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Convexification of Locally Connected Generalized Continua

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1. Introduction.

By a generalized continuum we mean a locally compact, connected, separable, metric space. If a generalized continuum is locally connected and compact, it is called a continuous curve or sometimes a Peano continuum. We say that a space R having a metric $d(x, y)$ is finitely compact if, for each positive number α and a point p in R, the closure of the set $\{x/d(p, x) < \alpha\}$ is compact. Finitely compact spaces are complete. It is known that each pair of points in a continuous curve can be joined by an arc. A metric $d(x, y)$ is called convex if p and q are points in R, there is a point r in R such that $d(p, r)=d(q, r)=d(p, q)/2$. If $d(x, y)$ is a convex metric for a compact (or more generally, complete) space R, then for each pair of points p, q of R there is an arc pq in R from p to q such that pq is isometric to a straight line interval. (Such arcs are called segments.)

In 1928, K. Menger [4] had proposed the following question: Whether or not each continuous curve has a convex metric that preserves its topology? By R. H. Bing [1] and E. E. Moise [5] the question was answered in the affirmative.

It this paper, we shall consider the convexification problem for the case where the space is locally compact, not necessarily compact, and prove the following theorem:

THEOREM. *Each locally connected generalized continuum has a convex metric and each pair of points can be joined by a segment.*

The method of this theorem is a modification of the method used by Bing.

2. Definitions.

For convenience we shall arrange the definitions used in this paper. Most of them were introduced in the papers [1] and [2].

Property S. A set has *property* S if, for each positive number ε, the set is the sum of a finite number of connected subsets each of diameter less than ε.

Local property S. A set M has *local property* S provided that, for each point p of M, there is a neighborhood of p which has property S.(1)

Chain. Let X and Y be sets. By a *chain from* X *to* Y is meant a finite sequence of sets $X_1, X_2, \ldots, X_n$ such that $X=X_1$, $Y=X_n$ and $X_i \cdot X_j \neq \phi$ if $|i-j| = 1$. X_i is called a *link* of the chain. A chain is said to be *simple* provided that $X_i \cdot X_j \neq \phi$ if and only if $|i-j|=1$. Let K and K' be chains from X to Y and from X' to Y' respectively such that $X \supset X'$, $Y \supset Y'$. If each link of K' is contained in a link of K, K' is said to be *in* K.

Partitioning. A *partitioning* $\mathfrak{P}$ of M is a collection of mutually exclusive connected open subsets of M whose sum is dense in M and each point p in M has a neighborhood which is covered by the closure of the sum of a finite subcollection of $\mathfrak{P}$.

Mesh of a partitioning. The *mesh* of a partitioning $\mathfrak{P}$ is the least upper bound of diameters of the elements in $\mathfrak{P}$.

Finite partitioning.(2) If the elements in a partitioning $\mathfrak{P}$ are finite in number, $\mathfrak{P}$ is called *finite*.

Refinement. If $\mathfrak{P}$ and $\mathfrak{Q}$ are two partitionings of a set, $\mathfrak{P}$ is called a *refinement* of $\mathfrak{Q}$ if each element of $\mathfrak{P}$ is a subset of an element of $\mathfrak{Q}$.

Regular. An open subset of a space is *regular* if it is the interior of its closure.

Regular partitioning. A partitioning is *regular* if each of its elements is regular.

Border and interior elements. If the partitioning $\mathfrak{P}$ is a refinement of the partitioning $\mathfrak{Q}$, the elements of $\mathfrak{P}$ which have a boundary point in common with the boundary of an element of $\mathfrak{P}$ are called *border elements*. Other elements are called *interior elements*.

Core refinement. Let $\mathfrak{P}$ and $\mathfrak{Q}$ be regular partitionings. We call $\mathfrak{P}$ a *core refinement of* $\mathfrak{Q}$ if $\mathfrak{P}$ is a refinement of $\mathfrak{Q}$, each border element of $\mathfrak{P}$ is adjacent to an interior element, and the sum of interior elements of $\mathfrak{P}$ in each element of $\mathfrak{Q}$ has a connected closure.

3. Preliminary Lemmas.

LEMMA 1. *If G is an open set, then $R-\bar{G}$ is regular* (cf. [3], p. 278).

(1) It is readily seen that if a neighborhood of p has property S, there exists an arbitrarily small one having property S.

(2) We use finite partitioning here in the sense of the paritioning of Bing.

LEMMA 2. *Each component M of a regular subset G of a locally connected space is regular.*

PROOF. M is open because of locally connectedness of the space. We denote the interior of $\bar{M}$ by $\mathcal{J}(\bar{M})$, then $M \subseteq \mathcal{J}(\bar{M}) \subseteq \bar{M} \subseteq \bar{G}$. Furthermore $\mathcal{J}(\bar{M}) \subseteq \mathcal{J}(\bar{G}) = G$. Since $\mathcal{J}(\bar{M})$ is connected, $M = \mathcal{J}(M)$

LEMMA 3. *Let $\mathfrak{P}$ be a partitioning of R. For each point p in R, an open neighborhood of p is covered by the closure of the sum of the elements of $\mathfrak{P}$ whose closure contains p.*

PROOF. By virtue of the definition of the partitioning, there exists an open neighborhood V of p and a finite subcollection $\mathfrak{Q} = \{g_1, g_2, \cdots, g_k\}$ of $\mathfrak{P}$ such that $U \subseteq \bigcup_{i=1}^{k} g_i$. Let $\mathfrak{Q}' = \{g'_1, g'_2, \cdots, g'_l\}$ be the subcollection of all elements in $\mathfrak{P}$ each of whose closure contains p. Suppose p is not an interior point of $\bigcap_{j=1}^{l} \overline{g'}_j$, then there is a sequence $\{p_m\}$ such that $\lim p_m = p$ and $p_m \notin \bigcup_{j=1}^{l} \overline{g'}_j$. Then there exists an infinite subsequence of $\{p_m\}$ in U which is contained in the closure of an element g of $\mathfrak{Q} - \mathfrak{Q}'$, and therefore $p \in \bar{g}$. This contradicts the definition of $\mathfrak{Q}'$.

LEMMA 4. *Let R be a locally connected generalized continuum. Then there exists an increasing sequence $E_1 \subset E_2 \subset E_3 \cdots$ as follows:*

(i) *E_n is a connected open subset.*

(ii) *$\bar{E}_n$ is locally connected, compact, connected.*

(iii) $\bar{E}_n \subset E_{n+1}$ $(n = 1, 2, 3, \cdots)$.

(iv) $\bar{E}_n - E_{n-1} = G_n$ $(E_0 = \phi)$ *has property S.*

(v) *If $\mathfrak{P}_0$ is the collection of all components of $E_n - \bar{E}_{n-1}$, it is a regular partitioning of R.*

(vi) $R = \bigcup_{n=1}^{\infty} E_n$.

PROOF. Since R has local proprety S (cf. [6], p. 22), there exists a countable sequence $\mathfrak{V} = \{V_1, V_2, \cdots\}$ covering R whose elements are all conditionally compact, connected and has property S. Set $F_1 = V_1$ and let V_{j2} be the first element of $\mathfrak{V}$ which is not contained in F_1. Then F_1 and V_{j2} can be joined by a chain C_1 whose links are the elements of $\mathfrak{V}$, because R is connected. Obviously $\overline{C_1{}^*}$ (which means the sum of links in C_1) is compact and connected. Hence there exists a finite subcollection $\mathfrak{U}_2$ of $\mathfrak{V}$ covering $\overline{C_1^*}$. Let $\mathfrak{V}_2^* = F_2$. In general, let V_{jn} be the first element of $\mathfrak{V}$ which is not contained in F_{n-1}. We can join F_{n-1} and V_{jn} by a chain C_n whose links are elements of $\mathfrak{V}$. Since $\overline{C_n^*}$ is compact, there is a finite subcollection $\mathfrak{U}_n$ of $\mathfrak{V}$ covering $\overline{C_n^*}$. Let $\mathfrak{U}_n^* = F_n$. Continuing this process infinitely we obtain a sequence F_1, F_2, F_3, $\cdots$ which has the following properties:

(i)' F_n is a connected open subset.

(ii)' $\bar{F}_n$ is locally connected, compact, connected.

(iii)' $\bar{F}_n \subset F_{n+1}$ $(n=1, 2, 3, \cdots)$.

(iv)' $R=\bigcup_{n=1}^{\infty} F_n$.

Since $\bar{F}_n$ and $\mathscr{F}(F_{n+1})$(1) are two disjoint closed subsets in $\bar{F}_{n+1}$, there are two mutually exclusive, relatively open subsets K and H of $\bar{F}_{n+1}$ containing $\bar{F}_n$ and $\mathscr{F}(F_{n+1})$ respectively such that each of K and H has property S and $F_{n+1}=\bar{K}+\bar{H}$. (See [1]). Let $E_n=\mathscr{I}(\bar{K})$. We note that K may be so chosen that it is connected.

It is readily seen that E_n's satisfy the conditions (i), (ii), (iii) and (vi). To prove that $\bar{E}_n-E_{n-1}$ has property S, it is sufficient to show that $E_n-\bar{E}_{n-1}$ has property S and $\overline{E_n-\bar{E}_{n-1}}=\bar{E}_n-E_{n-1}$. Suppose on the contrary that $\overline{E_n-\bar{E}_{n-1}} \neq \bar{E}_n-E_{n-1}$. Then there exists a point p of $\bar{E}_n-E_{n-1}$ not belonging to $\overline{E_n-\bar{E}_{n-1}}$. Let U be an open neighborhood of p such that $U \cap \overline{E_n-\bar{E}_{n-1}}=\phi$. Since $E_n-\bar{E}_{n-1} \subseteq \overline{E_n-\bar{E}_{n-1}}$ and $E_n \supset \bar{E}_{n-1}$, $U \cap E_n=\phi$ or $U \subset \bar{E}_{n-1}$. If $U \cap E_n=\phi$, contrary to $p \in \bar{E}_n$. If $U \subset E_{n-1}$ [because of regularity of E_{n-1}], then $p \in E_{n-1}$, contrary to $p \notin E_{n-1}$. It remains to show that $E_n-\bar{E}_{n-1}$ has property S. Since $\bar{K} \supset E_n \supset K$ and K has property S, E_n does so. Furthermore, since $E_n-\bar{E}_{n-1}=(E_n-\bar{F}_n)+(\bar{F}_n-\bar{E}_{n-1})=(E_n-\bar{F}_n)+H'$, where H' is obtained by division of $\bar{F}_n$ as above, it follows that $E_n-\bar{E}_{n-1}$ has property S.

Now, by Lemma 1, $E_n-\bar{E}_{n-1}$ is regular and therefore each component of $E_n-\bar{E}_{n-1}$ is regular on the basis of Lemma 2. Since $E_n-\bar{E}_{n-1}$ has property S, the components of $E_n-\bar{E}_{n-1}$ is finite in number. Thus $\mathfrak{P}_0$ is a regular partitioning of R.

4. Proof of Theorem.

In this section we shall use the same notations as in the previous section.

By Lemma 4, R has a sequence $\mathfrak{P}_0$, $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\cdots$ of partitionings such that (1) $\mathfrak{P}_i$ is a core refinement of $\mathfrak{P}_{i-1}$, (2) the mesh of $\mathfrak{P}_i$ is less than $1/2^i$, (3) the diameter of any element of $\mathfrak{P}_i$ in G_n is less than one third the minimum of the distance of any two non-adjacent elements of $\mathfrak{P}_{i-1}$ in G_n and the distance between E_{n-1} and the closure of the sum of the elements in $\mathfrak{P}_i$ whose boundary has no point in common with E_{n-1}, and (4) the diameters of elements of $\mathfrak{P}_1$ contained

(1) This means the boundary of F_{n+1}.

in G_{n+1} are less than $d(E_{n+1}, \mathcal{F}(E_n))/N_n$ where N_n is the number of elements of $\mathfrak{P}_1$ which is contained in $\bar{E}_n$. Here we note that the subcollection of $\mathfrak{P}_i$ whose elements are in E_n is a finte partitioning of $\bar{E}_n$ and each element of $\mathfrak{P}_i$ contains only finite elements of $\mathfrak{P}_{i+1}$.

The element of $\mathfrak{P}_1$ are assigned a size of 1. Each border element of $\mathfrak{P}_{i+1}$ is assigned a size equal to one-half the size of the element of $\mathfrak{P}_i$ containing it. Each interior element of $\mathfrak{P}_{i+1}$ which is contained in G_n is given a size equal to $1/(2^{i+1+n}\cdot N_{i+1,n})$ where $N_{i+1,n}$ is the number of elements of $\mathfrak{P}_{i+1}$ in G_n.

Suppose C is a chain each of whose links is the closure of an element of $\mathfrak{P}_i$. The sum of the sizes of the elements of $\mathfrak{P}_i$ in C is called the i-th length of C. There is a chain C from p to q whose length gives the minimum of the i-th length of the chains from p to q. The length of such a chain C is denoted by $d_i(p, q)$. If p and q are in $\bar{E}_n$, then C is contained in $\bar{E}_{n+1}$. For there exists a chain from p to q in $\bar{E}_n$ whose length is less than N_n+1. However, the length of each chain from p to q one of whose links is contained in $\overline{E_{n+l}}-E_{n+1}$ $(l>1)$ is greater than N_n+1 (by (4)).

It may be noted that if $d_i(p, q)$ is the i-th length of C, then C is a simple chain.

Suppose $p+q$ is contained in $\bar{E}_n$ and let C be a chain from p to q which has the length $d_i(p, q)$ and C', a chain in C such that C' has an $(i+1)$th length but does not contain three border elements of $\mathfrak{P}_{i+1}$ in the same element of $\mathfrak{P}_i$. Then the $(i+1)$th length of C' is not greater than $d_i(p, q)+1/2^{i+2}+1/2^{i+3}+\cdots+1/2^{i+n+2}$. Therefore $d_{i+1}(p, q)\leq d_i(p, q)+1/2^{i+2}+1/2^{i+3}+\cdots+1/2^{i+n+2}$ (by (3)). Hence $\lim d_i(p, q)=d(p, q)$ exists.

Obviously $d(p, q)\geq 0$ and $d(p, q)=0$ if $p=q$. Also $d(p, q)=d(q, p)$ because $d_i(p, q)=d_i(q, p)$. That $d(x, y)$ satisfies the triangle condition follows from the fact $d_i(p, q)+d_i(q, r)\geq d_i(p, r)$ where p, q, r are three points.

We now show that $d(x, y)$ preserves the topology of R. If p is a limit point of the subset A of R, then by Lemma 3 there is a point q of A such that $p+q$ is contained in the closure of the same element of $\mathfrak{P}_i$. Then $d_i(p, q)\leq 1/2^i$. Since $d(p, q)<d_i(p, q)+1/2^i$, we have $d(p, q)<1/2^{i-1}$ and therefore $d(p, A)=0$.

We next show that if p is not a point of the closed set A, $d(p, A)>0$. Since the mesh of $\mathfrak{P}_i$ is less than $1/2^i$ by (2), there exists an integer i such that any element of $\mathfrak{P}_i$ is adjacent neither to an element of $\mathfrak{P}_i$ whose closure contains p nor to an element of $\mathfrak{P}_i$ whose closure intersects A. If p is in $\bar{E}_n$ and ε is the least of the sizes of the elements of $\mathfrak{P}_i$ in $\bar{E}_{n+1}$, we assert that $d(p, A)\geq\varepsilon$.

To prove this, it will be noted that if C is a chain from p to a point a belonging to A and having an $(i+k)$th length, C contains 2^k elements g of $\mathfrak{P}_{i+k}$ such that g is a border element of $\mathfrak{P}_{i+k}$ and lies in the border elements of $\mathfrak{P}_{i+1}, \mathfrak{P}_{i+2}, \cdots, \mathfrak{P}_{i+k-1}$. The size of g is as much as $\varepsilon/2^k$ and the $(i+k)$th length of C is more than ε. Hence, for each point a of A, $d_{i+k}(p, a)>\varepsilon$ and $d(p, a)\geqq\varepsilon$.

Finally, we show that the metric $d(x, y)$ is convex. Let $C=[X_1, X_2, \cdots, X_m]$ be a chain from p to q having i-th length $d_i(p, q)$. Then there exists an integer $k(1\leqq k\leqq m)$ and a point r_i such that $r_i\in X_k$ and $C_p=[X_1, X_2, \cdots, X_k]$ and $C_q=[X_k, X_{k+1}, \cdots, X_m]$ are chains from p to r_i and from r_i to q respectively each of whose i-th length differs from one half the i-th length of C by less than $1/2^i$. Then r_i is in $\bar{E}_{n+1}$ if p and q are in $\bar{E}_n$. Since $\bar{E}_{n+1}$ is compact, there is a limit point r of $r_1, r_2, r_3, \cdots$. r is halfway between p and q. As $d(x, y)$ is finitely compact, there exists a segment for each pair of points.

It will be noticed that we have also proved :

COROLLARY. *Each locally connected generalized continuum is topologically complete.*

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