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A Theorem on Operator Algebras

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Let $\boldsymbol{A}$ be a B^*–algebra, that is, a Banach $*$–algebra with the property $\|A^*A\| =\|A\|^2$ for every $A\in \boldsymbol{A}$. Such an algebra is $*$–isomorphically representable as a uniformly closed algebra of operators on a Hilbert space $\mathfrak{H}$. In the sequel we assume that $\boldsymbol{A}$ is represented such an algebra of operetors on $\mathfrak{H}$. It is the purpose of this paper to prove the following theorem [1]:

THEOREM. *Let $\boldsymbol{A}$ be a B^*–algebra. Then for $A,\ B\in \boldsymbol{A}^+$*

(a) *if $A\geq B$, then $A^{\frac{1}{2}}\geq B^{\frac{1}{2}}$;*

(b) *if $A\geq B$ implies always $A^2\geq B^2$, then $\boldsymbol{A}$ is commutative.*

1. Proof of (a). First consider the case that $\mathfrak{H}$ is finite-dimensional. Suppose the contrary. Let $Tr(C)$ stand for an ordinary trace of operators C on $\mathfrak{H}$. It is a positive linear functional and has the property that $Tr(CD)\geq 0$ for $C\geq 0$ and $D\geq 0$. Put $S=A^{\frac{1}{2}}$ and $T=B^{\frac{1}{2}}$. Owing to the spectral resolution of $T-S$ there exists a non-zero projection P such that $P(T-S)=(T-S)P\geq \delta P>0$ for a positive number δ. Then $Tr(P(T-S)(T+S))\geq \delta Tr(P(T+S))\geq 0$. On the other hand $Tr(P(T-S)(T+S)) = \frac{1}{2}\{Tr(P(T-S)(T+S)) + Tr((T+S)(T-S)P)\} = \frac{1}{2}\{Tr(P(T-S)(T+S)) + Tr(P(T+S)(T-S))\} = Tr(P(B-A))\leq 0$. From these inequalities we have $Tr(P(T+S)) =0$ and therefore $P(T+S)P=0$, which entails that $PTP=PSP=0$. Then $P(T-S)=P(T-S)P=PTS-PSP=0$. It contradicts $P(T-S)\geq \delta P>0$.

Now we consider the general case. Without loss of generality we may assume that $\boldsymbol{A}$ is the algebra $\mathscr{B}$ of operators on $\mathfrak{H}$. For each finite-dimensional projection P_δ we designate by A_δ the greatest positive operator $\leq A$ such that $A_\delta P_\delta=P_\delta A_\delta=A_\delta$. Such an A_δ is determined by $\langle A_\delta f,f\rangle = \underset{P_\delta g=P_\delta f}{\text{g. l. b.}}\langle Ag,g\rangle$ (cf. [**1**]). $\{A_\delta\}$ is a directed set by the ordering "$\geq$" of operator algebras and it is easy to see that $\{A_\delta\}$ converges to A in the strong topology (cf. [**1**]). $P_\delta\geq P_{\delta'}$ entails $A_\delta\geq A_{\delta'}$ and therefore $A_\delta^{\frac{1}{2}}\geq A_{\delta}^{\frac{1}{2}}$ by the above discussion. Let T be the strong limit of the directed

1) Added in proof. (a) follows as a special case from a theorem due to E. Heinz (Math. Ann. 123 (1951), 415-438, §1 Satz 2). Cf. also T. Kato, Math. Ann. 125 (1952/53), 208-212, Theorem 2.

set $\{A_\delta^{\frac{1}{2}}\}$. $\langle T^2f, f\rangle = \langle Tf, Tf\rangle = \lim_\delta \langle A_\delta^{\frac{1}{2}}f, A_\delta^{\frac{1}{2}}f\rangle = \lim_\delta \langle A_\delta f, f\rangle = \langle Af, f\rangle$ for every $f \in \mathfrak{H}$. Hence $T = A^{\frac{1}{2}}$. Let $\{B_\delta\}$ be the corresponding directed set of positive finite-dimensional operators for B. Evidently $A_\delta \geq B_\delta$ and therefore $A_\delta^{\frac{1}{2}} \geq B_\delta^{\frac{1}{2}}$. This implies $A^{\frac{1}{2}} \geq B^{\frac{1}{2}}$.

2. Some partial order in $\mathscr{B}^+$. Let $\mathscr{B}^+$ be the set of all positive operators on $\mathfrak{H}$. Let $A = \int_0^\infty \lambda dE_\lambda$ and $B = \int_0^\infty \lambda dF_\lambda$ be the spectral resolutions of positive operators A and B respectively. After Dixmier [**2**] we write $A \gg B$ if $E_\lambda \leq F_\lambda$ for every $\lambda > 0$. $f \in E_\lambda \mathfrak{H}$ if and only if $\langle A^{2^n}f, f\rangle \leq \lambda^{2^n}\langle f, f\rangle$ holds for every non-negative integer n. Therefore $A \gg B$ is equivalent to $A^{2^n} \geq B^{2^n}$ $(n = 0, 1, 2, \cdots)$. The order "$\gg$" is evidently a partial order in $\mathscr{B}^+$. The l. u. b. of A and B exists and is given by $\int_0^\infty \lambda dG_\lambda$ where $G_\lambda = E_\lambda \cap F_\lambda$. For example, if P and Q are projections, then $P \vee Q$ coincides with the usually defined $P \cup Q$. We note that if A is any self-adjoint operator $\in \mathscr{B}$ and $A = \int_{-\infty}^{+\infty} \lambda dE_\lambda$ is its spectral resolution, then $|A| = A_+ \vee A_-$, where $A_+ = \int_0^\infty \lambda dE_\lambda$, $A_- = -\int_{-\infty}^0 \lambda dE_\lambda$.

3. Proof of (b). The condition of (b) is the same as the following: For any $A, B \in \boldsymbol{A}$, $A \geq B$ is equivalent to $A \gg B$. Let $\boldsymbol{A}'$ be the linear space of all self-adjoint operators of $\boldsymbol{A}$. We shall show that the order "$\geq$" is a lattice order. To this end it is sufficient to show that, for any $A \in \boldsymbol{A}'$, $|A|$ is the *l. u. b* of A and $-A$, where $|A|$ is the absolute of A in the usual sense. Let C be any operator $\in \boldsymbol{A}'$ such that A, $-A \leq C$. Then $A_+ \leq C + A_-$ and $A_- \leq C + A_-$. As the order "$\geq$" coincides with the order "$\gg$", we have $|A| = A_+ + A_- \leq C + A_-$ from the remark given in **2**, and therefore $A_+ \leq C$. Similarly $A_- \leq C$. Hence $|A| \leq C$. Since A, $-A \leq |A|$, it follows that $|A|$ is the *l. u. b.* of A and $-A$. Then by a result of Sherman [**4**] we can conclude that $\boldsymbol{A}$ is commutative. The proof is completed.

The above proof is based on a result of Sherman. Without using his result we can prove that $\boldsymbol{A}$ is commutative. For any $A, B \in \boldsymbol{A}^+$, $A + B \geq A - B$, $B - A$. Therefore $(A+B)^2 \geq |A-B|^2 = (A-B)^2$, which entails that $AB + BA = \frac{1}{2}\{(A+B)^2 - (A-B)^2\} \geq 0$. We can write $AB = C + iD$, $C \in \boldsymbol{A}^+$, $D \in \boldsymbol{A}'$. We have only to show that $D = 0$. Suppose the contrary. $A(BAB) = C^2 - D^2 + i(CD + DC)$. $C^2 \geq D^2$ since $BAB \geq 0$. Let α be the greatest positive number such that $C^2 \geq \alpha D^2$ for every

$A, B \in A^+$. $(C^2-D^2)^2 \geq \alpha(CD+DC)^2$, which is written as $C^4+D^4-C^2D^2-D^2C^2 \geq \alpha(CDCD+DCDC+CD^2C+DC^2D)$. $2D^4 \leq C^2D^2+D^2C^2$ since $(C^2-D^2)D^2+D^2(C^2-D^2) \geq 0$. $CDCD+DCDC \geq 0$ since $DCD=(DC^{\frac{1}{2}})(DC^{\frac{1}{2}})^*$. $CD^2C \geq 0$. $DC^2D \geq \alpha D^4$ since $DC^2D - \alpha D^4 = D(C^2-\alpha D^2)D \geq 0$. Therefore we have $C^4-D^4 \geq \alpha^2 D^4$, which implies $C^2 \geq \sqrt{\alpha^2+1}\, D^2$. This is a contradiction.

4. A generalization of (a). Hither-to we have been only concerned with bounded operators. We shall generalize (a) for unbounded operators. Let T and S be self-adjoint operators such thta $T \geq S \geq 0$. We show that $T^{\frac{1}{2}} \geq S^{\frac{1}{2}}$. We first assume that S is bounded. Let $T=\int_0^\infty \lambda dE_\lambda$ be the spectral resolution of T. Put $T_n = TE_n = \int_0^n \lambda dE_\lambda$. We have $T_n \geq E_nSE_n$ and therefore $T_n^{\frac{1}{2}} \geq (E_nSE_n)^{\frac{1}{2}}$ by (a). Since E_nSE_n converges strongly to S, it follows from a theorem of Kaplansky [**3**] that $\{E_nSE_n\}^{\frac{1}{2}}$ converges strongly to $S^{\frac{1}{2}}$. Therefore $T^{\frac{1}{2}} \geq S^{\frac{1}{2}}$. Next we omit the assumption that S is bounded. Let $S=\int_0^\infty \lambda dF_\lambda$ be the spectral resolution of S and put $S_n = SF_n = \int_0^n \lambda dF_\lambda$. $T \geq S \geq S_n$ implies that $T^{\frac{1}{2}} \geq S_n^{\frac{1}{2}}$ and therefore $T^{\frac{1}{2}} \geq S^{\frac{1}{2}}$. The proof is completed.

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