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On Some Properties of Non-Compact Peano Spaces

By

Akira TOMINAGA

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Introduction

By a Peano space we mean a locally compact, locally connected, separable metric space. If a Peano space is compact, it is called a continuous curve or Peano continuum. In §1 we shall introduce the notion "degree of a Peano space", and in §2 we shall consider characterization of non-compactness for Peano spaces by half-open or open arcs, closed in the spaces. A metric space R will be called convex provided it has a convex metric $\rho(x, y)$, that is, for each pair of points p, q in R, there exists a point r such that $\rho(p, r)=\rho(r, q)=\rho(p, q)/2$. R. H. Bing and E. E. Moise proved independently that each continuous curve has a convex metric ([1], [5]). We have shown that there exists a convex metric in each Peano space ([8]). However, the metric defined in [8] is complete but not bounded. We shall show that there exists a bounded convex metric in each Peano space (in §4).

We know several compactings of non-compact spaces. In §5 we shall consider a property of the socalled Freudenthal's compacting. In the last section the problem of extension of metric will be treated.

1. Degree of Peano spaces.

We shall recall that each locally compact Hausdorff space R has a compacting by adding a new point ξ; we denote the associated compact space by $R^*=R+\xi$, and then define the closure operator $—^*$ in R^* as follows:

$$\bar{M}^*=\bar{M}: \text{ if } M \not\ni \xi \text{ and } \bar{M} \text{ is compact,}$$

$$\bar{M}^*=\bar{M}+\xi: \text{ if } M \not\ni \xi \text{ and } \bar{M} \text{ is not compact and}$$

$$\bar{M}^*=(\overline{M-\xi})+\xi: \text{ if } M \ni \xi.$$

Hence each open set of R^* is either an open set of R or R^*-F, where F

is a compact subset of R. R^* is a Hausdorff space and compact; therefore it is normal. Furthermore, if R is perfectly separable, so is R^*. For let $\mathfrak{U}=\{U_i/i=1, 2, \cdots\}$ be a countable base of R consisting of open sets whose closure are compact. Let $V_k(\xi)=R^*-\overset{K}{\underset{i=1}{\cup}}\bar{U}_i$. $\mathfrak{V}=\{V_k(\xi)/k=1, 2, \cdots\}$ is a countable base of ξ. Thus R^* is metrizable. Since a locally compact, connected Hausdorff space can not be locally connected only at one point, we conclude that if R is a Peano space, then R^* is a Peano continuum.

Now H. Freudenthal showed a compacting of locally compact, locally connected, connected separable Hausdorff spaces ([3]). It is different from the compacting $R^*=R+\xi$ and is based on the notion "Endpunkt". However the notion starts from the original space R. We shall reformulate the notion by starting from the notion in the small, "degree at a point", and define the degree of R.

Assume that R is connected and locally connected at its point p. If p has a countable base of itself, there exists a sequence of regions U_1, U_2, U_3 $\cdots$ such that $U_{i+1}\subset U_i$ and $\overset{\infty}{\underset{i=1}{\cap}}U_i=p$. By $\mathfrak{U}$ we shall denote such sequence. Since each component of U_{i+1} is contained in a component of U_i aud each component of U_j $(j=1, 2, 3, \cdots)$ has p as a limit point, there exists a sequence of connected sets $C_1\supset C_2\supset\cdots$ where C_j is a component of U_j. The sequence is said to belong to $\mathfrak{U}$ and is denoted by $\mathfrak{C}_{\mathfrak{U}}$. Let $[\mathfrak{C}_{\mathfrak{U}}]$ be the collection of all sequences $\mathfrak{C}_{\mathfrak{U}}$. In $[\mathfrak{C}_{\mathfrak{U}}]$ we define the equivalence relation $\sim$ as follows: if $\mathfrak{C}_{\mathfrak{U}}=\{C_1\subset C_2\subset\cdots\}$ and $\mathfrak{C}_{\mathfrak{U}'}=\{C_1'\subset C_2'\subset\cdots\}$, then $\mathfrak{C}\sim\mathfrak{C}'$ if and only if for each n there exists an integer m such that $C'_m\subset C_n$. It is readily seen that $\mathfrak{C}_{\mathfrak{U}}\sim\mathfrak{C}_{\mathfrak{U}}$, and if $\mathfrak{C}_{\mathfrak{U}}\sim\mathfrak{C}_{\mathfrak{U}'}$ and $\mathfrak{C}_{\mathfrak{U}'}\sim\mathfrak{C}_{\mathfrak{U}''}$, then $\mathfrak{C}_{\mathfrak{U}}\sim\mathfrak{C}_{\mathfrak{U}''}$. If $\mathfrak{C}_{\mathfrak{U}}\sim\mathfrak{C}_{\mathfrak{U}'}$, then $\mathfrak{C}_{\mathfrak{U}'}\sim\mathfrak{C}_{\mathfrak{U}}$. Suppose, on the contrary, that for an integer m, $C'_m\not\supset C_n$ where n is arbitrary. Let U_n be an element of $\mathfrak{U}$ such that $U_n\subset U'_m$. The element C_n of $\mathfrak{C}_{\mathfrak{U}}$ which is a component of U_n fails to intersect C'_m. Thus $C_n\cap C'_k=\phi$ for each $k\geqq m$ and $\mathfrak{C}_{\mathfrak{U}}\not\sim\mathfrak{C}_{\mathfrak{U}'}$. Hence we can classify $[\mathfrak{C}_{\mathfrak{U}}]$ by the above defined relation and we shall denote the acquired classified system by $\mathfrak{B}$.

DEFINITION. The cardinal number of $\mathfrak{B}$ will be called the *degree of* R *at* p and will be denoted by $\mathscr{D}(R, p)$. Let M be a connected set containing p. By the component number of M at p will be meant the cardinal number of the components of $M-p$. We shall denote it by $\mathscr{M}(M, p)$.

The following properties are obvious:

A) $\mathscr{M}(U, p)\leqq\mathscr{D}(R, p)$ for each region of p.

B) If p is an end point of R, $\mathscr{D}(R, p) = 1$.

C) If p is a local separating point (or a cut point) of R, then $\mathscr{D}(R, p) \geqq 2$.

D) A necessary and sufficient condition that $\mathscr{D}(R, p)$ should be finite is that there exists a region U of p and an integer n such that for each region V of p in U, $\mathscr{M}(V, p) = n$.

E) If p has a finite order n in R, then $\mathscr{D}(R, p)$ is finite.

F) If $\mathscr{D}(R, p)$ is finite, then R is semi-locally connected at p.

DEFINITION. By the *degree of a Peano space* R is meant $\mathscr{D}(R^*, \xi)$, and we shall denote $\mathscr{D}(R) = \mathscr{D}(R^*, \xi)$. If R is compact, we make the convention that the degree of R is zero.

2. A Characterization of non-compactness

An arc α with end points p, q is called a segment if it is isometric with a closed interval of the real line R^1. By a ray γ we shall mean a half-open arc, closed in R, every subarc of which is a segment. The end point of a ray γ is said the initial point of γ. Furthermore, a straight line is an open arc, closed in R, every subarc of which is a segment. Myers has characterized non-compactness of a locally compact geodesic space by existence of a ray ([7]), and Montogomery and Zippin has proved that any non-compact group whose group-space is a Peano space has a (closed) subgroup which is isomorphic with R^1 ([6]). In this section we shall consider the analogous subjects in case of Peano space, particularly topological manifold.

THEOREM 1. *A necessary and sufficient condition that a Peano space R be non-compact is that R has a half-open arc, closed in R and having an arbitrary point in R as the end point.*

PROOF. Since R^* is a Peano continuum, ξ can be joined to arbitrary point p of R by a simple arc $p\xi$. $p\xi - \xi$ is closed in R.

Using of the Myers' result above and [8], the alternative proof is given.

We notice the following properties for the relation between ξ and R.

A) ξ is not a cut point.

B) If ξ is an end point, then two half-open arcs, closed in R, intersect.

C) R has at least $\mathscr{D}(R)$ half-open arcs, closed in R, which mutually exclusive.

We shall denote the cyclic element containing ξ by C_ξ. If $C_\xi = \xi$, then ξ is an end point of R^* and therefore $\mathscr{D}(R) = 1$ by §1, B). If $C_\xi \neq \xi$, $\mathscr{D}(R)$ may be either equal to 1 or $\geqq$ 2. The latter case is the only one in which ξ is a local separating point of R^*.

By a topological n-manifold M^n, we shall mean a connected separable metric space, every point of which has a neighborhood that is homeomorphic with the n-dimensional euclidean space, in other words, every point has a n-cell neighborhood. A n-manifold is obviously a special type of Peano space. And in case M^n is compact, we call it a closed n-manifold — otherwise, an infinite n-manifold.

THEOREM 2. *A necessary and sufficient condition that a topological n-manifold R be infinite is that R has an open arc, closed in R.*

PROOF. No infinite topological manifold R can have ξ as the end point of R^*. For if ξ would be an end point, then there might exist a cut point of R^*. However, the infinite topological manifold which has a cut point is only R^1. Since R^1 has degree 2, ξ is not an end point. Thus $C_\xi \neq \xi$.

(i) $n=1$. The statement of the theorem is true.

(ii) $n \geqq 2$. As C_ξ is closed in R^*, so is $C_\xi - \xi$ in R. On the other hand, for each point p in $C_\xi - \xi$ there exists a n-cell U containing p, and U is contained in $C_\xi - \xi$. Since R is connected, $C_\xi - \xi = R$. Thus there exists an arc $a\xi b$ for each pair of points in R such that $a \neq \xi$ and $a \neq b$. Let α be an arc joining a to b in R. Then $a\xi b + \alpha$ contains a simple closed curve ω containing ξ. $\omega - \xi$ is an open arc, closed in R.

THEOREM 3. *Let R be a locally compact convex metric space. If $\mathfrak{D}(R) \geqq 2$, there exists a straight line in R.*

PROOF. Let $\rho(x, y)$ be the convex metric on R and let ξ and η be two different boundary points of R (See § 5.). There exist sequences $\{x_i\}$ and $\{y_i\}$ consisting of points of R converging in R^{**} to ξ and η, respctively. Let $x_i y_i$ be a segment joining x_i to y_i in R. Since R^{**} is compact, then we can suppose them so chosen that $x_i y_i \to L$ where L is a subcontinum of R^{**} containing ξ and η. L contains a simply ordered subcontinunm L' which is order isomorphic with the closed unit interval [0, 1]. By the isomorphism ξ and η are correspond to 0 and 1, respectively. Since $R^{**} - R$ is 0-dimensional, there exists a point p of R such that $p \in L' - \{\xi + \eta\}$. Furthermore, since $R^{**} - R$ is closed in R^{**}, we can find a subcontinuum L'' of L' such that two end points ξ_1 and η_1 are points of $R^{**} - R$ and $K = L'' - \{\xi_1 + \eta_1\} \subset R$. K is a straight line in R. For let a, b, c be points in K, then there exists sequences $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ consisting of points of R such that $\lim a_i = a$, $\lim b_i = b$ and $\lim c_i = c$. Noting that $\rho(x, y)$ is contnuous in R, $\rho(a, c) = \rho(a, b) + \rho(b, c)$ and each closed subarc of K is a segment.

Theorem 3 and the result in [8] imply,

COROLLARY. *If R is a Peano space whose degree $\geqq 2$, there exists an open arc, closed in R.*

We can also prove the corollary by the same manner as in Theorem 1.

3. Relations between compactness and metric

In this section we shall assume that spaces are all metrizable. The properties for metric, for example boundedness, completeness, property S, etc. are not always preserved under topological mappings. However, compact spaces can be characterized by the terms of metric.

Definition. Let $\rho(x, y)$ be a metric of R. R will be said to be *totally bounded*, when for each positive number ε, R is the sum of a finite number of sets N_i $(i = 1, 2, \cdots. k)$ each of diameter less than ε. If N_i is connected, R is said to have *property S.*

Proposition 1. *If R is compact, it is totally bounded and complete for any metric on R. Conversely, if R is totally bounded and complete for a metric, it is compact.*

Corollary. *If R is locally connected and compact, it has property S and is complete. Conversely if R has property S and is complete, it is locally connected and compact.*

Proposition 2. *In order that R might be compact, it is necessary and sufficient that any metric on R is bounded.*

For if R is not compact, there exists a countable discrete subset $M=\{p_1, p_2, \cdots\}$. If we define a metric σ on M such that $\sigma(p_i, p_j) = |i-j|$, σ is not bounded. By Bing's result [2], we can extend σ over R.

4. Existence of a bounded convex metric.

As we saw in §3, no non-compact Peano spaces have a both complete and bounded metric. In the previous paper [8] we have shown that each Peano space R has a complete (but not bounded) convex metric (for which the space has not property S) if R is not compact. Here it will be proved that there exists a bounded (but not complete) convex metric for each non-compact Peano space. A slightly different argument is required for the case; however the method used is the one by partitioning of Bing. The same notation as in [8] will be used.

Lemma 1. *Let $\{B_1, B_2, \cdots\}$ be a sequence consisting of subsets of a locally connected, connected set B such that the number of the components of B_1 is finite, each component of B_{i+1} contains* *a component of* B_i, $B_i \subset \mathfrak{I}(B_{i+1})$, *and* $\bigcup_{i=1}^{\infty} B_i = B$. *Then for some integer* n, B_n *is connected.*

Proof. Let k be the number of the components of B_1, then the number of components of B_i, α_i, is not larger than k. We shall denote the least integer of α_i's by α and $C_{1,m}, C_{2,m}, \cdots, C_{\alpha,m}$ are the components of B_m such that $\alpha_m = \alpha$.

For each i, there exists a sequence of connected sets such that $C_{i,m} \subset C_{i,m+1} \subset \cdots$ where $C_{i,m+k}$ is a component of B_{m+k}. Let $C_i = \bigcup_{k=1}^{\infty} C_{i,m+k}$, then $C_1, C_2, \cdots, C_\alpha$ are mutually exclusive connected sets and $B = \bigcup_{i=1}^{\alpha} C_i$. If C_i were not closed, $\bar{C}_i - C_i$ would have a point p of C_j, $i \neq j$. Since p is a point of $C_{j,m+l}$ for some l, $C_{j,m+l} \subset \mathfrak{I}(C_{j,m+l+1})$ and a neighborhood containing p is contained in $C_{j,m+l+1}$, we conclude $C_i \cap C_j \neq \phi$. This contradicts the fact that $C_1, C_2, \cdots, C_\alpha$ are mutually exclusive.

Now we can approximate non-compact Peano spaces by continuous curves.

Lemma 2. *Let* R *be a non-compact Peano space. Then there exists an increasing sequence* $E_1 \subset E_2 \subset E_3 \subset \cdots$ *as follows:*

(i) E_n *is a connected open subset.*

(ii) $\bar{E}_n$ *is a Peano continuum.*

(iii) $\bar{E}_n \subset E_{n+1}$ $(n = 1, 2, \cdots)$.

(iv) $\bar{E}_n - E_{n-1} = G_n (E_0 = \phi)$ *is a locally connectd.*

(v) *If* $\mathfrak{P}_0$ *is the collection of all components of* $E_n - \bar{E}_{n-1}$, *it is a regular partitioning of* R.

(vi) $R = \bigcap_{n=1}^{\infty} E_n$.

(vii) *Distinct components of* G_n *are contained in distinct components of* $R - E_{n-1}$.

Proof. In [8] it has been shown that there exists a sequence $E_1', E_2', E_3', \cdots$ satisfying the conditions (i)$\sim$(vi). Since the boundary of each component of $R - \bar{E}_1'$ is a nonempty subset of $\bar{E}_1'$ and the components of G_2 are finite in number, there exist only finite components of $R - E_1'$, being denoted by $C_{1,1}, C_{1,2}, \cdots, C_{1,n(1)}$. It is obvious that the components of $C_{1,k} \cap (\bar{E}_2' - E_1')$ are finite in number, because $C_{1,k} \cap (\bar{E}_2' - E_1') = C_{1,k} \cap (M_1 \cup \cdots \cup M_l)$, where M_i $(i = 1, 2, \cdots)$ is a component of $\bar{E}_2' - E_1'$, and $M_i \subset C_{1k}$ or $M_i \cap C_{1,k} = \phi$. Since $C_{1,k} = C_{1,k} \cap (R - E_1')$ $= C_{1,k} \cap ((\bigcup_{i=1}^{\infty} \bar{E}_{1i}) - E_1') = \bigcup_{i=1}^{\infty} (C_{1,k} \cap (\bar{E}_i' - E'_1))$, by Lemma 1, there exists an integer $k(1)$ $(\geqq 2)$ such that $C_{1,k} \cap (\bar{E}_i' - E_1')$ is contained in a component of $C_{1,k} \cap (\bar{E}'_{k(1)} - E_1')$ Since $\bar{E}'_{k(1)}$ is connected, each component of $C_{1,k} \cap (\bar{E}'_{k(1)} - E_1')$ contains a point of a component of $C_{1,k} \cap (\bar{E}_2' - E_1')$, $C_{1,k} \cap (\bar{E}'_{k(1)} - E_1')$ is connected. Let $E_1 = E_1'$ and $E_2 = E_1' \cup \{\bigcup_{k=1}^{n(1)} \{C_{1,k} \cap (\bar{E}'_{k(1)} - E_1')\}\}$. Let us continue this manner. In general, let

$C_{i,1}, C_{i,2}, \cdots, C_{i,n(i)}$ be the components of $R-\bar{E}_i$ and let E_j' be an element such that $E_j' \supset \bar{E}_i$. By the same process as in case $i=1$, we shall get E_{i+1} satisfying the conditions in the lemma.

Theorem 4. *Each Peano space R has a bounded convex metric for which the space has property S.*

Proof. In case where R is compact, this is the result of Bing and Moise. Assume that R is non-compact.

Let $\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2, \cdots$ be a sequence of partitionings such that (1) $\mathfrak{P}_i$ is a core refinement of $\mathfrak{P}_{i-1}$, (2) the mesh of $\mathfrak{P}_i$ is less than $1/2^i$, (3) the diameter of any element of $\mathfrak{P}_i$ in G_n is less than one third the minimum of the distance of any two non-adjacent elements of $\mathfrak{P}_{i-1}$ in G_n and the distance between E_{n-1} and the closure of the sum of the elements in $\mathfrak{P}_i$ whose boundary has no point in common with E_{n-1}.

Now we assign a size for each element g of $\mathfrak{P}_i$ as follows:

$$\text{size of } g=\begin{cases} 1/2 \text{ if } g\subset E_1 \text{ and } g \text{ is a border element of } \mathfrak{P}_1, \\ 1/(2^2\cdot N_{1,1}) \text{ if } g\subset E_1 \text{ and } g \text{ is an interior element of } \mathfrak{P}_1, \\ 1/2^i \text{ if } g\subset E_i-\bar{E}_{i-1} \text{ and } g \text{ is a border element of } \mathfrak{P}_1, \\ 1/(2^{2+i}\cdot N_{1,i}) \text{ if } g\subset E_i-\bar{E}_{i-1} \text{ and } g \text{ is an interior ele-} \\ \quad \text{ment of } \mathfrak{P}_1, \\ 1/2 \text{ of the size of the element of } \mathfrak{P}_{k-1} \text{ which contains} \\ \quad g: \text{ if } g \text{ is a border element of } \mathfrak{P}_k, \\ 1/(2^{k+i}\cdot N_{k,i}) \text{ if } g \text{ is an interior element in } E_i-\bar{E}_{i-1}, \end{cases}$$

where $N_{k,i}$ is the number of interior elements of $\mathfrak{P}_k$ in E_i. If $p\cup q\subset \bar{E}_l$, the chain C which length $=d_k(p,q)$ is not empty. Let $X_1, X_2, \cdots, X_n$ be links of C such that $p\in X_1$ and $q\in X_n$. Suppose that $X_i\subset \bar{E}_{l+1}$, $X_{i+p}\subset \bar{E}_{l+1}$ and $X_{i+j}\subset R-E_{l+1}$ $(1\leq j\leq p-1)$. By the condition (vii) and connectedness of $\overline{X_i\cup\cdots\cup X_{i+p}}$, X_i and X_{i+p} are contained in the same component K of $\bar{E}_{l+1}-E_l$. It is easily seen there exists a chain in E_{l+1} joining p to q, and having a shorter length than C. This contradicts the definition of $d_k(p,q)$. Hence we conclude that C is in E_{l+1} and $\lim C\subset \bar{E}_{l+1}$.

Now we estimate the value of $\lim d_k(p,q)=d(p,q)$. Since $d_{k+1}(p,q)\leqq d_k(p,q)$ $+1/2^{k+1}+1/2^{k+2}+\cdots+1/2^{k+l+1}$, $d(p,q)=\lim d_k(p,q)\leqq d_1(p,q)+1/2+1/2^2+\cdots$ $+1/2^{l+1}<d_1(p,q)+1<4+1/2$. $d(p,q)$ has all properties for metric.

5. Compactings of Peano spaces

In §1 we have seen one of well known compactings of locally compact spaces. Let R be a simply connected Peano space, then the associated compact space R^* given by the compacting is not always simply connected; for example, if R is R^1, R^* is a simple closed curve. We shall prove that the socalled Freudenthal's compacting preserves simply connectedness.

Definition. By the Freudenthal's compacting will be meant the compacting as follow: let R^{**} be the set $R+\mathfrak{B}$ where $\mathfrak{B}$ is the same notation as in §1. In order to define a topology of R^{**}, we shall assign to each point p of R^{**} in the following way: if p is in R, it has the same neighborhorhoods as in the original space R, and if p is in $\mathfrak{B}$, its neighborhoods are $\{C_i+Q\}$ where C_i is an element of a sequence defining p. Q is a set of all points q of $\mathfrak{B}$ such that C_i is an element of a sequence defining q. We shall also denote the obtained topological space by the same notation, R^{**}. Then it is a Peano continuum. Each point of $R^{**}-R$ is called a *boundary point of* R.

Definition. A Peano space is said to be a *tree* provided it contains no simple closed curve. If it is compact, it will be said to be a *dendrite* (or *acyclic curve*).

Definition. A topological space R is said to be *shrinkable* provided that there exists a continuous mapping f of $R\times[0, 1]$ onto R such that $f(x, 0)=x$ and $f(x, 1)=p$ for each point x and for a fixed point p. It is readily seen that if R is arcwise connected, then p can be arbitrarily chosen.

It is known that in order that a Peano space might be a tree (or dendrite), it is necessary and sufficient that one and only one arc exists between any two points (and the space is compact).

Lemma. *Each tree T is shrinkable.*

For by the convexification theorem, T has a bounded convex metric $\rho(x, y)$. Let p be a fixed point in T. There exists one and only one segment px joining p to each point x in R. For each number $t\in[0, 1]$, we shall define $f(x, t)=x_t$ where $x_t\in px$ and $\rho(x_t, p)=t\rho(x, p)$. Then T is shrinkable by f.

Theorem 5. *Let R be a simply connected Peano space. Then there exists a compacting such that the associated compact space is simply connected.*

Proof. We shall show that the Freudenthal's compacting preserves simply connectedness. To do this, first, we shall construct a tree in R. The notations used in proof are the same as in §4. Let $E_{j_1j_2}(j_1=1)$ be an element of $\mathfrak{P}_0$ in $E_3-\bar{E}_2$ such that $E_{j_1j_2j_3}$ and $E_{j_1j_2}$ are contained in the same component of $R-\bar{E}_1$.

In general, let $E_{j_1j_2\cdots jm-1}$ be an element of $\mathfrak{P}_0$ in $E_m-\bar{E}_{m-1}$ such that $E_{j_1j_2\cdots im-1}$ and $E_{j_1j_2\cdots jm}$ are contained in the same component of $R-\bar{E}_{m-2}$. Let $p_{j_1j_2\cdots jm}$ be a point in $E_{j_1j_1\cdots jm}$. We can easily construct a tree T in R by joining $p_{j_1j_2\cdots jm}$ to $p_{j_1j_2\cdots jm-1}$ in $E_m-\bar{E}_{m-2}$ and we can see that the closure of T in R^{**} is a dendrite T' and contains $R^{**}-R$.

Now let ω be a closed path in R^{**}, that is, a subcontinuum of R^{**} which is an image of a continuous mapping f of $[0, 1]$ such that $f(0)=f(1)$. As $R^{**}-R$ is closed, so is $(R^{**}-R)\frown\omega$. If $(R^{**}-R)\frown\omega=\phi$, then $\omega\subset R$ and $\omega\sim 0$. If $(R^{**}-R)\frown\omega$ is a point $f(t)$, there exists $f(t_i^1)$ and $f(t_i^2)$ such that $\lim t_i^1=t=\lim t_i^2$ and $t_i^1<t<t_i^2$ (where we can assume that $0<t<1$ without loss of generarity). Since $\lim f(t_i^1)=f(t)=\lim f(t_i^2)$, for each neighborhood U of $f(t)$ there exists an integer i_0 such that $f(t_{i0}^1)\smile f(t_{i0}^2)\subset U$. Further f_{i0}^1, f_{i0}^2 are joined by an arc α in $U\frown R$. There exists a closed path through $f(t)$ in U which is a subset of $\omega\smile\alpha$ and is homotopic to ω. By $U\to f(t)$, it is seen that $\omega\sim 0$.

Now let $f(t_1)$, $f(t_2)$ be two different points on $\omega\frown(R^{**}-R)$ such that $f(t)\subset R$ for $t_1<t<t_2$ and let ω_1' be an arc in T' joining $f(t_1)$ to $f(t_2)$. Then $\omega_1'\sim\{f(t)/t_1\leq t\leq t_2\}=\omega_1$. Thus each such subpath ω_1 is homotopic to a subarc in T'. Sinch $R^{**}-R$ is 0-dimensional, the number of ω_1 is at most countable. Thus ω is homotopic to a subcontinuum L of T'. Since L is a dendrite, by lemma $L\sim 0$, and thus $\omega\sim 0$.

Remark. The complete enclosure of the bounded convex metric space obtained in Theorem 3 is identical with Freudenthal's compacting.

6. Extensions of a convex metric

Let R be a metric space whose metric is $\rho(x, y)$ and let M be a subset of R on which a metric $\sigma(x, y)$ is defined. We shall say that $\rho(x, y)$ is an extension of $\sigma(x, y)$ if $\rho(x, y)=\sigma(x, y)$ on M. It has been shown by Bing ([2]) that if M is a closed subset of a metrizable space R and $\sigma(x, y)$ is a metric on M, then there is a metric $\rho(x, y)$ on R which is an extension of $\sigma(x, y)$.

We shall consider the case where $\sigma(x, y)$ is convex and R is a Peano space. The proof we give is due to that given by Bing.

Theorem 6. *Suppose R is a Peano space, M is a closed subset of R and $\sigma(x, y)$ is a convex metric on M. Then there exists a convex metric $\rho(x, y)$ on R which is an extension of $\sigma(x, y)$. If σ is bounded, $\rho(x, y)$ may be bounded.*

The assumption of the above theorem that $\sigma(x, y)$ is convex is necessary.

For example, let R be a simple arc and let M be three different points p, q, r on R arranged in natural oder. Let $\sigma(p, q)=\sigma(q, r)=\sigma(p, r)=1$. Then there exists no convex metric on R which is an extension of $\sigma(x, y)$.

Proof of Theorem. Case 1. M is compact. By the lemma it suffices to prove it for the case where R is compact. We may assume that the diameter of M under the metric σ is less than $1/2$, without loss of generality. Let α_1 be the glb $\rho'(p, q)$ for all pairs of points in M that $\sigma(p, q) \geqq 1/2^2$. Then $\alpha_1>0$. For if $\alpha_1=0$, there would exist two sequences of points $\{p_n\}$ and $\{q_n\}$ in M such that for each n, $\rho'(p_n, q_n)<1/n$, but $\sigma(p_n, q_n) \geqq 1/2^2$. Since M is compact we can suppose them so chosen that $\{p_n\} \rightarrow p$ where $p \in M$. But then also $\{q_n\} \rightarrow p$ by the equivalency of $\rho'(x, y)$ and $\sigma(x, y)$ on M, for n sufficiently large, $\sigma(p_n, q_n)<1/2^2$, contrary to the definition of $\{p_n\}$ and $\{q_n\}$.

There exists a regular partitioning of R whose mesh is less than $\alpha_1/3$. Each element of $\mathfrak{P}_1$ is assigned a size eqal to $1/2$. In general, we obtain a sequence of partitionings of R, $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \cdots$ as follows:

(i) $\mathfrak{P}_{i+1}$ is a core refinement of $\mathfrak{P}_i$ $(i=1,2, \cdots)$.

(ii) The mesh of $\mathfrak{P}_i$ is less than $\min[1/2^i, \alpha_i/3]$, where α_i is the glb $\rho'(p, q)$ for all pairs of points in M that $\sigma(p, q) \geqq 1/2^{i+1}$ and $\alpha_i>0$.

(iii) Each element of $\mathfrak{P}_{i+1}$ is of diameter less than one-third the distance of non-adjacent elements of $\mathfrak{P}_i$.

Then the element g of $\mathfrak{P}_i$ $(i \geqq 2)$ is assigned a size as follows:

$$\text{size of } g=\begin{cases} 1/2^i \text{ if } g \cap M \neq \phi \text{ or } g \cap g' \neq \phi,\ g' \cap M \neq \phi,\ g' \ni \mathfrak{P}_i . \\ 1/(2^i \cdot N_i) \text{ if } g \text{ is an interior element of } \mathfrak{P}_1, \text{ where } N_i \\ \text{is the number of interior elements of } \mathfrak{P}_{i-1}. \\ \text{one-half the size of the element of } \mathfrak{P}_1 \text{ containing } g \text{ if} \\ g \text{ is a border element of } \mathfrak{P}_i . \end{cases}$$

By the i-th chain $\mathfrak{C}=[C_1, C_2 \cdots, C_n]$ from p to q we shall mean the chain such that $p \in C_1$, $q \in C_n$, C_i being either an element of $\mathfrak{P}_i$ or a segment in M, and then both $C_{j-1} \cap C_j$ and $C_j \cap C_{j+1}$ consisting of only one point. Further by the length of $\mathfrak{C}$ we shall mean the sum of the sizes of C_i's, where if C_j is a segment in M, the size of C_j is length for σ. By $\rho_i(p, q)$ we shall denote the lower bounded value of length of i-th chains from p to q whose length is $\rho_i(p, q)$. We note that $\mathfrak{C}_0$ can have at most one link which is a segment in M and see that $\lim \rho_i(p, q)$ is bounded. $\rho(p, q)=\lim \rho_i(p, q)$ is an extension of σ.

Case 2. M is non-compact. Let $M_i = E_i \cap M$. We shall construct a sequence of partitionings $\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2, \cdots$ as follows : (1) $\mathfrak{P}_i$ is a core refinement of $\mathfrak{P}_{i-1}$, (2) mesh of $\mathfrak{P}_i$'s converges to zero, (3) each element of $\mathfrak{P}_{i+1}$ is of diameter less than one-third the distance of non-adjacent elements of $\mathfrak{P}_i$, (4) let d_k be the diameter of M_k for σ and let N_k be the number of elements of $\mathfrak{P}_1$ in E_k. We shall assign a size to the element of $\mathfrak{P}_1$ in $E_{k+1} - \bar{E}_k$ such as less than $\rho(E_k, \mathcal{F}(E_{k+1}))/3(d_k + N_k)$ and the same process as in Case 1 is carried on.

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