JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 17, NO. DCE., 1953

A Note on Semi-Local Rings

By

Michio YOSHIDA and Motoyoshi SAKUMA

(Received July 31, 1953)

The object of this note is to give the complete information on the indecomposable components of the completions of semi-local rings.

Let $\mathfrak{o}$ be a commutative ring, and let $\mathfrak{m}$ be an ideal in $\mathfrak{o}$ such that $\bigcap_{n=1}^{\infty}\mathfrak{m}^n=(0)$. The metrisable, uniform structure, defined in $\mathfrak{o}$ by adopting the set $\{\mathfrak{m}^n;\ n=1,2,\cdots\}$ as a fundamental system of neighbourhoods of zero, shall be called an $\mathfrak{m}$-adic topology. If we give this topology to $\mathfrak{o}$, it becomes a topological ring and we shall call thus topologized ring an $\mathfrak{m}$-adic ring.

The completion of an $\mathfrak{m}$-adic ring $\mathfrak{o}$ shall be called an $\mathfrak{m}$-adic completion of $\mathfrak{o}$, and shall be denoted by $\bar{\mathfrak{o}}$. If we denote by $\overline{\mathfrak{m}^\sigma}$ the adherence of $\mathfrak{m}^\sigma$ in $\bar{\mathfrak{o}}$, the set $\{\overline{\mathfrak{m}^\sigma};\ \sigma=1,2,\cdots\}$ is a fundamental system of neighbourhoods of zero in $\bar{\mathfrak{o}}$. If $\mathfrak{m}$ has a finite base, then we have $\overline{\mathfrak{m}^\sigma}=\bar{\mathfrak{m}}^\sigma$, and $\bar{\mathfrak{o}}$ is an $\bar{\mathfrak{m}}$-adic ring. If moreover $\mathfrak{o}$ has a unit element, then $\bar{\mathfrak{m}}=\mathfrak{m}\bar{\mathfrak{o}}$. If $\mathfrak{o}$ is a Noetherian ring (that is, a commutative ring with the maximal condition for ideals) with a unit element, so is $\bar{\mathfrak{o}}$ too.

DEFINITION (D). Let $\mathfrak{m}$, $\mathfrak{m}_1$ be ideals in $\mathfrak{o}$, such that $\mathfrak{m}\subseteq\mathfrak{m}_1$, and $\bigcap\mathfrak{m}^n=(0)$. Set $\bigcap\mathfrak{m}_1{}^n=\mathfrak{m}_1{}^\infty$, $\mathfrak{o}/\mathfrak{m}_1{}^\infty=\mathfrak{o}_1'$, $\mathfrak{m}_1/\mathfrak{m}_1{}^\infty=\mathfrak{m}_1'$, and let $\bar{\mathfrak{o}}$, $\bar{\mathfrak{o}}_1$ be the $\mathfrak{m}$-adic and the $\mathfrak{m}_1'$-adic completion of $\mathfrak{o}$ and $\mathfrak{o}_1'$ respectively. Let x^* be any element in $\bar{\mathfrak{o}}$, and let $x^*=\lim x_n$ ($x_n\in\mathfrak{o}$). Then if we denote by x'_n the residue class modulo $\mathfrak{m}_1{}^\infty$ which contains x_n, $\{x'_n\}$ is a Cauchy sequence in the $\mathfrak{m}_1'$-adic ring $\mathfrak{o}_1'$. If we denote by x^{**} the limit of $\{x'_n\}$ in $\bar{\mathfrak{o}}_1$, then the mapping $\tau_1: x^*\to x^{**}$ is clearly a homomorphism (that is, a continuous ring-homomorphism) of $\bar{\mathfrak{o}}$ into $\bar{\mathfrak{o}}_1$. This shall be called the canonical homomorphism of $\bar{\mathfrak{o}}$ into $\bar{\mathfrak{o}}_1$.

Now, let $\mathfrak{m}_i$ ($i=1,2,\cdots,r$) be ideals in $\mathfrak{o}$ such that $\mathfrak{m}\subseteq\mathfrak{m}_i$, and define $\bar{\mathfrak{o}}_i$, τ_i similarly as $\mathfrak{o}_1'$, τ_1. Then the mapping τ of $\bar{\mathfrak{o}}$ into the direct sum $\tilde{\mathfrak{o}}$ of $\bar{\mathfrak{o}}_1,\cdots,\bar{\mathfrak{o}}_r$ defined by setting $\tau x^*=\tau_1x^*+\cdots+\tau_rx^*$, shall be called the canonical homomorphism of $\bar{\mathfrak{o}}$ into $\tilde{\mathfrak{o}}$.

THEOREM I. *Let $\mathfrak{o}$ be a commutative ring with a unit element, and let $\mathfrak{m}$ be an ideal in $\mathfrak{o}$ such that $\bigcap\mathfrak{m}^n=(0)$. Suppose that*

$$\mathfrak{m}=\mathfrak{m}_1\cap\cdots\cdots\cap\mathfrak{m}_r,$$

where $\mathfrak{m}_i$ ($i=1,2,\cdots,r$) are ideals in $\mathfrak{o}$, such that $(\mathfrak{m}_i,\mathfrak{m}_j)=(1)$ for $i\neq j$. Then (with the same notations as in (D)), the canonical homomorphism τ is an isomorphism of $\bar{\mathfrak{o}}$ onto $\tilde{\mathfrak{o}}$.

PROOF. We shall first prove that τ is a mapping on $\tilde{\mathfrak{o}}$. Let x_i^* be any element in $\bar{\mathfrak{o}}_i$, and let $x_i^*=\lim x_\nu''$ ($x_\nu''\in\mathfrak{o}_i'$). Let x'_ν be any element in the residue class x_ν'', then there exists an element x_ν in $\mathfrak{o}$ such that

$$x_\nu \equiv 0 \quad \text{mod. } \mathfrak{m}_j^\nu \quad \text{for } j \neq i,$$

$$x_\nu \equiv x'_\nu \quad \text{mod. } \mathfrak{m}_i^\nu .$$

Since we have $\mathfrak{m}^n = \mathfrak{m}_1^n \cap \cdots \cap \mathfrak{m}_r^n$ for any positive integer n, $\{x_\nu\}$ is a Cauchy sequence in $\mathfrak{o}$. Let x^* be the limit of x_ν in $\bar{\mathfrak{o}}$, then we have $\tau x^* = x_i^*$.

We shall next show that τ is one-to-one. Assume $\tau x^*=0$, and let $x^* = \lim x_\nu$ $(x_\nu \in \mathfrak{o})$. Let $x_{i,\nu}$ denote the residue class modulo $\mathfrak{m}_i^\infty$ which contains x_ν, and let x_i^* be the limit of $x_{i,\nu}$ in $\bar{\mathfrak{o}}_i$. Then our assumption says that $x_i^*=0$, that is $x_{i,\nu} \in \mathfrak{m}_i'^{\sigma_i(\nu)}$ with $\lim_{\nu\to\infty} \sigma_i(\nu) = \infty$, hence $x_\nu \in \mathfrak{m}_i^{\sigma_i(\nu)}$, whence follows $x^*=0$.

Finally we shall show that the inverse mapping τ^{-1} is continuous. Set $U_\rho = \overline{\mathfrak{m}_1'^\rho} + \cdots + \overline{\mathfrak{m}_r'^\rho}$, then $\{U_\rho;\ \rho=1,2,\cdots\}$ is a fundamental system of the neighbourhoods of zero in $\tilde{\mathfrak{o}}$. Assume that τ^{-1} is not continuous at zero. Then for some positive integer σ, there exists x_ρ^* in $\bar{\mathfrak{o}}$ for every positive integer ρ, such that

$$x_\rho^* \notin \overline{\mathfrak{m}^\sigma}$$

$$\tau x_\rho^* \in U_\rho .$$

Since τ is continuous and U_ρ is open, we may assume that $x_\rho^* \in \mathfrak{o}$, whence

$$\tau x_\rho^* \in U_\rho \cap (\mathfrak{o}'_1 + \cdots + \mathfrak{o}'_r) = (\overline{\mathfrak{m}_1'^\rho} \cap \mathfrak{o}'_1) + \cdots + (\overline{\mathfrak{m}_r'^\rho} \cap \mathfrak{o}'_r) = \mathfrak{m}_1'^\rho + \cdots + \mathfrak{m}_r'^\rho .$$

Hence $\tau_i x_\rho^* \in \mathfrak{m}_i'^\rho$, that is $x_\rho^* \in \mathfrak{m}_i^\rho$. Therefore we have $x_\rho^* \in \mathfrak{m}^\rho = \mathfrak{m}_1^\rho \cap \cdots \cap \mathfrak{m}_r^\rho$, which is a contradiction. q. e. d.

Let $\mathfrak{o}$ be a commutative ring with a unit element and let $\mathfrak{m}$ be an ideal in $\mathfrak{o}$. $\mathfrak{m}$ is called dually indecomposable, if $\mathfrak{m} \neq \mathfrak{o}$ and there exist no ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$ in $\mathfrak{o}$ such that

$$\mathfrak{m} = \mathfrak{m}_1 \cap \mathfrak{m}_2, \qquad \mathfrak{m}_1 + \mathfrak{m}_2 = \mathfrak{o}, \qquad \mathfrak{m}_1 \neq \mathfrak{o}, \qquad \mathfrak{m}_2 \neq \mathfrak{o}.$$

$\mathfrak{o}$ is called directly indecomposable, if there exists no ideals $\mathfrak{a}_1$, $\mathfrak{a}_2$ in $\mathfrak{o}$ such that

$$\mathfrak{o} = \mathfrak{a}_1 + \mathfrak{a}_2 \ \ (\text{direct sum}), \qquad \mathfrak{a}_1 \neq (0),\ \ \mathfrak{a}_2 \neq (0).$$

With this terminologies, $\mathfrak{o}/\mathfrak{m}$ is directly indecomposable if and only if $\mathfrak{m}$ is dually indecomposable.

THEOREM II. *Let $\mathfrak{o}$ be a commutative ring with a unit element, and let $\mathfrak{m}$ be an ideal in $\mathfrak{o}$ such that $\cap \mathfrak{m}^n = (0)$. Then if $\mathfrak{m}$ is dually indecomposable, the $\mathfrak{m}$-adic completion of $\mathfrak{o}$ is directly indecomposable.*

PROOF. Assume that $\bar{\mathfrak{o}}$ is directly decomposable, and set $\bar{\mathfrak{o}} = \mathfrak{o}_1 + \mathfrak{o}_2$ where $\mathfrak{o}_1$ and $\mathfrak{o}_2$ are ideals not equal to (0). Then, we have $\bar{\mathfrak{o}}/\bar{\mathfrak{m}} = (\mathfrak{o}_1 + \bar{\mathfrak{m}})/\bar{\mathfrak{m}} + (\mathfrak{o}_2 + \bar{\mathfrak{m}})/\bar{\mathfrak{m}}$ (direct sum). If $\mathfrak{o}_1 \subseteq \bar{\mathfrak{m}}$, then $\mathfrak{o}_1 = \mathfrak{o}_1^\rho \subseteq \bar{\mathfrak{m}}^\rho \subseteq \overline{\mathfrak{m}^\rho}$ for any positive integer ρ. Hence we have $\mathfrak{o}_1 = (0)$ by the relation $\cap \overline{\mathfrak{m}^\rho} = (0)$, which is a contradiction. On the other hand we have $\bar{\mathfrak{o}}/\bar{\mathfrak{m}} \cong \mathfrak{o}/\mathfrak{m}$, which shows that $\mathfrak{m}$ is dually decomposable. q. e. d.

From now on we shall confine our consideration to the case where $\mathfrak{o}$ is

Noetherian. If $\mathfrak{o}$ has a unit element, any ideal $\mathfrak{m}(\neq\mathfrak{o})$ in $\mathfrak{o}$ is represented uniquely as an intersection of such ideals $\mathfrak{m}_i$ $(i=1,2,\cdots,r)$ that $(\mathfrak{m}_i, \mathfrak{m}_j)=(1)$ for $i\neq j$, $\mathfrak{m}_i\neq\mathfrak{o}$ $(i=1,2,\cdots,r)$, and $\mathfrak{m}_i$ is dually indecomposable. Dually $\mathfrak{o}$ is, as is well known, uniquely represnted as a direct sum of directly indecomposable ideals which we shall call the indecomposable components of $\mathfrak{o}$. Let $\mathfrak{o}$ be an $\mathfrak{m}$-adic ring, $\bar{\mathfrak{o}}$ its completion, then Theorems I and II give us the complete information on indecomposable components of $\bar{\mathfrak{o}}$. This is the main result we wanted to establish in this note.

Let $\mathfrak{m}_i$ $(i=1,2)$ be ideals in $\mathfrak{o}$ such that $\bigcap\mathfrak{m}_i^n=(0)$ $(i=1,2)$. Then in order that the $\mathfrak{m}_1$-adic and the $\mathfrak{m}_2$-adic topologies coincide, it is necessary and sufficient that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ have the same radical. Hence in the consideration of $\mathfrak{m}$-adic Noetherian rings we may assume, without loss of generality, that $\mathfrak{m}$ is semi-prime.

Such being the case, let $\mathfrak{m}$ be semi-prime and let $\mathfrak{m}=\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_r$ be the irredundant representation of $\mathfrak{m}$. We divide the set $\{\mathfrak{p}_1,\cdots,\mathfrak{p}_r\}$ into classes such that $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are in the same class if and only if there exists a chain $\mathfrak{p}_i, \mathfrak{p}_{\nu_1},\cdots,\mathfrak{p}_{\nu_s},\mathfrak{p}_j$, connecting $\mathfrak{p}_i$ and $\mathfrak{p}_j$ and satisfying the conditions $(\mathfrak{p}_i,\mathfrak{p}_{\nu_1})\neq(1)$, $(\mathfrak{p}_{\nu_1},\mathfrak{p}_{\nu_2})\neq(1)$, $\cdots$, $(\mathfrak{p}_{\nu_s},\mathfrak{p}_j)\neq(1)$. Let $\mathfrak{m}_i$ be the intersection of prime ideals in the same classes. Then we have $\mathfrak{m}=\mathfrak{m}_1\cap\cdots\cap\mathfrak{m}_t$ and $\mathfrak{m}_i$ is dually indecomposable. Therefore in the investigation of the $\mathfrak{m}$-adic completion, we may confine ourselves to the case where $\mathfrak{m}$ is dually indecomposable.

On this point we have the following

THEOREM III. *Let $\mathfrak{o}$ be a Noetherian ring (not assumed to have a unit element), and let $\mathfrak{m}$ be a semi-prime ideal in $\mathfrak{o}$ such that $\bigcap\mathfrak{m}^n=(0)$. Let $\mathfrak{m}=\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_r$ be the irredundant representation, and set $\mathfrak{p}_i^\infty=\bigcap\mathfrak{p}_i^n$, $\mathfrak{o}/\mathfrak{p}_i^\infty=\mathfrak{o}_i'$, $\mathfrak{p}_i/\mathfrak{p}_i^\infty=\mathfrak{p}_i'$ and denote by $\bar{\mathfrak{o}}_i$ the $\mathfrak{p}_i'$-adic completion of $\mathfrak{o}_i'$. Then the canonical homomorphism of the $\mathfrak{m}$-adic completion of $\mathfrak{o}$ into the direct sum of $\bar{\mathfrak{o}}_1,\cdots,\bar{\mathfrak{o}}_r$ is an isomorphism.*

This will be proved similary as Theorem I, by utilizing the following

LEMMA I. *With the same notations as in Theorem III, set $\mathfrak{a}_n=\mathfrak{p}_1^n\cap\cdots\cap\mathfrak{p}_r^n$, then we have $\mathfrak{m}^{\sigma(n)}\supseteq\mathfrak{a}_n\supseteq\mathfrak{m}^n$ with $\sigma(n)\to\infty$ as $n\to\infty$.*

PROOF. Since $\mathfrak{p}_1,\cdots,\mathfrak{p}_r$ is the set of all the isolated prime divisors of $\mathfrak{m}^\sigma$, we have, $\mathfrak{m}^\sigma\supseteq\mathfrak{a}_\rho$, if ρ denotes the maximum of exponents of primary ideals which appear in an irredundant representation of $\mathfrak{m}^\sigma$ as an intersection of primary ideals.

Finally we shall remark that the following Corollary, due to Chevalley, is an immediate consequence of Theorem I and Lemma II given below.

COROLLARY. *If $\mathfrak{m}$ is the intersection of r maximal ideals in a Noetherian ring $\mathfrak{o}$ with a unit element such the $\bigcap\mathfrak{m}^n=(0)$, then the $\mathfrak{m}$-adic completion of $\mathfrak{o}$ is the direct sum of r complete local rings.*

LEMMA II. *If $\mathfrak{p}$ is a maximal ideal in a Noetherian ring $\mathfrak{o}$ with a unit element, such that $\bigcap\mathfrak{p}^n=(0)$. Then we can form the quotient ring $\mathfrak{o}_\mathfrak{p}$ of $\mathfrak{p}$ with respect to $\mathfrak{o}$, and the $\mathfrak{p}$-adic ring $\mathfrak{o}$ is a subspace of the $\mathfrak{p}\cdot\mathfrak{o}_\mathfrak{p}$-adic ring $\mathfrak{o}_\mathfrak{p}$ and is everywhere dense in $\mathfrak{o}_\mathfrak{p}$. The $\mathfrak{p}$-adic ring $\mathfrak{o}$ and the $\mathfrak{p}\cdot\mathfrak{o}_\mathfrak{p}$-adic ring $\mathfrak{o}_\mathfrak{p}$ have the same completion.*

PROOF. That $\mathfrak{o}$ has no zero divisors outside of $\mathfrak{p}$, follows from $\bigcap \mathfrak{p}^n=(0)$. If we set $\mathfrak{p}^*=\mathfrak{p}\cdot\mathfrak{o}_{\mathfrak{p}}$, then we have $\mathfrak{p}^{*n}\cap\mathfrak{o}=\mathfrak{p}^n$, which proves the second assertion. Let x/s ($x\in\mathfrak{o}$, $s\notin\mathfrak{p}$) be any element in $\mathfrak{o}_{\mathfrak{p}}$, then we have $(\mathfrak{p}^n, s\mathfrak{o})=(1)$ for any positive integer n, whence $1\equiv su \pmod{\mathfrak{p}^n}$ for some u in $\mathfrak{o}$. Hence $x\equiv sux \pmod{\mathfrak{p}^n}$, that is $x/s\equiv ux \pmod{\mathfrak{p}^{*n}}$. q. e. d.

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Department of Mathematics,
Hiroshima University