $m\to\infty$. Therefore it must be $Q=Q'$. Now, by our assumption, C_i is simply closed, consequently S_m tends to the center of curvature at Q as $m\to\infty$. Now $Q_mS_m\to 0$ as $m\to\infty$, therefore the curvature of C_i at Q must be infinity. This contradicts our assumption on differentiability of C_i. Thus we see that, for sufficiently small ε, the curve $\bar{C}'_i$ is simply closed.

By our assumption, C_1 and C_2 do not intersect each other, consequently, for sufficiently small δ, there exist the disjoint δ-neighborhoods of C_1 and C_2. Now $C'_i \to C_i$ for $0 \leqq s \leqq L_i$ as $\varepsilon \to 0$, consequently, for sufficiently small ε, $\bar{C}'_i$ lies in the δ-neighborhood of C_i. Therefore, for such ε, $\bar{C}'_1 \frown \bar{C}'_2$, $\bar{C}'_1 \frown C_2$ and $C_1 \frown \bar{C}'_2$ are all empty. When one of the curves C_1 and C_2 does not contact in any point with the field vectors, we adopt the curve C_i itself as $\bar{C}'_i$. Then, by the above discussion, we see that $\bar{C}'_1$ and $\bar{C}'_2$ bound a closed region R' which tends to R as $\varepsilon \to 0$.

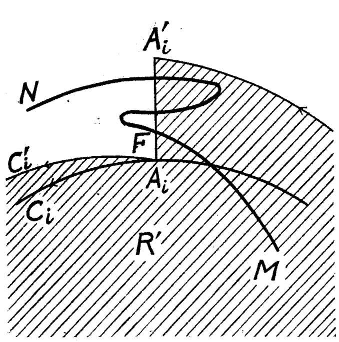
We consider the characteristic passing through the point A_i. From (2.4), and (2.6), in A_i, it follows that

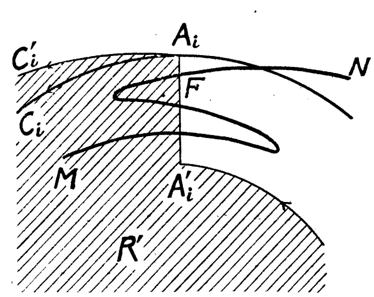
(i) $$\text{when } \theta=0, \quad \theta_1=\pm\varepsilon\frac{\pi}{2};$$

(ii) $$\text{when } \theta=\pm\pi, \quad \theta_1=\pm\left(1-\frac{\varepsilon}{2}\right)\pi.$$

Namely, in A_i, the field vector crosses C'_i prolonged backwards beyond A_i from the exterior side of R' to its interior side. Consequently, for sufficiently small ε, when $\theta=0$, the characteristic followed from A_i goes into the interior of R', and when $\theta=\pm\pi$, the characteristic followed backward from A_i goes into the exterior of R'. Thus we see that the characteristic can never cross $\bar{C}'_i$ in A_i from the interior of R' to its exterior.

We take a point M in the interior of the region R and consider the characteristic C starting from M. We assume that, for sufficiently large t, C reaches the point N in the exterior of R. Then, for sufficiently small ε, M lies in the interior of R' and N lies in the exterior of R'. Now, when the curve C_i does not contact in any point with the field vector, it is valid that, on any point of C_i, either $0<\theta<\pi$ or $-\pi<\theta<0$. Consequently C followed from M can not go out of R' meeting such $C_i=\bar{C}'_i$. Therefore C followed from M must go out of R' meeting such $\bar{C}'_i$ that the corresponding C_i contacts in some points with the field vectors. Let the first or last point of crossing or touching $\bar{C}'_i$ from the interior of R' to its exterior





be F. Now, on any point of C_i', it is valid that either $0<\theta_1<\pi$ or $-\pi<\theta_1<0$. Consequently F can not lie on C_i' except for A_i and A_i', namely F must lie on A_iA_i'. From the property of the point A_i, it must be $F \neq A_i$. Then, by taking sufficiently small value ε_0 of ε, we can make F lie in the exterior or interior of the closed region R'' where R'' is the region R' corresponding to ε_0. Let the point A_i' corresponding to ε_0 be A_i''. Then, again, the arc MN of C must cross the segment A_iA_i'' in at least one point, say in F', from the interior of R'' to its exterior, and it must be $F' \neq A_i$. Now, from the way of constructing C_i', it is readily seen that the interior and exterior sides of R'' with regard to A_iA_i'' are the same as those of R' with regard to A_iA_i' respectively. Then, it follows that the arc MN of C crosses the segment A_iA_i' in F' from the interior of R' to its exterior. This contradicts the assumption that F is the first or last point of crossing the segment A_iA_i' from the interior of R' to its exterior. Thus we see that, when C is followed from M, C can never go out of the region R.

Then, as in the proof of Poincaré-Bendixson theorem, we can conclude by means of Poincaré theorem[1] that R contains at least one limit-cycle.

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1) The Poincaré theorem quoted here runs as follows: if the closed bounded region free of critical points contains a half-path, then it contains a limit-cycle.