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# *On Homogeneous Ideals of Graded Noetherian Rings*

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## Introduction

Given a polynomial ring over a field K, $A = K[x_0, x_1, \cdots, x_n; y_0, y_1, \cdots, y_m]$ we may indicate by $A_{i,j}$ the set of forms of degree $i$ in the $x$, $j$ in the $y$. Then A is the direct sum of K-submodules $A_{i,j}$:

$$A = \sum_{0 \leqq i, j < \infty} A_{i,j},$$

and we have moreover the relation: $A_{i,j} A_{k,l} = A_{i+k, j+l}$. Thus this ring may be regarded to have a graded structure in a sense.

Let us note here another ring which has a similar graded structure. Given a commutative ring R, and an ideal $\mathfrak{a}$ of R such that $\bigcap_{n=1}^{\infty} \mathfrak{a}^n = (0)$, we form the formring [(1)] of the ideal $\mathfrak{a}$:

$$F(\mathfrak{a}) = \sum_{n=0}^{\infty} \mathfrak{a}^n / \mathfrak{a}^{n+1},$$

where holds again the relation:

$$(\mathfrak{a}^n / \mathfrak{a}^{n+1}) \cdot (\mathfrak{a}^m / \mathfrak{a}^{m+1}) = \mathfrak{a}^{n+m} / \mathfrak{a}^{n+m+1}.$$

As is well known, this ring plays an important role in the theory of local rings.

The object of this paper is to introduce the notion of graded Noetherian rings, which include above mentioned manyfold projective coordinate rings as well as formrings, and to formulate some elementary properties of homogeneous ideals of such rings.

DEFINITION. Let A be a commutative ring with a unit element, and J be an ordered additive semi-group with zero element 0. (J is linearly ordered

(1) For the precise definition, see e. g. P. Samuel: Algèble Locale, mémorial des sc. Math., 1953.

such that $\alpha<\beta$ implies $\alpha+\delta<\beta+\delta$). Suppose given a decomposition of A as a direct sum of a family $(A_\alpha)_{\alpha\in J}$ of submodules $A_\alpha$, the set of indices of the family being J and the following condition being satisfied:

$$A_\alpha \cdot A_\beta \subseteq A_{\alpha+\beta}.$$

We say this decomposition defines on A a structure of a graded ring. An element of A is called homogeneous if it belongs to one of the sets $A_\alpha$ and homgeneous of degree $\alpha$ if it belongs to $A_\alpha$. Now let $f$ be an arbitrary element of A, and put $f=\sum_{\alpha\in J} f_\alpha$, with $f_\alpha\in A_\alpha$ for all $\alpha\in J$. We say then the $f_\alpha$ are the homogeneous components of $f$ and that $f_\alpha$ is the homogeneous component of degree $\alpha$ of $f$. An ideal $\mathfrak{a}$ of A is called homogeneous if the homogeneous components of all elements of $\mathfrak{a}$ belong to $\mathfrak{a}$.

In the following sections, we shall assume, in addition to a graded structure on A, the maximal condition on ideals of A, and investigate the properties of homogeneous ideals of A.

## I. Decompositions of homogeneous ideals

In this section we shall examine the homogeneity of the prime divisors and the primary components of homogeneous ideals. For an arbitrary ideal $\mathfrak{a}$ of A, let $\mathfrak{a}_H$ denote the largest homogeneous ideal contained in $\mathfrak{a}$. With this notation we state the following:

LEMMA 1. (i) $(\mathfrak{a}\cap\mathfrak{b})_H=\mathfrak{a}_H\cap\mathfrak{b}_H$, *for any ideals* $\mathfrak{a}$, $\mathfrak{b}$ *of* A.

(ii) *Let* $\mathfrak{q}$ *be a primary ideal and* $\mathfrak{p}$ *be the prime ideal associated to* $\mathfrak{q}$. *Then* $\mathfrak{q}_H$ *is a primary ideal and* $\mathfrak{p}_H$ *is the prime ideal associated to* $\mathfrak{q}_H$.

PROOF. (i) will be obvious and the fact that $\mathfrak{p}_H$ is prime will be easily seen. By induction with respect to the exponent of $\mathfrak{q}$, we shall prove that $\mathfrak{q}_H$ is a primary ideal belonging to $\mathfrak{p}_H$. Let $\rho$ be the exponent of $\mathfrak{q}$, and suppose $\rho\geqq 2$. Form the quotient $\mathfrak{q}'=\mathfrak{q}:\mathfrak{p}$, then $\mathfrak{q}'$ is a primary ideal of exponent $\rho-1$ and belongs to $\mathfrak{p}$. Now assume that our assertion is valid for primary ideals of exponent $\rho-1$. Since obviously $\mathfrak{q}_H\subseteq\mathfrak{p}_H$, $\mathfrak{p}_H{}^\rho\subseteq\mathfrak{q}_H$ hold, we have only to prove that

$$a\cdot b\in\mathfrak{q}_H \text{ and } a\notin\mathfrak{p}_H \text{ imply } b\in\mathfrak{q}_H.$$

Express $a$, $b$ as the sums of homogeneous elements:

$$a=\sum_{i=1}^{n} a_{\alpha_i},\quad \alpha_1<\alpha_2<\cdots<\alpha_n;$$

$$b=\sum_{j=1}^{m} b_{\beta_j},\quad \beta_1<\beta_2<\cdots<\beta_m.$$

Now assume $b\notin \mathfrak{q}_H$, then there exist integers $\nu$, $\mu$ such that

$$a_{\alpha_1},\ \cdots,\ a_{\alpha_{\nu-1}}\in \mathfrak{p}_H,\quad a_{\alpha_\nu}\notin \mathfrak{p}_H\,;$$

$$b_{\beta 1},\ \cdots,\ b_{\beta_{\mu-1}}\in \mathfrak{q}_H,\quad b_{\beta_\mu}\notin \mathfrak{q}_H.$$

Then the homogeneous component of degree $\alpha_\nu+\beta_\mu$ of the product $ab$ is written as follows:

$$a_{\alpha_\nu}\cdot b_{\beta_\mu}+\sum_{\substack{\alpha_i+\beta_j=\alpha_\nu+\beta_\mu\\ \alpha_i<\alpha_\nu}} a_{\alpha_i}\cdot b_{\beta_j}+\sum_{\substack{\alpha_i+\beta_j=\alpha_\nu+\beta_\mu\\ \alpha_i>\alpha_\nu}} a_{\alpha_i}\cdot b_{\beta_j}.$$

Since $a\cdot b\in \mathfrak{q}'_H$, $a\notin \mathfrak{p}_H$, we have $b\in \mathfrak{q}'_H$ by the inductive assumption. Therefore if $\alpha_i<\alpha_\nu$, then $a_{\alpha_i}b_{\beta_j}\in \mathfrak{p}\cdot\mathfrak{q}'\subseteq \mathfrak{q}$. Hence follows $a_{\alpha_\nu}\cdot b_{\beta_\mu}\in \mathfrak{q}$. But this contradicts our previous assumption $a_{\alpha_\nu}\notin \mathfrak{p}$, $b_{\beta_\mu}\notin \mathfrak{q}$.

With the help of this lemma, we can easily prove the following theorem,(2) which is well known to hold in projective coordinate rings.

THEOREM 1. *Let $\mathfrak{a}$ be a homogeneous ideal of a graded Noetherian ring. Then the prime ideals associated to $\mathfrak{a}$ and the isolated primary components of $\mathfrak{a}$ are all homogeneous. Moreover in any irredundant representation of $\mathfrak{a}$ as an intersection of primary ideals, imbedded components can be replaced by homogeneous ones.*

PROOF. Let $\mathfrak{a}=\mathfrak{q}_1\cap\mathfrak{q}_2\cap\cdots\cap\mathfrak{q}_r$ be an irredundant representation of $\mathfrak{a}$, and let $\mathfrak{p}_i$ be the radical of $\mathfrak{q}_i$. Then from lemma 1 follows $\mathfrak{a}=\mathfrak{a}_H=\mathfrak{q}_{1H}\cap\cdots\cap\mathfrak{q}_{rH}$ and that $\mathfrak{p}_{iH}$ is the radical of the primary ideal $\mathfrak{q}_{iH}$. Now these facts and the uniqueness of prime divisors and the isolated primary components of $\mathfrak{a}$ imply our assertions.

## II. Ranks of homogeneous prime ideals

THEOREM 2. *Let $\mathfrak{p}$, $\mathfrak{p}_0$ be two homogeneous prime ideals of a graded Noetherian ring such that $\mathfrak{p}\supset\mathfrak{p}_0$. If there lies no homogeneous prime ideal strictly between $\mathfrak{p}$ and $\mathfrak{p}_0$. Then there lies no prime ideal strictly between $\mathfrak{p}$ and $\mathfrak{p}_0$.*

PROOF. Select any homogeneous element $f$ such that $f\in\mathfrak{p}$, $f\notin\mathfrak{p}_0$. Then by theorem 1 and our assumption on $\mathfrak{p}$, $\mathfrak{p}_0$, it is seen that $\mathfrak{p}$ is an isolated prime divisor of the ideal $(\mathfrak{p}_0, f)$. The principal ideal theorem (3) implies

(3) See the remark at the end of the paper.

(2) O. Zariski: Foundations of a general theory of birational correspondence. (Trans. Amer. Math. Soc., t. 45, 1944), Theorem 2, p. 492. His method of proof is not applicable to our general case.

then that there lies no prime ideal between $\mathfrak{p}$ and $\mathfrak{p}_0$.

We shall next give two definitions and a lemma which is well known to hold in some special rings.

DEFINITION. A graded ring $A=\sum_{\alpha\in J}A_\alpha$ is called a *specially* graded ring, if J the set of indices satisfies the following additional condition: Given any element $\alpha$ of J, every sufficiently large element $\beta\in J$ can be represented such that $\beta=\alpha+\gamma$.

DEFINITION. Homogeneous ideal $\mathfrak{a}$ of a specially graded ring $A=\sum_{\alpha\in J}A_\alpha$ is called relevant if it contains no $A_\alpha$.

The denomination "relevant ideals" may find some justification in the geometric meaning of such ideals. In the rest of this section, we shall deal with specially graded rings. However, in the other places, we shall never assume that the rings we are concerned with are specially graded.

LEMMA 2.(4) *Let $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n$ be a finite set of homogeneous, relevant prime ideals and $\mathfrak{a}$ be a homogeneous ideal of a specially graded Noetherian ring. Suppose $\mathfrak{a}$ is not contained in any of $\mathfrak{p}_i$ $(i=1, 2, \dots, n)$. Then $\mathfrak{a}$ contains a homogeneous element not contained in any of $\mathfrak{p}_i$ $(i=1, 2, \dots, n)$.*

PROOF. We may assume $\mathfrak{p}_i\not\subseteq\mathfrak{p}_j$ for $i\neq j$. Then $\mathfrak{a}\cdot\mathfrak{p}_1\cdots\mathfrak{p}_{i-1}\cdot\mathfrak{p}_{i+1}\cdots\mathfrak{p}_n\not\subseteq\mathfrak{p}_i$, hence there exists a homogeneous element $a_i$ such that

$$a_i\in\mathfrak{a}, \mathfrak{p}_1, \dots, \mathfrak{p}_{i-1}, \mathfrak{p}_{i+1}, \dots, \mathfrak{p}_n;\ a_i\notin\mathfrak{p}_i.$$

Denote by $\alpha_i$ the degree of $a_i$ and take $\alpha\in J$ sufficiently large that there exist $\gamma_i\in J$ such that $\alpha_i+\gamma_i=\alpha$ $(i=1, 2, \dots, n)$. Since $A_{\gamma_i}\not\subseteq\mathfrak{p}_i$, there exists a homogeneous element $b_i$ of degree $\gamma_i$ not contained in $\mathfrak{p}_i$. Put $a=\sum_{i=1}^{n}b_ia_i$, than $a$ is a homogeneous element and satisfies the condition: $a\in\mathfrak{a}$, $a\notin\mathfrak{p}_i$ $(i=1,2,\dots,n)$,

THEOREM 3. *Let $\mathfrak{p}$ be a relevant homogeneous prime ideal of a specially graded Noetherian ring, and let $r$ be its rank. Then there exists a chain of length $r$,*

$$\mathfrak{p}\supset\mathfrak{q}_{r-1}\supset\mathfrak{q}_{r-2}\supset\cdots\supset\mathfrak{q}_0$$

*where $\mathfrak{q}_i$ $(0\leq i\leq r-1)$ are homogeneous prime ideals.*

PROOF. By the very definition of the rank of $\mathfrak{p}$, there exists a chain of prime ideals of length $r$ beginning with $\mathfrak{p}$, $\mathfrak{p}\supset\mathfrak{p}_{r-1}\supset\mathfrak{p}_{r-2}\supset\cdots\supset\mathfrak{p}_0$, where $\mathfrak{p}_0$ must be a minimal prime ideal (The symbol $\supset$ will always indicate *proper* inclusion). Therefore theorem 1 implies that $\mathfrak{p}_0$ is homogeneous. Suppose $\mathfrak{p}\supset\mathfrak{p}_0$ and take any homogeous elemet $a_1$ such that $a_1\in\mathfrak{p}$, $a_1\notin\mathfrak{p}_0$. Let $\mathfrak{q}_{11}, \mathfrak{q}_{12}, \dots, \mathfrak{q}_{1\nu_1}$

(4) See e.g. P. Samuel: Algèbre locale, loc. cit., p. 22.

be the isolated prime divisors of $(\mathfrak{p}_0, a_1)$, which are contained in $\mathfrak{p}$. If $\mathfrak{p} \supset \mathfrak{q}_{11}$, then by lemma 2 we can select a homogeneous element $a_2$ of $\mathfrak{p}$ which is not contained in any of $\mathfrak{q}_{1i}$ $(i = 1, 2, \ldots, \nu_1)$. Now let $\mathfrak{q}_{21}, \mathfrak{q}_{22}, \ldots, \mathfrak{q}_{2\nu_2}$ be the isolated prime divisors of $(\mathfrak{q}_0, a_1, a_2)$ which are contained in $\mathfrak{p}$. If $\mathfrak{p} \supset \mathfrak{q}_{21}$, we repeat this process till we reach the stage: $\mathfrak{p}$ is an isolated prime divisor of $(\mathfrak{p}_0, a_1, \ldots, a_s)$. Then set $\mathfrak{q}_{s-1} =$ any of $\mathfrak{q}_{s-1,1}, \ldots, \mathfrak{q}_{s-1, \nu_{s-1}}$; $\mathfrak{q}_{s-2} =$ any of $\mathfrak{q}_{s-2,1}, \ldots, \mathfrak{q}_{s-2, \nu_{s-2}}$ which is contained in $\mathfrak{q}_{s-1}$ and so on. Thus we obtain a chain of homogeneous prime ideals, $\mathfrak{p} \supset \mathfrak{q}_{s-1} \supset \mathfrak{q}_{s-2} \supset \cdots \supset \mathfrak{q}_0 = \mathfrak{p}_0$. Since $\mathfrak{p}$ is an isolated prime divisor of the ideal $(\mathfrak{p}_0, a_1, \ldots, a_s)$, the theorem of Krull quoted before implies $r \leq s$, that is $r = s$.

COROLLARY. *Let $\mathfrak{p} \supset \mathfrak{p}_{r-1} \supset \cdots \supset \mathfrak{p}_0$ be a chain of prime ideals, where $\mathfrak{p}$ and $\mathfrak{p}_0$ are homogeneous and $\mathfrak{p}$ is relevant. Then there exists a chain $\mathfrak{p} \supset \mathfrak{q}_{r-1} \supset \cdots \supset \mathfrak{q}_0 = \mathfrak{p}_0$, where $\mathfrak{q}_i$ $(i = 0, 1, \ldots, r-1)$ are all homogeneous prime ideals.*

This corollary follows from theorem 3 and the fact that the residue ring $A/\mathfrak{a}$ of $A$ by a homogeneous ideal $\mathfrak{a}$ is also a graded Noetherian ring: $A/\mathfrak{a} = \sum_{\alpha \in J} A_\alpha/\mathfrak{a}_\alpha$, where $\mathfrak{a}_\alpha = A_\alpha \cap \mathfrak{a}$ and $A_\alpha/\mathfrak{a}_\alpha$, are canonically identified with $(A_\alpha + \mathfrak{a})/\mathfrak{a}$.

## III. Lengths [6] of homogeneous primary ideals

The following lemma is analogous to theorem 2. However, this lemma may not be expected to be proved so easily as that theorem, for this time we have no such a convenient tool as the principal ideal theorem which was used essentially to prove the theorem 2.

LEMMA. 3. *Let $\mathfrak{q} \supset \mathfrak{q}_0$ be two primary homogeneous ideals which belong to the same prime ideal $\mathfrak{p}$ and suppose there exists no homogeneous primary ideal strictly between $\mathfrak{q}$ and $\mathfrak{q}_0$. Then there exists no primary ideal strictly between $\mathfrak{q}$ and $\mathfrak{q}_0$.*

PROOF. By passing from $A$ to its residue ring $A/\mathfrak{q}_0$, we may assume $\mathfrak{q}_0 = (0)$. Let $f(\neq 0)$ be any homogeneous element of $\mathfrak{q}$, then by theorem 1, it is seen that $\mathfrak{q}$ is the primary component belonging to $\mathfrak{p}$ of the principal ideal $(f)$. Hence there exists a homogeneous element $b$ such that

$$b \cdot \mathfrak{q} \subseteq (f), \quad b \notin \mathfrak{p}.$$

(5) For the notion of ranks of prime ideals, see e. g. D. G. Northcott: Ideal Theory, (Cambridge Tracts, No. 40, 1953), p. 56.

(6) For the notions such as the length and a composition series of a primary ideal, see e. g. Van der Waerden: On Hilbert's Function. (Proc. Kon. Acad. Amsterdam, t. 31, 1928), Section III, p. 759.

Therefore, if $g$ is any element of $\mathfrak{q}$, we have such a relation as $b\cdot g=a\cdot f$. Now let $h$ be any non-homogeneous element of $\mathfrak{q}$, and let $h=h_0+h_1+\cdots+h_n$ $(n\geqq 1)$ be the representation of $h$ as the sum of nonzero homogeneous elements. And assume that $h_0$ has the smallest degree among $h_0, h_1, \dots, h_n$. Applying the above observation to the pairs $(h_0, h_i)$, we have $b_i h_i=a_i h_0$, where $b_i$ and $a_i$ are homogeneous and $b_i \notin \mathfrak{p}$. Whence $bh=ah_0$, where $b=b_1\dots b_n \notin \mathfrak{p}$, and $a=b_1\dots b_n+a_1\cdot b_2\dots b_n+\cdots+b_1\cdot b_2\dots b_n\cdot a_n$. Since deg. of $b=b_1\dots b_n<$ deg. of $b_1\dots b_{i-1}a_i b_{i+1}\dots b_n$, we have $a\notin\mathfrak{p}$. Thus we see any primary ideal, which belongs to $\mathfrak{p}$ and contains $h$, necessarily contains $h_0$, hence contains $\mathfrak{q}$.

This lemma implies immediately the following:

THEOREM 4. *Let $\mathfrak{q}$ be a homogeneous primary ideal and let $\mathfrak{p}$ be the prime ideal associated to $\mathfrak{q}$. And let*

$$\mathfrak{p}=\mathfrak{q}_1\supset\mathfrak{q}_2\supset\cdots\supset\mathfrak{q}_l=\mathfrak{q}$$

*be a chain of homogeneous primary ideals which begins with $\mathfrak{p}$ and ends with $\mathfrak{q}$ such that no homogeneous primary ideal can be inserted strictly between $\mathfrak{q}_i$ and $\mathfrak{q}_{i+1}$ $(i=1, 2, \dots, l-1)$. Then this is a composition series of $\mathfrak{q}$, and $l$ is the length of $\mathfrak{q}$.*

REMARK. The following theorem due to Krull really marked a turning point in the recent development of the theory of Noetherian rings.

THEOREM. *Let* R *be a Noetherian ring with a unit element. Let $(a_1, a_2, \dots, a_m)$ be a proper ideal of* R, *and $\mathfrak{p}$ be an isolated prime divisor of $(a_1, a_2, \dots, a_m)$. Then the rank of $\mathfrak{p}$ is at most $m$.*

Krull proved this theorem by induction with respect to $m$. The special case of this theorem, in which $m=1$, is usually referred to as the principal ideal theorem. The part of his proof, which concerns the principal ideal theorem, is quite elegant, but the rest of his proof is rather long and seems to comprise superfluous complexity, which appears not yet excluded even in recently appeared tracts quoted in footnotes (4) and (5).

So it may not be inutile to note here a easier way leading to the general theorem from the principal ideal theorem. We first state the following lemma which is an immediate consequence of the principal ideal theorem.

LEMMA. *Let* R *be a Noetherian ring, and $\mathfrak{p}=\mathfrak{p}_0\supset\mathfrak{p}_1\supset\cdots\supset\mathfrak{p}_{r-1}\supset\mathfrak{p}_r$ be a chain of prime ideals of* R. *Let $a$ be an arbitrary element of $\mathfrak{p}$. Then there exists a chain of prime ideals such that*

$$\mathfrak{p}\supset\mathfrak{p}_1^*\supset\cdots\supset\mathfrak{p}_{r-1}^*\supset\mathfrak{p}_r, \text{ with } a\in\mathfrak{p}_{r-1}^*.$$

PROOF. This is easily seen if we notice the following fact. Suppose given a chain of prime ideals $\mathfrak{q} \supset \mathfrak{q}_1 \supset \mathfrak{q}_2$ and an element $a$ such that $a \in \mathfrak{q}$, $a \notin \mathfrak{q}_1$. Then there exists a prime ideal $\mathfrak{q}_1{}^*$ such that $\mathfrak{q} \supset \mathfrak{q}_1{}^* \supset \mathfrak{q}_2$, $a \in \mathfrak{q}_1{}^*$. Any isolated prime divisor of the ideal $(\mathfrak{q}_2, a)$, which is contained in $\mathfrak{q}$, may be taken as $\mathfrak{q}_1{}^*$. This is seen by applying the principal ideal theorem to the principal ideal $(a, \mathfrak{q}_2)/\mathfrak{q}_2$ of the residue ring $R/\mathfrak{q}_2$.

Proof of the theorem. Let $\mathfrak{p} \supset \mathfrak{q}_1 \supset \cdots \supset \mathfrak{q}_r$ be an arbitrary chain of prime ideals, which begins with $\mathfrak{p}$. Then there exists a chain of prime ideals

$$\mathfrak{p} \supset \mathfrak{p}_1{}^* \supset \cdots \supset \mathfrak{p}^*_{r-1} \supset \mathfrak{p}_r, \quad a_m \in \mathfrak{p}^*_{r-1}.$$

Then in $R/(a_m)$, $\mathfrak{p}/(a_m)$ is an isolated prime divisor of the ideal $(\bar{a}_1, \dots, \overline{a_{m-1}})$, $\bar{a}_i$ denoting the residue of $a_i$ in $R/(a_m)$; and we have

$$\mathfrak{p}/(a_m) \supset \mathfrak{p}_1{}^*/(a_m) \supset \cdots \supset \mathfrak{p}^*_{r-1}/(a_m).$$

Hence by the induction hypothesis, we have $r-1 \leq m-1$, that is, $r \leq m$.

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