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Extension of ♮-Application to Unbounded Operators

By

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In the previous paper [13], the present authors developed the so-called "non-commutative theory" of integration for rings of operators from a point of view resumed as follows. Every semi-finite ring of operators $\mathbb{M}$ with a normal, faithful and essential pseudo-trace m is normally $*$-isomorphic to the left ring $\mathbb{L}$ of an H-system $\boldsymbol{H}$ such that m corresponds to the canonical pseudo-trace of $\boldsymbol{H}$ [13]. We have shown that this $*$-isomorphism can be uniquely extended to a $*$-isomorphic mapping between the sets of measurable operators with respect to $\mathbb{M}$ and $\mathbb{L}$ respectively. Thus the theory of integration for $\mathbb{M}$ can be reduced to that for $\mathbb{L}$. But in $\boldsymbol{H}$ the set of all square-integrable measurable operators is given *a priori*, basing on which our whole theory was built.

In his investigation on ♮-applications in a ring of operators, Dixmier has shown ([4], Theorem 3) that every normal, faithful and essential pseudo-trace defined on a semi-finite ring $\mathbf{M}$ has the form $m(A)=\varphi(A^\natural)$, where ♮ is a fixed normal, faithful and essential pseudo-♮-application defined on $\mathbf{M}^+$ and φ is a normal, faithful and essential pseudo-measure on the spectre Ω of the center $\mathbf{M}^\natural$. This leads us to another formulation of the theory which is divided into two parts: the classical theory of pseudo-measure on the spectre Ω of $\mathbf{M}^\natural$ and the extension of ♮-application to unbounded operators $\eta\mathbf{M}$. The main purpose of this paper is to develop this theory of extension. The pseudo-♮-application defined on $\mathbf{M}^+$, $\mathbf{M}^+\ni A\rightarrow A^\natural\in \boldsymbol{Z}$, will be extended over the set of all positive, closed, densely defined operators $T\eta\mathbf{M}$, $T\rightarrow T^\natural\in\boldsymbol{Z}$,

$$(\natural)\qquad T^\natural = \operatorname*{l.u.b.}_{\mathbf{M}^+\ni A\leq T} A^\natural.$$

If we wish the integral of T to be finite, $T^\natural$ must be finite except on a nowhere dense subset of Ω. Such a T will be measurable in the sense of Segal ([15], [13]) and the set of all such T forms the positive part of an invariant linear system $\mathfrak{S}$, which will play a fundamental rôle in our present theory.

§1 is devoted to the proof of a theorem concerning the least upper bound of an increasing directed set $\{T_\delta\}$ of positive, closed and densely defined operators

$T_\delta \eta \mathbf{M}$. Then l.u.b. $T_\delta = T_0$ exists if and only if $\mathfrak{D} = \{x ; \{\|T_\delta^{\frac{1}{2}} x\|\}$ is bounded$\}$ is dense, and if this is satisfied $\mathfrak{D}_{T_0^{\frac{1}{2}}} = \mathfrak{D}$ and $\|T_\delta^{\frac{1}{2}} x - T_0^{\frac{1}{2}} x\| \to 0$ for every $x \in \mathfrak{D}_{T_0^{\frac{1}{2}}}$ (Theorem 1).

In §2 the properties of extended pseudo-$\natural$-application defined by ($\natural$) will be discussed. It is a normal, faithful and essential application if so is the original one. It will be proved that the set $\mathfrak{S}^+$ of all positive operators $\eta \mathbf{M}$ such that $T^\natural$ is finite except on a nowhere dense subset of Ω forms the positive part of an invariant linear system $\mathfrak{S}$ which satisfies the conditions $(\ll)_1$ and $(\ll)_2$ introduced in [13]. Then the invariant linear system $\mathfrak{S}^\alpha (\alpha > 0)$ will conveniently be defined, and hold the relations $(\mathfrak{S}^\alpha)^\beta = \mathfrak{S}^{\alpha\beta}$, $\mathfrak{S}^\alpha \cdot \mathfrak{S}^\beta = \mathfrak{S}^{\alpha+\beta}$ for every $\alpha, \beta > 0$. Besides we shall prove that our extended pseudo-$\natural$-application defined on $\mathfrak{S}^+$ can be uniquely extended to an "extended $\natural$-application" on $\mathfrak{S}$. It is noted that the extended pseudo-$\natural$-application is an application onto the set of all functions $\in \boldsymbol{Z}$, finite except on a nowhere dense set. We show that $\mathfrak{S}$ is an algebra if and only if $\mathbf{M}$ is of type I. Various special properties concerning the extended $\natural$-application are proved. Finally, as an example, the canonical $\natural$-application of an H-system (=Ambrose space [14]) will be considered.

As an application of these results, the theory of integration will be developed in §3. $\mathfrak{S}$ contains every "integrable operator" with respect to a normal, faithful and essential pseudo-trace. We shall define, as usual, the space $\boldsymbol{L}_1$ of all integrable operators and the space $\boldsymbol{L}_2$ of all square-integrable operators. The monotone convergence theorems for them will be proved, and by using these results we show that $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are complete. Finally the Radon-Nikodym theorem in the sense of Segal [15] will be proved anew.

§ 1. Preliminaries

Throughout this paper the following conventions will be used. Let $\mathfrak{H}$ be a Hilbert space of arbitrary dimension. Unless otherwise stated, operator will always mean a linear closed operator on $\mathfrak{H}$ with dense domain. The domain of an operator T will be denoted by $\mathfrak{D}_T$. A ring of operators $\mathbf{M}$ on $\mathfrak{H}$ will mean an algebra of bounded everywhere defined operators which is self-adjoint (i.e. closed under adjunction), closed in the weak (operator) topology and contains the identity operator I. $\mathbf{M}_U$ and $\mathbf{M}_P$ denote the set of all unitary operators and the set of all projections in $\mathbf{M}$ respectively. $\mathbf{M}^+$ and $\mathbf{M}^\natural$ stand for the positive part of $\mathbf{M}$ and the center of $\mathbf{M}$ respectively. $P^\perp$ is the orthocomplement of a projection P. If A is a bounded operator, $\|A\|$ will denote the operator norm of A. The strong sum, strong difference and strong product of two measurable operators S and T are denoted as $S \dotplus T$, $S \dot{-} T$ and $S \cdot T$ respectively ([13], [15]).

DEFINITION 1. (cf. [8]). Let S and T be positive operators. We write $S \leq T$ if $\mathfrak{D}_{T^{\frac{1}{2}}} \subset \mathfrak{D}_{S^{\frac{1}{2}}}$, and $\|S^{\frac{1}{2}}x\| \leq \|T^{\frac{1}{2}}x\|$ for every $x \in \mathfrak{D}_{T^{\frac{1}{2}}}$.

We note that this condition is equivalent to that $\mathfrak{D}_T \subset \mathfrak{D}_{S^{\frac{1}{2}}}$ and $\|S^{\frac{1}{2}}x\| \leq \|T^{\frac{1}{2}}x\|$ for every $x \in \mathfrak{D}_T$.

In our previous paper [13], we have defined the order between two self-adjoint measurable operators S and T as follows: $S \leq T$ if and only if the strong difference $T \dot{-} S$ is positive. But in case of positive measurable operators it can be easily seen that these two notions are identical. Moreover, in this case $S \leq T$ if and only if $\langle Sx, x\rangle \leq \langle Tx, x\rangle$ holds on a dense set $\mathfrak{D}$ contained in $\mathfrak{D}_S \cap \mathfrak{D}_T$. For, let S' and T' be the respective restriction of S and T on $\mathfrak{D}$, then $(T' - S')^{**}$ exists and agrees on $\mathfrak{D}$ with $T \dot{-} S$, and hence $(T' - S')^{**} = T \dot{-} S$ ([13], Lemma 1.2). Thus $T \dot{-} S$ is the closure of $T' - S'$. From this we can easily see $T \dot{-} S \geq 0$.

Before stating Theorem 1, we cite the following two propositions which will be used repeatedly in the proof.

1. (Lemma of E. Heinz [8]). Let S and T be operators such that $S \geq c$ and $T \geq c$ for some positive constant c. Then $T \leq S$ and $T^{-1} \geq S^{-1}$ are equivalent.

2. (Theorem of I. Kaplansky [9]). Let $h(t)$ be a continuous bounded real-valued function of the real variable t. Then the mapping $A \rightarrow h(A)$ is strongly continuous on the set of all bounded self-adjoint operators.

THEOREM 1. *Let $\{T_\delta\}$ be an increasing directed set of positive operators $\eta \mathbf{M}$. Then the following conditions* (1), (2), (3), (4) *and* (5) *are equivalent:*

(1) *There exists a positive operator T such that $T_\delta \leq T$ for every δ;*

(2) $\text{l.u.b.}_{\delta}\, T_\delta = T_0$ *exists in the sense of the ordering of the positive operators on $\mathfrak{H}$;*

(3) $\mathfrak{D} = \{x ; \{\|T_\delta^{\frac{1}{2}}x\|\}$ is bounded$\}$ *is dense in $\mathfrak{H}$;*

(4) *There exists a positive operator T' such that $T_\delta^{\frac{1}{2}} \leq T'$ for every δ;*

(5) $\text{l.u.b.}_{\delta}\, T_\delta^{\frac{1}{2}} = S_0$ *exists in the sense of the ordering of the positive operators on $\mathfrak{H}$.*

Moreover, if any one of these conditions is satisfied, then $T_0^{\frac{1}{2}} = S_0 \eta \mathbf{M}$ and $T_0^{\frac{1}{2}}$ is characterized as the operator S_1 such that $\mathfrak{D}_{S_1} = \mathfrak{D}$ and $\|T_\delta^{\frac{1}{2}}x - S_1 x\| \rightarrow 0$ for every $x \in \mathfrak{D}$.

PROOF. First we shall prove the equivalence of (1)–(5).

Ad (1) $\rightarrow$ (2): By the lemma of E. Heinz cited above, we have $(I + T_\delta)^{-1} \geq (1 + T)^{-1}$ for every δ, and $\{(I + T_\delta)^{-1}\}$ is a decreasing directed set of bounded positive operators. Hence by a theorem of Dixmier [4], $\underset{\delta}{\text{g.l.b.}}(I+T_\delta)^{-1}=A$ exists with $A\in\mathbf{M}$ and $(I+T_\delta)^{-1}$ converges strongly to A. Since $A\geq(I+T)^{-1}$, it is easy to see that A^{-1} has a dense domain and $T_0=A^{-1}-I\eta\mathbf{M}$ is the desired least upper bound. This proves (1)→(2).

Ad (2)→(3): $\mathfrak{D}$ is dense, since $\mathfrak{D}\supset\mathfrak{D}_{T_0^{\frac{1}{2}}}$ and $\mathfrak{D}_{T_0^{\frac{1}{2}}}$ is dense. This proves (2)→(3).

Ad (3)→(4): Construct the filter of sections $\mathcal{F}_0$ on the directed set $\{\delta\}$ of indices, and inflate it to an ultrafilter $\mathcal{F}$. For every $x\in\mathfrak{D}$ and $y\in\mathfrak{H}$, we have $|\langle T_\delta^{\frac{1}{2}}x, y\rangle|\leq\|T_\delta^{\frac{1}{2}}x\|\ \|y\|\leq c\|y\|$ for some positive constant c depending on x. Therefore by the Riesz representation theorem for bounded linear functionals, we can write $\lim\limits_{\mathcal{F}}\langle T_\delta^{\frac{1}{2}}x, y\rangle=\langle Sx, y\rangle$, where S is a linear, positive operator whose closedness is not assured for the present. As the domain $\mathfrak{D}=\mathfrak{D}_S$ of S is dense and hence S is symmetric, it has Freudenthal's self-adjoint extension $\tilde{S}$ ([16], p. 35). $\tilde{S}$ is the restriction of S^* on $\tilde{\mathfrak{D}}=\mathfrak{D}_{S^*}\cap\mathfrak{D}'$, where $\mathfrak{D}'$ is the completion of $\mathfrak{D}$ by the norm $\|x\|_1=\langle(I+S)x, x\rangle^{\frac{1}{2}}$ and is considered as a linear subset of $\mathfrak{H}$ in an obvious way. For any $x\in\tilde{\mathfrak{D}}=\mathfrak{D}_{\tilde{S}}$, we select a sequence $\{x_n\}$ from $\mathfrak{D}$ such that $\|x_n-x\|_1\to0$. Then $\|x_n-x\|\leq\|x_n-x\|_1\to0\,(n\to\infty)$, and from the inequality

$$\|x_n-x_m\|_1{}^2=\langle(I+S)(x_n-x_m), x_n-x_m\rangle\geq\langle S(x_n-x_m), x_n-x_m\rangle$$
$$\geq\langle T_\delta^{\frac{1}{2}}(x_n-x_m), x_n-x_m\rangle=\|T_\delta^{\frac{1}{4}}(x_n-x_m)\|^2,$$

we see that $x\in\mathfrak{D}_{T_\delta^{\frac{1}{4}}}$ and $\|\tilde{S}^{\frac{1}{2}}x\|\geq\|T_\delta^{\frac{1}{4}}x\|$. Thus $\mathfrak{D}_{\tilde{S}}\subset\mathfrak{D}_{T_\delta^{\frac{1}{4}}}$ and $\|\tilde{S}^{\frac{1}{2}}x\|\geq\|T_\delta^{\frac{1}{4}}x\|$ for every $x\in\mathfrak{D}_{\tilde{S}}$. Hence by the remark after Definition 1, it follows that $\tilde{S}\geq T_\delta^{\frac{1}{2}}$ for every δ. This proves (3)→(4) with $T'=\tilde{S}$. Later we will show that $S=\tilde{S}$.

Ad (4)→(5): We need only to apply (1)→(2), already proved, to the increasing directed set $\{T_\delta^{\frac{1}{2}}\}$.

Ad (5)→(1): Since $\text{l.u.b.}(I+T_\delta^{\frac{1}{2}})=I+S_0$, we have $\text{g.l.b.}(I+T_\delta^{\frac{1}{2}})^{-1}=(I+S_0)^{-1}$ by a further application of the lemma of E. Heinz. Hence $(I+T_\delta^{\frac{1}{2}})^{-1}$ converges strongly to $(I+S_0)^{-1}$. By the theorem of I. Kaplansky, applied to the continuous bounded function $h(t)=\dfrac{t^2}{t^2+(1-t)^2}$, $(I+T_\delta)^{-1}=h((I+T_\delta^{\frac{1}{2}})^{-1})$ converges strongly to $h((I+S_0)^{-1})=(I+S_0^2)^{-1}$. Hence $\text{g.l.b.}\,(I+T_\delta)^{-1}=(I+S_0)^{-1}$. Thus $\text{l.u.b.}\,(I+T_\delta)=I+S_0{}^2$ by the lemma of E. Heinz, and hence $\text{l.u.b.}\,T_\delta=S_0{}^2$. This proves (5)→(1). And the equivalence of (1)—(5) is thus established.

Next we show the last statements. $T_0^{\frac{1}{2}}=S_0\eta\mathbf{M}$ is seen from the proof of (1)→(2) and that of (5)→(1). To obtain the characterization of $T_0^{\frac{1}{2}}$, we proceed as follows. First, using the notations in the proof of (3)→(4), we will prove

$\tilde{S}=T_0^{\frac{1}{2}}$. We have already seen that $\tilde{S}\geq T_\delta^{\frac{1}{2}}$ for every δ. Hence $\tilde{S}\geq S_0=T_0^{\frac{1}{2}}$. The proof of $\tilde{S}\leq T_0^{\frac{1}{2}}$ goes as follows. Let x be any element of $\mathfrak{D}_{T_0^{\frac{1}{2}}}$. Since $\mathfrak{D}_{T_0^{\frac{1}{2}}}\subset\mathfrak{D}$ by the proof of (2)$\rightarrow$(3), it follows that $x\in\mathfrak{D}=\mathfrak{D}_S\subset\mathfrak{D}_{\tilde{S}}$. Hence

$$\|\tilde{S}^{\frac{1}{2}}x\|^2=\langle\tilde{S}x, x\rangle=\langle Sx, x\rangle=\lim_{\mathcal{F}}\langle T_\delta^{\frac{1}{2}}x, x\rangle=\lim_{\mathcal{F}}\|T_\delta^{\frac{1}{4}}x\|^2\leq\|T_0^{\frac{1}{4}}x\|^2.$$

Thus $\mathfrak{D}_{T_0^{\frac{1}{2}}}\subset\mathfrak{D}_{\tilde{S}^{\frac{1}{2}}}$ and $\|\tilde{S}^{\frac{1}{2}}x\|\leq\|T_0^{\frac{1}{4}}x\|$ for every $x\in\mathfrak{D}_{T_0^{\frac{1}{2}}}$. This shows us that $\tilde{S}\leq T_0^{\frac{1}{2}}$ by the remark after Definition 1. Therefore $\tilde{S}=T_0^{\frac{1}{2}}$. Since $\mathfrak{D}_{T_0^{\frac{1}{2}}}\subset\mathfrak{D}$, it results that $\mathfrak{D}_{\tilde{S}}=\mathfrak{D}_{T_0^{\frac{1}{2}}}\subset\mathfrak{D}=\mathfrak{D}_S$. This and the fact that $\tilde{S}$ is a extension of S imply $S=\tilde{S}$. In particular, $\mathfrak{D}_{S_0}=\mathfrak{D}_{T_0^{\frac{1}{2}}}=\mathfrak{D}_{\tilde{S}}=\mathfrak{D}_S=\mathfrak{D}$. Since $\lim_{\mathcal{F}}\langle T_\delta^{\frac{1}{2}}x, y\rangle=\langle Sx, y\rangle=\langle S_0x, y\rangle$ for every ultrafilter $\mathcal{F}$ containing the filter of sections $\mathcal{F}_0$, we see that, along the given directed set $\{\delta\}$. $\lim_{\delta}\langle T_\delta^{\frac{1}{4}}x, y\rangle=\langle S_0x, y\rangle$ for every $x\in\mathfrak{D}$ and $y\in\mathfrak{H}$. Let $x\in\mathfrak{D}$. Then

$$\overline{\lim_{\delta}}\|T_\delta^{\frac{1}{2}}x-T_0^{\frac{1}{2}}x\|^2=\overline{\lim_{\delta}}(\|T_\delta^{\frac{1}{2}}x\|^2-\langle T_\delta^{\frac{1}{2}}x, T_0^{\frac{1}{2}}x\rangle-\langle T_0^{\frac{1}{2}}x, T_\delta^{\frac{1}{2}}x\rangle+\|T_0^{\frac{1}{2}}x\|^2)$$
$$\leq\|T_0^{\frac{1}{2}}x\|^2-\langle T_0^{\frac{1}{2}}x, T_0^{\frac{1}{2}}x\rangle-\langle T_0^{\frac{1}{2}}x, T_0^{\frac{1}{2}}x\rangle+\|T_0^{\frac{1}{2}}x\|^2=0.$$

That is, $\lim_{\delta}\|T_\delta^{\frac{1}{2}}x-T_0^{\frac{1}{2}}x\|=0$ for every $x\in\mathfrak{D}$. Conversely, if S_1 has the property that $\mathfrak{D}_{S_1}=\mathfrak{D}$ and $\|T_\delta^{\frac{1}{2}}x-S_1x\|\rightarrow 0$ for every $x\in\mathfrak{D}$, then $\lim_{\delta}\langle T_\delta^{\frac{1}{2}}x, y\rangle=\langle S_1x, y\rangle$ for every $x\in\mathfrak{D}$ and $y\in\mathfrak{H}$. Hence $S_1=S=\tilde{S}=T_0^{\frac{1}{2}}$. This proves the last statement. The theorem is thus completely proved.

From this theorem it follows easily that every increasing directed set $\{T_\delta\}$ of self-adjoint measurable operators with a measurable upper bound $T\eta\mathbf{M}$ has the measurable l.u.b. $T_\delta=T_0\eta\mathbf{M}$ in the sense of the ordering of the measurable operators. Similar statement holds for a decreasing directed set $\{T_\delta\}$.

COROLLARY. *Let $\{T_\delta\}$ be an increasing directed set of measurable operators $\eta\mathbf{M}$ with the measurable operator T_0 as its least upper bound in the sense of the ordering of the measurable operators. Let T be an arbitrary measurable operator $\eta\mathbf{M}$. Then $\text{l.u.b.}_{\delta}\, T^*\cdot T_\delta\cdot T=T^*\cdot T_0\cdot T$ in the sense of the ordering of the measurable operators. Similar statement holds for a decreasing directed set $\{T_\delta\}$.*

PROOF. With no loss of generalities, we may restrict ourselves to the case $T_\delta\geq 0$, so that the ordering in question may be identified with that in the sense of the positive operators. By the remark after Definition 1, $\{T^*\cdot T_\delta\cdot T\}$ is an increasing directed set of positive measurable operators with a measurable upper bound $T^*\cdot T_0\cdot T$. Hence l.u.b. $T^*\cdot T_\delta\cdot T=S_0$ exists with measurable S_0. The proof of $S_0 = T^* \cdot T_0 \cdot T$ goes as follows. If $x \in \mathfrak{D}_{T^*T_\delta T}$, then

$$\|(T^* \cdot T_\delta \cdot T)^{\frac{1}{2}} x\|^2 = \langle T^* \cdot T_\delta \cdot Tx, x\rangle = \langle T_\delta^{\frac{1}{2}} Tx, T_\delta^{\frac{1}{2}} Tx\rangle = \|T_\delta^{\frac{1}{2}} Tx\|^2.$$

Since $\mathfrak{D}_{T^*T_\delta T}$ is (strongly) dense, we may easily see that $\|(T^* \cdot T_\delta \cdot T)^{\frac{1}{2}} x\|^2 = \|T_\delta^{\frac{1}{2}} \cdot Tx\|^2$ for every $x \in \mathfrak{D}_{T_\delta^{\frac{1}{2}} \cdot T} = \mathfrak{D}_{(T^* \cdot T_\delta \cdot T)^{\frac{1}{2}}}$. As $\mathfrak{D}_{T_0^{\frac{1}{2}}} \subset \mathfrak{D}_{T_\delta^{\frac{1}{2}}}$ and $\mathfrak{D}_{S_0^{\frac{1}{2}}} \subset \mathfrak{D}_{(T^* \cdot T_\delta \cdot T)^{\frac{1}{2}}}$ for every δ, we have $\|(T^* \cdot T_\delta \cdot T)^{\frac{1}{2}} x\|^2 = \|T_\delta^{\frac{1}{2}} Tx\|^2$ for every $x \in \mathfrak{D}_{T_0^{\frac{1}{2}}T} \cap \mathfrak{D}_{S_0^{\frac{1}{2}}}$ and δ. Thus by Theorem 1, $\|S_0^{\frac{1}{2}} x\|^2 = \|T_0^{\frac{1}{2}} Tx\|$ for every $x \in \mathfrak{D}_{T_0^{\frac{1}{2}}T} \cap \mathfrak{D}_{S_0^{\frac{1}{2}}}$. In particular $\langle S_0 x, x\rangle = \langle TT_0Tx, x\rangle$ for every $x \in \mathfrak{D}_{S_0} \cap \mathfrak{D}_{T^*T_0T}$, and hence $\langle S_0 x, y\rangle = \langle T^*T_0Tx, y\rangle$ for every $x, y \in \mathfrak{D}_{S_0} \cap \mathfrak{D}_{T^*T_0T}$. As $\mathfrak{D}_{S_0} \cap \mathfrak{D}_{T^*T_0T}$ is dense in $\mathfrak{H}$, we have $S_0 x = T^*T_0Tx$ for every $x \in \mathfrak{D}_{S_0} \cap \mathfrak{D}_{T^*T_0T}$. Thus $S_0 = T^* \cdot T_0 \cdot T$ [13].

REMARK 1. Let $\{T_\delta\}$ be an increasing directed set of positive operators, and p be an arbitrary real number such that $0 < p \leq 1$. Then the following conditions (1) and (2) are equivalent:

(1) $\operatorname{l.u.b.}_\delta T_\delta = T_0$ exists in the sense of the ordering of the positive operators on $\mathfrak{H}$;

(2) $\operatorname{l.u.b.}_\delta T_\delta^p = S_0$ exists in the sense of the ordering of the positive operators on $\mathfrak{H}$.

Moreover, in this case $S_0 = T_0^p$. The proof is sketched as follows. Ad (1)$\rightarrow$(2): Since $0 < p \leq 1$, we have $T_\delta^p \leq T_0^p$ for every δ [8]. Hence Theorem 1 assures the existence of S_0. Ad (2)$\rightarrow$(1): In this case the proof is quite similar to that of (5)$\rightarrow$(1) for Theorem 1. Let $h_p(t)$ be the continuous function defined as follows:

$$\begin{aligned} h_p(t) &= \frac{t^{\frac{1}{p}}}{t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}} && \text{for } 0 \leq t \leq 1, \\ &= 0 && \text{for } t < 0, \\ &= 1 && \text{for } t > 1. \end{aligned}$$

Then $h_p(t)$ will serve for $h(t)$ in the proof (5)$\rightarrow$(1) cited above, and details are omitted.

REMARK 2. Let $\{T_\delta\}$ be an increasing directed set of positive operators, and p be an arbitrary real number such that $0 < p \leq 1$. Let $\mathfrak{D} = \{x;\ \{\|T_\delta^p x\|\}$ is bounded$\}$ is dense in $\mathfrak{H}$. Then $\operatorname{l.u.b.}_\delta T_\delta = T_0$ exists in the sense of the ordering of the positive operators on $\mathfrak{H}$. It is proved in much the same way as in the proof of (3)$\rightarrow$(4) for Theorem 1. Take the ultrafilter $\mathcal{F}$ in that proof, and construct the operator S with $\mathfrak{D} = \mathfrak{D}_S$ such that $\lim_{\mathcal{F}} \langle T_\delta^p x, y\rangle = \langle Sx, y\rangle$ for every

$x \in \mathfrak{D}$ and $y \in \mathfrak{H}$. Then S has Freudenthal's self-adjoint extension $\tilde{S}$. It is easy to see that $\tilde{S} \geq T_\delta{}^p$ for every δ so that $\text{l.u.b.}\, T_\delta{}^p = S_0$ exists by Theorem 1. Hence $\text{l.u.b.}\, T_\delta = T_0$ exists by Remark 1.

REMARK 3. As for a decreasing directed set of positive operators $\eta \mathbb{M}$, we mention the following fact. Let $\{T_\delta\}$ be such a directed set. Then $\underset{\delta}{\text{g.l.b.}}\, T_\delta = T_0$ always exists in the sense of the ordering of the positive operators on $\mathfrak{H}$. $T_0 \eta \mathbb{M}$ and $\underset{\delta}{\text{g.l.b.}}\, T_\delta{}^p = T_0{}^p$ for every real number p such that $0 < p \leq 1$. Let $\mathfrak{D}$ be the set-theoretic union of all $\mathfrak{D}_{T_\delta{}^{\frac{1}{2}}}$. Then $\lim\limits_{\delta} \langle T_\delta{}^{\frac{1}{2}} x, y\rangle$ exists for every $x \in \mathfrak{D}$ and $y \in \mathfrak{H}$. Hence this limit defines a linear, symmetric, positive and not necessarily closed operator S with dense domain $\mathfrak{D}_S = \mathfrak{D}$: $\lim\limits_{\delta} \langle T_\delta{}^{\frac{1}{2}} x, y\rangle = \langle Sx, y\rangle$ for every $x \in \mathfrak{D}$ and $y \in \mathfrak{H}$. Let $\tilde{S}$ be Freudenthal's self-adjoint extension of S. Then $\tilde{S} = T_0{}^{\frac{1}{2}}$ and $T_\delta{}^{\frac{1}{2}} x \to T_0{}^{\frac{1}{2}} x$ weakly for every $x \in \mathfrak{D}$.

§ 2 Extended pseudo-♮-application

Let $\mathbb{M}$ be a ring of operators on $\mathfrak{H}$, and Ω, a hyperstonian space [3], be the spectre of the center $\mathbb{M}^\natural$. In the canonical fashion $\mathbb{M}^\natural$ will be identified with the ring $C(\Omega)$ of all continuous, finite- and complex-valued functions on Ω. Following Dixmier [4] we denote by $\boldsymbol{Z}$ the set of all continuous, non-negative, finite- or infinite-valued functions on Ω. $\boldsymbol{Z}$ admits the operations: sum and product of two elements, and multiplication by non-negative constants. More precisely, if $f, g \in \boldsymbol{Z}$ and $\alpha > 0$, then $f+g$ and αf are defined in the ordinary manner. fg is defined as follows. Under the convention $0 \cdot (+\infty) = 0$, the function $\omega \to f(\omega) g(\omega)$ is defined everywhere on Ω. As is easily verified it is lower semi-continuous. Hence there is a uniquely determined function $h \in \boldsymbol{Z}$ such that $h(\omega) = f(\omega) g(\omega)$ except on a nowhere dense set. We will define fg by h. In particular, if $f=0$, then $0 \cdot g = 0$.

An application ♮ of $\mathbb{M}^+$ into $\boldsymbol{Z}$, $\mathbb{M}^+ \ni A \to A^\natural \in \boldsymbol{Z}$, will be called *pseudo-♮-application* [4] if the following conditions are satisfied:

1. If $A \in \mathbb{M}^+$ and $A_1 \in \mathbb{M}^+$, then $(A + A_1)^\natural = A^\natural + A_1{}^\natural$;
2. If $A \in \mathbb{M}^+$ and λ is a constant ≥ 0, then $(\lambda A) = \lambda A^\natural$;
3. If $A \in \mathbb{M}^+$ and $U \in \mathbb{M}_U$, then $(UAU^*)^\natural = A^\natural$;
4. If $A \in \mathbb{M}^{\natural +}$ and $B \in \mathbb{M}^+$, then $(AB)^\natural = AB^\natural$.

A pseudo-♮-application ♮ is called *normal*, provided that for every increasing directed set $\{A_\delta\} \subset \mathbb{M}^+$ with the least upper bound $A \in \mathbb{M}^+$, $A^\natural = \text{l.u.b.}\, A_\delta{}^\natural$ holds.

$\natural$ is called *faithful*, if $A^\natural=0$ implies $A=0$, and is called *essential*, if for every $A\in \mathbf{M}^+$, $A\neq 0$, an $A'\neq 0$, $0\leq A'\leq A$, exists such that $A'^\natural\in C(\Omega)$. A ring of operators $\mathbf{M}$ will be called *semi-finite* [7] provided that every non-zero projection $\in \mathbf{M}$ contains a non-zero finite projection $\in \mathbf{M}$. It is known that a ring of operators $\mathbf{M}$ is semi-finite if and only if $\mathbf{M}$ has a normal, faithful and essential pseudo-$\natural$-application [4].

In the sequel we always assume, unless otherwise stated, that $\mathbf{M}$ is a semi-finite ring of operators and $\natural$ is a fixed, normal, faithful and essential pseudo-$\natural$-application.

DEFINITION 2. Let T be a positive operator $\eta\mathbf{M}$. We define

$$(\natural) \qquad T^\natural = \underset{\mathbf{M}^+\ni A\leq T}{\text{l. u. b.}}\ A^\natural,$$

where l.u.b. is taken in $\boldsymbol{Z}$.

Clearly, for every $T\in\mathbf{M}^+$, $T^\natural$ defined by ($\natural$) is the same as the original $T^\natural$ and hence ($\natural$) is an extension of the pseudo-$\natural$-application $\natural$ on $\mathbf{M}^+$ to the set of all positive operators $\eta\mathbf{M}$. Put

$\mathfrak{S}^+=\{T;\ T$ is a positive operator, and $T^\natural(\omega)$ is finite except on a nowhere dense subset of $\Omega\}$,

$$\mathfrak{s}^+ = \mathfrak{S}^+\cap\mathbf{M},$$

and

$$\mathfrak{m}^+ = \{A;\ A\in\mathbf{M}^+ \text{ and } A^\natural \in C(\Omega)\}.$$

It is known, by Dixmier [4], that $\mathfrak{s}^+$ and $\mathfrak{m}^+$ are, respectively, positive parts of ideals $\mathfrak{s}$ and $\mathfrak{m}$. As $\natural$ is essential we have $\overline{\mathfrak{m}^r}=\bar{\mathfrak{m}}=\overline{\mathfrak{s}^r}=\bar{\mathfrak{s}}=\mathbf{M}$, where $\mathfrak{m}^r$ and $\mathfrak{s}^r$ are restricted ideals associated with $\mathfrak{m}$ and $\mathfrak{s}$, respectively, and "—" is the closure in the strong topology.

LEMMA 1. $T^\natural = \underset{\mathfrak{m}^{r+}\ni A\leq T}{\text{l. u. b.}}\ A^\natural$.

PROOF. Put $g=\underset{\mathfrak{m}^{r+}\ni A\leq T}{\text{l. u. b.}}\ A^\natural\in\boldsymbol{Z}$. Clearly $g\leq T^\natural$. Now for any $A\in\mathbf{M}^+$, $A\leq T$, we have $A\in\mathbf{M}^+=\overline{\mathfrak{m}^{r}}^+=\overline{\mathfrak{m}^{r+}}$, and $A=\underset{B\in\mathcal{F}_A}{\text{l.u.b.}}\ B$, where $\mathcal{F}_A=\{B;\ \mathfrak{m}^{r+}\ni B\leq A\}$. As $\mathcal{F}_A$ is an increasing directed set we get $A^\natural=\underset{B\in\mathcal{F}_A}{\text{l.u.b.}}\ B^\natural$ by the normality of $\natural$. Thus $T^\natural\leq g$. The proof is complete.

The set of all continuous, finite except on nowhere dense sets, and complex-valued functions defined on Ω will be denoted by $\boldsymbol{Z}'$. If $f\in\boldsymbol{Z}'$ and $g\in\boldsymbol{Z}'$ then $f(\omega)+g(\omega)$ is defined and finite on a dense open set $\subset\Omega$. Hence there is a unique function $h \in \boldsymbol{Z}'$ such that $f(\omega)+g(\omega)=h(\omega)$ except on a nowhere dense set ([12], p. 57). We define $f+g$ by h. Similarly fg and αf, where α is a constant, are defined. With these operations $\boldsymbol{Z}'$ has a structure of an algebra over the complex number field. In an obvious manner we can regard $\boldsymbol{Z}'$ as the set of all (measurable) operators $\eta \mathbb{M}^{\natural}$. It is to be noted that for any $f, g \in \boldsymbol{Z} \cap \boldsymbol{Z}'$, fg defined on $\boldsymbol{Z}'$ coincides with that defined on $\boldsymbol{Z}$. The same will hold for $f+g$ and αf $(\alpha \geq 0)$. As Dixmier [4] observed we have the following

LEMMA 2. *The application* ♮ *defined on* $\mathfrak{S}^+$, $\mathfrak{S}^+ \ni A \rightarrow A^{\natural} \in \boldsymbol{Z}$, *can be uniquely extended on* $\mathfrak{S}$, $\mathfrak{S} \ni A \rightarrow A^{\natural} \in \boldsymbol{Z}'$, *so as to have the following properties:*

(1) *If* $A \in \mathfrak{S}$ *and* $A_1 \in \mathfrak{S}$, *and* α, α_1 *are complex numbers, then* $(\alpha A+\alpha_1 A_1)^{\natural}=\alpha A^{\natural}+\alpha_1 A_1{}^{\natural}$;

(2) *If* $A \in \mathfrak{S}$ *and* $B \in \mathbb{M}$, *then* $(AB)^{\natural}=(BA)^{\natural}$;

(3) *If* $A \in \mathfrak{S}^+$, *then* $A^{\natural} \geq 0$;

(4) *If* $A \in \mathbb{M}^{\natural}$ *and* $B \in \mathfrak{S}$, *then* $(AB)^{\natural}=AB^{\natural}$.

PROOF. The proof goes in a similar manner as that of Lemma 4.7 of Dixmier [4], and the details are omitted.

REMARK 4. From this lemma we can show that $(AA^*)^{\natural}=(A^*A)^{\natural}$ for every $A \in \mathbb{M}$. The proof is sketched as follows. First we infer that if $AA^* \in \mathfrak{S}^+$ then $A^*A \in \mathfrak{S}^+$ and $(AA^*)^{\natural}=(A^*A)^{\natural}$. In the general case, put $O=\overline{\{\omega\,;\,(AA^*)^{\natural}(\omega)<+\infty\}}$, $O'=\overline{\{\omega\,;\,(A^*A)^{\natural}(\omega)<+\infty\}}$, and denote the corresponding central projections by P and P' respectively. It follows at once that $PAA^* \in \mathfrak{S}$ and

$$P(A^*A)^{\natural}=(PA^*A)^{\natural}=((PA^*)\,(PA))^{\natural}=((PA)\,(PA^*))^{\natural}=(PAA^*)^{\natural}.$$

Hence $P \leq P'$. By symmetry $P' \leq P$ and so we have $P=P'$ or $O=O'$. Hence $(AA^*)^{\natural}=(A^*A)^{\natural}$. Note that this remark holds as well for every not necessarily normal, faithful and essential pseudo-♮-application.

We can now prove the normality of the extended pseudo-♮-application in the most general form.

THEOREM 2. *If an increasing directed set* $\{T_\delta\}$ *of positive operators* $\eta \mathbb{M}$ *has the least upper bound* T_0 *in the sense of the ordering of the positive operators, then*

$$T_0{}^{\natural}=\operatorname*{l.u.b.}_{\delta} T_\delta{}^{\natural}.$$

PROOF. Let $\boldsymbol{Z} \ni g=\text{l.u.b.}\, T_\delta{}^{\natural}$. Then it follows from the definition of $T^{\natural}$ that $g \leq T^{\natural}$. The opposite inequality is proved as follows. Let $T_0=\int_0^\infty \lambda\, dE_\lambda$ be the spectral resolution. By Theorem 1, $\|T_\delta{}^{\frac{1}{2}}E_\lambda x\| \uparrow \|T_0{}^{\frac{1}{2}}E_\lambda x\|$ for every $E_\lambda x$. In particular $(T_\delta{}^{\frac{1}{2}}E_\lambda)^*(T_\delta{}^{\frac{1}{2}}E_\lambda) \leq (T_0{}^{\frac{1}{2}}E_\lambda)^*(T_0{}^{\frac{1}{2}}E_\lambda)=T_0E_\lambda$, and hence $(T_\delta{}^{\frac{1}{2}}E_\lambda)^*(T_\delta{}^{\frac{1}{2}}E_\lambda) \in \mathbb{M}^+$.

Since $\langle (T_\delta^{\frac{1}{2}}E_\lambda)^*(T_\delta^{\frac{1}{2}}E_\lambda)x, x\rangle = \|T_\delta^{\frac{1}{2}}E_\lambda x\|^2 \uparrow \|T_0^{\frac{1}{2}}E_\lambda x\|^2 = \langle T_0 E_\lambda x, x\rangle$ for every $x \in \mathfrak{H}$, we see that $(T_\delta^{\frac{1}{2}}E_\lambda)^*(T_\delta^{\frac{1}{2}}E_\lambda) \uparrow T_0 E_\lambda$. By normality of $\natural$ in $\mathbb{M}^+$, we have $((T_\delta^{\frac{1}{2}}E_\lambda)^*(T_\delta^{\frac{1}{2}}E_\lambda))^\natural \uparrow (T_0 E_\lambda)^\natural$. But by Remark 4, we have $((T_\delta^{\frac{1}{2}}E_\lambda)^*(T_\delta^{\frac{1}{2}}E_\lambda))^\natural = ((T_\delta^{\frac{1}{2}}E_\lambda)(T_\delta^{\frac{1}{2}}E_\lambda)^*)^\natural$. And $\|(T_\delta^{\frac{1}{2}}E_\lambda)^* x\| \leq \|T_\delta^{\frac{1}{2}}x\|$ for every $x \in \mathfrak{D}_{T_\delta^{\frac{1}{2}}}$, since $(T_\delta^{\frac{1}{2}}E_\lambda)^* x = E_\lambda T_\delta^{\frac{1}{2}} x$ for every $x \in \mathfrak{D}_{T_\delta^{\frac{1}{2}}}$. Hence $(T_\delta^{\frac{1}{2}}E_\lambda)(T_\delta^{\frac{1}{2}}E_\lambda)^* \leq T_\delta$ and consequently $((T_\delta^{\frac{1}{2}}E_\lambda)^*(T_\delta^{\frac{1}{2}}E_\lambda))^\natural \leq T_\delta^\natural$. Thus we see that $(T_0 E_\lambda)^\natural \leq \text{l.u.b.}\, T_\delta^\natural = g$. Let $\mathbb{M}^+ \ni C \leq T$. Then

$$g \geq (T_0 E_\lambda)^\natural \geq (E_\lambda C E_\lambda)^\natural = ((E_\lambda C^{\frac{1}{2}})(E_\lambda C^{\frac{1}{2}})^*)^\natural = ((E_\lambda C^{\frac{1}{2}})^*(E_\lambda C^{\frac{1}{2}}))^\natural = (C^{\frac{1}{2}}E_\lambda C^{\frac{1}{2}})^\natural$$

for every λ. But $C = \underset{\lambda}{\text{l.u.b.}}\, C^{\frac{1}{2}}E_\lambda C^{\frac{1}{2}}$. Hence $C^\natural = \underset{\lambda}{\text{l.u.b.}}\, (C^{\frac{1}{2}}E_\lambda C^{\frac{1}{2}})^\natural \leq g$. This shows $T^\natural \leq g$. Thus $T^\natural = g = \text{l.u.b.}\, T_\delta^\natural$. The proof is complete.

REMARK 5. This proof shows us that Theorem 2 holds as well for every normal, but not necessarily faithful and essential, pseudo-$\natural$-application.

LEMMA 3. *If $T \in \mathfrak{S}^+$ and $T = \int_0^\infty \lambda dE_\lambda$ is the spectral resolution, then $E_\lambda^\perp$ is a finite projection for every $\lambda > 0$, and hence T is a measurable operator.*

PROOF. For every $\lambda > 0$, $\lambda E_\lambda^\perp \leq T$. Hence $(E_\lambda^\perp)^\natural(\omega)$ is finite except on a nowhere dense subset of Ω, and therefore $E_\lambda^\perp$ is finite. Hence T is measurable (cf. [13] Lemma 1.1).

REMARK 6. From Dixmier's construction of $\natural$-application [4] a projection $P \in \mathbb{M}$ is finite if and only if $P \in \mathfrak{S}^+$. Therefore $\mathfrak{S}^r = \mathfrak{m}_0$ (= the ideal generated by all finite projections in $\mathbb{M}$ [13]).

The set of all measurable operators $\eta\mathbb{M}$ forms a $*$-algebra with respect to the strong sum $S \dotplus T$ and strong product $S \cdot T$, the scalar multiplication (except that $0 \cdot T = 0$) and adjunction S^* [15]. Relations between these operations and our extended pseudo-$\natural$-application are given in the next

LEMMA 4. *If T and T_1 are positive measurable operators $\eta\mathbb{M}$, then*

(1) $(T \dotplus T_1)^\natural = T^\natural + T_1^\natural$;

(2) $(\lambda T)^\natural = \lambda T^\natural$ *for every non-negative consant* λ;

(3) $(UTU^*)^\natural = T^\natural$ *for every* $U \in \mathbb{M}_U$;

(4) $(A \cdot T)^\natural = AT^\natural$ *for every* $A \in \mathbb{M}^{\natural +}$.

PROOF. Ad (1): Let $A \leq T \dotplus T_1$ and $A \in \mathfrak{m}^{r+}$, then $A = C \cdot TC^* \dotplus C \cdot T_1 C^*$ for some $C \in \mathbb{M}$ with $\|C\| \leq 1$ ([13], [5]). Since

$$(C \cdot T^{\frac{1}{2}}) \cdot (C \cdot T^{\frac{1}{2}})^* = C \cdot TC^* \leq C \cdot TC^* \dotplus C \cdot T_1 C^* = A \in \mathbb{M}^+,$$

we have $C \cdot T^{\frac{1}{2}} \in \mathbb{M}$. And $(T^{\frac{1}{2}}C^*)(C \cdot T^{\frac{1}{2}}) \leq T$ follows from $\|C\| \leq 1$. Hence

$$T^\natural \geq ((T^{\frac{1}{2}}C^*)(C \cdot T^{\frac{1}{2}}))^\natural = ((C \cdot T^{\frac{1}{2}})(T^{\frac{1}{2}}C^*))^\natural = (C \cdot TC^*)^\natural.$$

Similarly $T_1^\natural \geq (C \cdot T_1 C^*)^\natural$. Therefore we have

$$A^\natural = (C \cdot TC^*)^\natural + (C \cdot T_1 C^*)^\natural \leq T^\natural + T_1^\natural$$

This shows $(T \dotplus T_1)^\natural \leq T^\natural + T_1^\natural$. Evidently $(T \dotplus T_1)^\natural \geq T^\natural + T_1^\natural$, and we have $(T \dotplus T_1)^\natural = T^\natural \dotplus T_1^\natural$.

(2) is clear.

Ad (3): It is sufficient to remember that, $A \leq UTU^*$ and $U^* A U \leq T$ are equivalent for every $A \in \mathbf{M}^+$.

Ad (4): Put $A^{\frac{1}{2}} = B \in \mathbf{M}^{\natural +}$. Then for any $C \in \mathfrak{m}^{r+}$, $C \leq T$, we have $BCB \leq B \cdot TB$, so that $AC^\natural = (BCB)^\natural \leq (B \cdot TB)^\natural = (A \cdot T)^\natural$. This shows that $AT^\natural \leq (A \cdot T)^\natural$. On the other hand if $\mathfrak{m}^{r+} \ni C_1 \leq B \cdot TB = A \cdot T$, then $C_1 = (DB) \cdot TBD^* = (DA) \cdot TD^* = A \cdot D \cdot TD^*$ for some $D \in \mathbf{M}$ with $\|D\| \leq 1$. If P_n is the central projection corresponding to the open-closed set $\overline{\{\omega ; A(\omega) > 1/n\}}$, then $C_1 P_n = (T^{\frac{1}{2}} BD^* P_n)^* (T^{\frac{1}{2}} BD^* P_n) \in \mathfrak{m}^r$ and hence

$$\begin{aligned}(C_1 P_n)^\natural &= ((T^{\frac{1}{2}} BD^* P_n)^* (T^{\frac{1}{2}} BD^* P_n))^\natural = ((T^{\frac{1}{2}} BD^* P_n)(T^{\frac{1}{2}} BD^* P_n)^*)^\natural \\ &= (A \cdot P_n \cdot T^{\frac{1}{2}} \cdot D^* D \cdot T^{\frac{1}{2}})^\natural.\end{aligned}$$

But $P_n \cdot D \cdot TD^* \in \mathbf{M}^+$. Therefore $P_n \cdot T^{\frac{1}{2}} \cdot D^* D \cdot T^{\frac{1}{2}} \in \mathbf{M}^+$. So we see that

$$(C_1 P_n)^\natural = A(P_n \cdot T^{\frac{1}{2}} \cdot D^* D \cdot T^{\frac{1}{2}})^\natural \leq A(T^{\frac{1}{2}} T^{\frac{1}{2}})^\natural = AT^\natural.$$

And as $C_1 P_n = (AP_n) \cdot D \cdot TD^* \uparrow A \cdot D \cdot TD^* = C_1$, it follows from the normality of the mapping ♮ that l.u.b. $(C_1 P_n)^\natural = C_1^\natural$. This leads to the inequality $C_1^\natural \leq AT^\natural$. Hence $(A \cdot T)^\natural \leq AT^\natural$, completing the proof.

LEMMA 5. *$\mathfrak{S}^+$ has the following properties:*

(1) *If $T \in \mathfrak{S}^+$ and $U \in \mathbf{M}_U$, then $UTU^* \in \mathfrak{S}^+$ and $(UTU^*)^\natural = T^\natural$;*

(2) *If $T \in \mathfrak{S}^+$ and S is an opeator, $0 \leq S \leq T$, then $S \in \mathfrak{S}^+$;*

(3) *If $T \in \mathfrak{S}^+$ and $T_1 \in \mathfrak{S}^+$, then $T \dotplus T_1 \in \mathfrak{S}^+$ and $(T \dotplus T_1)^\natural = T^\natural + T_1^\natural$.*

PROOF. It is evident from the previous lemma.

A linear set $\mathfrak{L}$ of measurable operators $\eta \mathbf{M}$ is called an *invariant linear system* of $\mathbf{M}$ if $T \in \mathfrak{L}$ implies UT, $TU \in \mathfrak{L}$ for every $U \in \mathbf{M}_U$. We have shown [13] that a set $\mathfrak{L}^\times$ of positive measurable operators $\eta \mathbf{M}$ is the positive part of an invariant linear system if and only if $\mathfrak{L}^\times$ satisfies the following conditions:

1. If $T \in \mathfrak{L}^\times$ and $U \in \mathbf{M}_U$, then $UTU^* \in \mathfrak{L}^\times$;
2. If $T \in \mathfrak{L}^\times$ and S is a measurable operator such that $0 \leq S \leq T$, then $S \in \mathfrak{L}^\times$;
3. If $S \in \mathfrak{L}^\times$ and $T \in \mathfrak{L}^\times$, then $S \dotplus T \in \mathfrak{L}^\times$.

Hence Lemma 5 shows that $\mathfrak{S}^+$ is the positive part of an invariant linear system $\mathfrak{S}$. More precisely,

THEOREM 3. *There is a unique invariant linear system $\mathfrak{S}$ whose positive part is $\mathfrak{S}^+$. And the application $\natural$ defined on $\mathfrak{S}^+$, $\mathfrak{S}^+ \ni T \to T^\natural \in \boldsymbol{Z}$, can be uniquely extended on $\mathfrak{S}$, $\mathfrak{S} \ni T \to T^\natural \in \boldsymbol{Z}'$, so as to have the following properties:*

(1) *If $T \in \mathfrak{S}$ and $T_1 \in \mathfrak{S}$, and α, α_1 are complex numbers, then $(\alpha T + \alpha_1 T_1)^\natural = \alpha T^\natural + \alpha_1 T_1^\natural$;*

(2) *If $T \in \mathfrak{S}$ and $A \in \mathbf{M}$, then $(A \cdot T)^\natural = (TA)^\natural$;*

(3) *If $T \in \mathfrak{S}^+$, then $T^\natural \geq 0$;*

(4) *If $A \in \mathbf{M}^\natural$ and $T \in \mathfrak{S}$, then $(A \cdot T)^\natural = AT^\natural$;*

(5) *$(T^*)^\natural = \overline{T^\natural}$ for every $T \in \mathfrak{S}$;*

(6) *If $SS^* \in \mathfrak{S}$ for an operator S, then $S^*S \in \mathfrak{S}$ and $(SS^*)^\natural = (S^*S)^\natural$.*

PROOF. As pointed out in [13] (p. 320), existence and uniqueness of $\mathfrak{S}$ can be proved in much the same way as Dixmier ([4], Lemma 4.7). Thus details are omitted. Every $T \in \mathfrak{S}$ can be expressed as a linear combination of elements in $\mathfrak{S}^+$. Hence $\natural$ can be uniquely extended on $\mathfrak{S}$ so as to satisfy (1). (3), (4) and (5) are evident from the way of extension. (2) is proved as in a usual fashion: first by $A \in \mathbf{M}_U$, next by self-adjoint $A \in \mathbf{M}$ and lastly by general $A \in \mathbf{M}$. (6) is proved as follows: Let $S = U|S|$ be the polar decomposition of S. Then $SS^* = US^*SU^*$. Hence

$$(SS^*)^\natural = (US^*SU^*)^\natural = (U^*US^*S)^\natural = (S^*S)^\natural.$$

The proof is complete.

REMARK 7. From the property (6) of this theorem, we can show, more generally, that $(SS^*)^\natural = (S^*S)^\natural$ for every operator S. The proof goes in much the same way as in Remark 4.

In our previous paper [13] we defined the powers $\mathfrak{L}^\alpha$ $(\alpha > 0)$ of an invariant linear system $\mathfrak{L}$ as the invariant linear system generated by all T^α with $T \in \mathfrak{L}^+$. But, in general, it was an open question whether or not the set $\{T^\alpha;\ T \in \mathfrak{L}^+\}$ coincides with $\mathfrak{L}^{\alpha+}$. Hence we were forced to give the sufficient conditions, $(\lll)_1$ and $(\lll)_2$. To state this, we need the following notation [5], [13]. Let S and T be positite operators $\eta\mathbf{M}$, and $S = \int_0^\infty \lambda dE_\lambda$, $T = \int_0^\infty \lambda dF_\lambda$ be their spectral resolutions respectively. Put $G_\lambda = E_\lambda \cap F_\lambda$, then $\{G_\lambda\}$ defines an operator $\int_0^\infty \lambda dG_\lambda$ which will be denoted by $S \vee T$.

$(\lll)_1$ If $T = \int_0^\infty \lambda dF_\lambda \in \mathfrak{L}^+$ *and if* $0 \leq S = \int_0^\infty \lambda dE_\lambda$ is an operator such that

$E_\lambda^\perp \lesssim F_\lambda^\perp$ for every positive λ, then $S \in \mathfrak{L}^+$.

$(\ll)_2$ If $S \in \mathfrak{L}^+$ and $T \in \mathfrak{L}^+$, then $S \vee T \in \mathfrak{L}^+$.

THEOREM 4. *$\mathfrak{S}$ satisfies $(\ll)_1$ and $(\ll)_2$. Hence*

(1) $\mathfrak{S}^{\alpha+} = \{T^\alpha\,;\ T \in \mathfrak{S}^+\}$;

(2) $(\mathfrak{S}^\alpha)^\beta = \mathfrak{S}^{\alpha\beta}$, $\mathfrak{S}^\alpha \cdot \mathfrak{S}^\beta = \mathfrak{S}^{\alpha+\beta}$ *for every* $\alpha,\ \beta > 0$;

(3) *If $\mathfrak{S}^\alpha$ is an algebra for some $\alpha > 0$, then so are all the other $\mathfrak{S}^\beta$.*

PROOF. Let $T = \int_0^\infty \lambda dF_\lambda \in \mathfrak{S}^+$, $0 \leq S = \int_0^\infty \lambda dE_\lambda$, and assume $E_\lambda^+ \lesssim F_\lambda^\perp$ for every $\lambda > 0$. Put $S_n = (1/2^n)(E_{1/2^n}^\perp + E_{2/2^n}^\perp + \cdots + E_{n2^n/2^n}^\perp)$ and $T_n = (1/2^n)(F_{1/2^n}^\perp + F_{2/2^n}^\perp + \cdots + F_{n2^n/2^n}^\perp)$. Then $S_n \leq S_{n+1}$, $T_n \leq T_{n+1}$, l.u.b. $S_n = S$, and l.u.b. $T_n = T$. Then by normality (Theorem 2) it follows that l.u.b. $S_n^\natural = S^\natural$ and l.u.b. $T_n^\natural = T^\natural$, while from the assumption $E_\lambda^\perp \lesssim F_\lambda^\perp$ we obtain that $(E_\lambda^\perp)^\natural \leq (F_\lambda^\perp)^\natural$ for every $\lambda > 0$. Hence $S_n^\natural \leq T_n^\natural$, so that $S^\natural \leq T^\natural$. This proves that $S \in \mathfrak{S}^+$. Thus $\mathfrak{S}$ satisfies $(\ll)_1$. Next we turn to the proof of $(\ll)_2$. Let $S, T \in \mathfrak{S}^+$ and $S = \int_0^\infty \lambda dE_\lambda$, $T = \int_0^\infty \lambda dF_\lambda$ be their spectral resolutions.

$$S \vee T = \int_0^\infty \lambda dG_\lambda = \int_0^\infty G_\lambda^\perp d\lambda = \int_0^\infty (E_\lambda^\perp \cup F_\lambda^\perp) d\lambda.$$

Now for any projections P, Q in $\mathbb{M}$ we have $(P \cup Q)^\natural \leq P^\natural + Q^\natural$ because $(P \cup Q)^\natural = P^\natural + (P \cup Q - P)^\natural \leq P^\natural + Q^\natural$ since $P \cup Q - P \lesssim Q$ [10]. Hence $G_\lambda^{\perp\natural} \leq E_\lambda^{\perp\natural} + F_\lambda^{\perp\natural}$. From this inequality we have $(S \vee T)^\natural \leq S^\natural + T^\natural$, so that $S \vee T \in \mathfrak{S}^+$. That is, $\mathfrak{S}$ satisfies $(\ll)_2$. The rest of the statements were proved previously [13].

For the later use we put $\mathfrak{S}^0 = \mathbb{M}$.

Next we show that the mapping $T \to T^\natural$ of $\mathfrak{S}^+$ into $\boldsymbol{Z} \cap \boldsymbol{Z}'$ is onto.

THEOREM 5. *For each function $f \in \boldsymbol{Z}$, finite except on a nowhere dense set, there exists an operator $T \in \mathfrak{S}^+$ such that $T^\natural = f$.*

PROOF. From the proof of the existence theorem of the pseudo-$\natural$-application given by Dixmier ([4] Theorem 1), we may assume that $E^\natural(\omega) \equiv 1$ for a finite projection E with I as its central envelope. Under this assumption we may construct an operator T of the theorem as follows. For every $\lambda \geq 0$, $\{\omega\,; f(\omega) \leq \lambda\}$ is an open-closed set O_λ modulo a nowhere dense subset of Ω. The central projections corresponding to O_λ are denoted by P_λ. Put $E_\lambda = EP_\lambda + E^\perp$ for $\lambda \geq 0$ and $E_\lambda = 0$ for $\lambda < 0$. Then $\{E_\lambda\}$ is a spectral resolution of the identity, and defines an operator $T = \int_0^\infty \lambda dE_\lambda$. We show that T is a desired operator. To this end we put

$$T_n=(1/2^n)(E_{1/2^n}^{\perp}+E_{2/2^n}^{\perp}+\cdots+E_{n2^n/2^n}^{\perp})=(1/2^n)\,E(P_{1/2^n}^{\perp}+P_{2/2^n}^{\perp}+\cdots+P_{n2^n/2^n}^{\perp}).$$

Then l.u.b. $T_n=T$. Hence from the normality of $\natural$ (Theorem 2) we have l.u.b.$T_n^{\natural}=T^{\natural}$. On the other hand,

$$T_n^{\natural}=(1/2^n)E^{\natural}(P_{1/2^n}^{\perp}+P_{2/2^n}^{\perp}+\cdots+P_{n2^n/2^n}^{\perp})=(1/2^n)(P_{1/2^n}^{\perp}+P_{2/2^n}^{\perp}+\cdots+P_{n2^n/2^n}^{\perp}).$$

It is not difficult to see that $T_n^{\natural}\uparrow f$ as $n\uparrow\infty$. Thus $T^{\natural}=f$, completing the proof.

The invariant linear system $\mathfrak{S}$ is not in general an algebra. It is the case if and only if $\mathbb{M}$ is of type I ([10], [11], [2]). To the proof we need the following lemma.

LEMMA 6. *Let $\mathbb{M}$ be a ring of type I, and let $\{P_n\}$ be a decreasing sequence of finite projections in $\mathbb{M}$ such that $P_n\downarrow 0$. If we denote the central envelope of P_n by Q_n, then $Q_n\downarrow 0$.*

PROOF. First we remark that, in a ring of type I, the $\natural$-application can be normalized as follows: $P^{\natural}(\omega)\geq 1$ and $P^{\natural}(\omega)>0$ are equivalent for each projection P in the ring. This follows from Dixmier's construction of $\natural$-application (cf. [4] Theorem 1 and [1], [2]). Now we turn to the proof of the lemma. If the contrary holds, we may assume that $Q_n=I$ for $n=1, 2, 3, \cdots$. As the support of $P_n^{\natural}$ becomes Ω, we have $P_n^{\natural}(\omega)\geq 1$ everywhere on Ω. While P_n are finite and $P_n\downarrow 0$, so that by the normality of $\natural$ we obtain $P_n^{\natural}\downarrow 0$, a contradiction. The proof is complete.

THEOREM 6. *The following statements for a semi-finite ring $\mathbb{M}$ are equivalent:*

(1) *$\mathbb{M}$ is of type I;*

(2) *$\mathfrak{S}^2\subset\mathfrak{S}$, that is, $\mathfrak{S}$ is an algebra.*

PROOF. Ad (1)$\rightarrow$(2): Let $T=\int_0^{\infty}\lambda dE_{\lambda}$ be any operator in $\mathfrak{S}^+$. Put $T_1=\int_0^1\lambda dE_{\lambda}$ and $T_2=\int_1^{\infty}\lambda dE_{\lambda}$. Then $T_1^2\leq T_1$ so that $T_1^2\in\mathfrak{S}$. Denote the central envelope of $E_{\lambda}^{\perp}$ by Q_{λ}. Then by the preceding lemma, $Q_{\lambda}\downarrow 0$. But $Q_{\lambda}^{\perp}\leq E_{\lambda}$. Hence $Q_{\lambda}^{\perp}T_2$ is a bounded operator. Thus $Q_{\lambda}^{\perp}T_2^2=(Q_{\lambda}^{\perp}T_2)\cdot T_2\in\mathfrak{S}^+$, that is $Q_{\lambda}^{\perp}(T_2^2)^{\natural}(\omega)<+\infty$ except on a nowhere dense set. By letting $\lambda\rightarrow\infty$, we have $(T_2^2)^{\natural}(\omega)<+\infty$ except on a nowhere dense set, that is $T_2^2\in\mathfrak{S}^+$. Thus $T^2=T_1^2+T_2^2\in\mathfrak{S}^+$. This proves (1)$\rightarrow$(2).

Ad (2)$\rightarrow$(1): It is sufficient to show a contradiction under the assumption that $\mathbb{M}$ is of type II. Then there is a finite projection P with central envelope I [2]. Let $\mathfrak{M}$ be the range of P. $\mathbb{M}_{\mathfrak{M}}$, the reduction of $\mathbb{M}$ on $\mathfrak{M}$, is finite and of type II. There is a partition $\{\mathfrak{M}_n\}$ of $\mathfrak{M}$ such that $P^{\natural}{}_{\mathfrak{M}_n}(\omega)=(1/2^n)P^{\natural}(\omega)$. Let

$T=\sum_{n=1}^{\infty} 2^{\frac{n}{2}} P_{\mathfrak{M}_n}$. Then T becomes a positive operator ηM by Theorm 1. $T^\natural(\omega)<+\infty$ by the construction of T. On the other hand $T^2=\sum_{n=1}^{\infty} 2^n P_{\mathfrak{M}_n}$, and $(T^2)^\natural(\omega)\equiv+\infty$ identically, that is, $T^2\notin\mathfrak{S}$, a contradiction as desired.

Next we prove

THEOREM 7. *The following statements for a semi-finite ring* $\mathbf{M}$ *are equivalent:*

(1) $\mathbf{M}$ *is finite;*

(2) $\mathfrak{S}^2\supset\mathfrak{S}$.

PROOF. Ad (1)$\rightarrow$(2): As $\mathbf{M}$ is finite, we normalize the ♮-application so that $I^\natural(\omega)\equiv 1$ identically. Let T be any operator in $\mathfrak{S}^+$. Then $(T^{\frac{1}{2}})^\natural\leq(T^\natural)^{\frac{1}{2}}$ by a usual calculation [1]. This shows us that $T\in\mathfrak{S}^2$.

Ad (2)$\rightarrow$(1): If the contrary holds, we may assume that $\mathbf{M}$ is properly infinite. Then there exists an orthogonal sequence $\{P_n\}$ of finite projections such that $P_n\sim P_m$ and $P_n^\natural(\omega)\equiv 1$ $(m, n=1, 2, 3, \cdots)$ [2]. Put $T=\sum_{n=1}^{\infty}(1/n^2)P_n$. Then T is a positive operator ηM by Theorem 1, and $T^{\frac{1}{2}}=\sum_{n=1}^{\infty}(1/n)P_n$. Normality of ♮ shows us that $T^\natural(\omega)=\sum 1/n^2<+\infty$ and $(T^{\frac{1}{2}})^\natural(\omega)=\sum 1/n=+\infty$. That is $T\in\mathfrak{S}$ and $T^{\frac{1}{2}}\notin\mathfrak{S}$. This proves that $\mathfrak{S}\not\supset\mathfrak{S}^{\frac{1}{2}}$ or $\mathfrak{S}^2\not\supset\mathfrak{S}$, a contradiction.

Combining the last two theorems we obtain the following

THEOREM 8. *The following statements for a semi-finite ring* $\mathbf{M}$ *are equivalent:*

(1) $\mathbf{M}$ *is finite and of type I;*

(2) $\mathfrak{S}=\mathfrak{S}^2$.

PROOF. Clear.

Here we will mention some special properties concerning the extended ♮-application. Some of them will interest us directly in their own nature, and others will reveal their meaning more clearly when applied to the theory of integration in the next §.

LEMMA 7. *If* $A\in\mathbf{M}^+$ *and* $T\in\mathfrak{S}^+$, *then* $(A\cdot T)^\natural=(TA)^\natural=(A^{\frac{1}{2}}\cdot TA^{\frac{1}{2}})^\natural=(T^{\frac{1}{2}}\cdot A\cdot T^{\frac{1}{2}})^\natural\geq 0$.

PROOF. The first two equalities are clear from Theorem 3. It remains only to prove that $(S_1\cdot S_2)^\natural=(S_2\cdot S_1)^\natural$ for every $S_1\in\mathfrak{S}^{\frac{1}{2}}$ and $S_2\in\mathfrak{S}^{\frac{1}{2}}$. With no loss of generalities, we may assume that $S_1\geq 0$ and $S_2\geq 0$. Then the equality:

$$((S_1+iS_2)\cdot(S_1-iS_2))^\natural=((S_1+iS_2)(S_1+iS_2)^*)^\natural$$

$$=((S_1+iS_2)^*(S_1+iS_2))^\natural=((S_1-iS_2)\cdot(S_1+iS_2))^\natural,$$

shows us that $(S_1\cdot S_2)^\natural=(S_2\cdot S_1)^\natural$, as desired.

THEOREM 9. *If* $T\in\mathfrak{S}^+$, *then the mapping* $A\rightarrow(A\cdot T)^\natural$ *of* $\mathbf{M}$ *into* $\boldsymbol{Z}'$ *is normal.*

PROOF. Let $\{A_\delta\}$ be an increasing directed set of operators $\in \mathbf{M}^+$ with the least upper bound $A \in \mathbf{M}^+$. Then

$$(A_\delta \cdot T)^\natural = (T^{\frac{1}{2}} \cdot A_\delta \cdot T^{\frac{1}{2}})^\natural \uparrow (T^{\frac{1}{2}} \cdot A \cdot T^{\frac{1}{2}})^\natural = (A \cdot T)^\natural,$$

by Lemma 7 and Corollary of Theorem 1, completing the proof.

COROLLARY. *If $T \in \mathfrak{S}$, then the mapping $P \to (P \cdot T)^\natural$ of $\mathbf{M}_P$ into $\boldsymbol{Z}'$ is completely additive.*

PROOF. Since T can be expressed as a linear combination of operators $\in \mathfrak{S}^+$, the statement is clear from the preceding theorem.

LEMMA 8. *Let α and β be non negative real numbers such that $\alpha + \beta = 1$. If $S \in \mathfrak{S}^\alpha$ and $T \in \mathfrak{S}^\beta$, then the following statements hold:*

(1) *If $S \geq 0$ and $T \geq 0$, then $(S \cdot T)^\natural = (S^{\frac{1}{2}} \cdot T \cdot S^{\frac{1}{2}})^\natural \geq 0$;*

(2) $(S \cdot T)^\natural = (T \cdot S)^\natural$.

PROOF. In case that $\alpha = 0$ or $\beta = 0$, the statements are already proved in Lemma 7 and Theorem 3; (Note that $\mathfrak{S}^0 = \mathbf{M}$). Hence we may assume that $\alpha > 0$ and $\beta > 0$.

Ad (1): Let $T = \int_0^\infty \lambda dE_\lambda$ be the spectral resolution. Then, as $P_n = E_n E_{1/n}^\perp$ is a projection in $\mathfrak{S}^\beta$, it is also a projection in $\mathfrak{S}^\gamma$ for every $\gamma > 0$. Since $S \cdot T \in \mathfrak{S}$ and $\text{l.u.b.}\, P_n = E_0^\perp$ we have $\lim (P_n \cdot S \cdot T)^\natural = (S \cdot T)^\natural$ by Corollary of Theorem 9, and $\text{l.u.b.}\, S^{\frac{1}{2}} \cdot TP_n \cdot S^{\frac{1}{2}} = S^{\frac{1}{2}} \cdot T \cdot S^{\frac{1}{2}}$ by Corollary of Theorem 1. On the other hand, as $TP_n \in \mathbf{M}^+ \cap \mathfrak{S}^\gamma$ for every $\gamma > 0$ and hence $S \cdot (TP_n)^{\frac{1}{2}} \in \mathfrak{S}$, it follows that

$$\begin{aligned}(P_n \cdot S \cdot T)^\natural &= (S \cdot TP_n)^\natural = (S \cdot (TP_n)^{\frac{1}{2}} (TP_n)^{\frac{1}{2}})^\natural = ((TP_n)^{\frac{1}{2}} \cdot S \cdot (TP_n)^{\frac{1}{2}})^\natural \\ &= ((S^{\frac{1}{2}} \cdot (TP_n)^{\frac{1}{2}})^* (S^{\frac{1}{2}} \cdot (TP_n)^{\frac{1}{2}}))^\natural = (S^{\frac{1}{2}} \cdot (TP_n)^{\frac{1}{2}} \cdot (S^{\frac{1}{2}} \cdot (TP_n)^{\frac{1}{2}})^*)^\natural = (S^{\frac{1}{2}} \cdot TP_n \cdot S^{\frac{1}{2}})^\natural,\end{aligned}$$

by Lemma 7. Thus $(P_n \cdot S \cdot T)^\natural = (S^{\frac{1}{2}} \cdot TP_n \cdot S^{\frac{1}{2}})^\natural \uparrow (S^{\frac{1}{2}} \cdot T \cdot S^{\frac{1}{2}})^\natural$, $(n \to \infty)$, whence $(S \cdot T)^\natural = (S^{\frac{1}{2}} \cdot T \cdot S^{\frac{1}{2}})^\natural$. This proves (1).

Ad (2): Since S and T are linear combinations of positive elements of $\mathfrak{S}^\alpha$ and $\mathfrak{S}^\beta$ respectively, it suffices to assume that $S \geq 0$ and $T \geq 0$. Then (1) yields the equality (2), completing the proof.

LEMMA 9. *Let α and β be non-negative real numbers such that $\alpha + \beta = 1$. Let $\{S_\delta\}$ and $\{T_\delta\}$ be increasing directed sets of positive operators in $\mathfrak{S}^\alpha$ and $\mathfrak{S}^\beta$ respectively. If $\text{l.u.b.}_\delta\, S_\delta = S \in \mathfrak{S}^\alpha$ and $\text{l.u.b.}_\delta\, T_\delta = T \in \mathfrak{S}^\beta$ exist, then $\text{l.u.b.}_\delta\, (S_\delta \cdot T_\delta)^\natural = (S \cdot T)^\natural$.*

PROOF. Let $g = \text{l.u.b.}\, (S_\delta \cdot T_\delta)^\natural$. Since $(S_\delta \cdot T_\delta)^\natural \leq (S_\delta \cdot T_{\delta'})^\natural \leq (S_{\delta'} \cdot T_{\delta'})^\natural \leq (S \cdot T)^\natural$ for $\delta < \delta'$ (Lemma 7), it follows that $g \leq (S \cdot T)^\natural$ and $g \geq (S_\delta \cdot T_{\delta'})^\natural$ for every δ and δ'. Thus

$$g \geq \underset{\delta'}{\text{l.u.b.}} (S_\delta \cdot T_{\delta'})^\natural = \underset{\delta'}{\text{l.u.b.}} (S_\delta^{\frac{1}{2}} \cdot T_{\delta'} \cdot S_\delta^{\frac{1}{2}})^\natural = (S_\delta^{\frac{1}{2}} \cdot T \cdot S_\delta^{\frac{1}{2}})^\natural = (S_\delta \cdot T)^\natural$$

for every δ. It is not difficult to see that $g \geq (S \cdot T)^\natural$. This completes the proof.

Concerning the invariant linear system $\mathfrak{S}$ and $\mathfrak{S}^{\frac{1}{2}}$, we obtain the following properties summed up in

THEOREM 10. *In $\mathfrak{S}$ and $\mathfrak{S}^{\frac{1}{2}}$, the following statements hold:*

(1) *If $T \in \mathfrak{S}$, then $\underset{\|A\| \leq 1,\, A \in \mathbf{M}}{\text{l.u.b.}} |(A \cdot T)^\natural| = |T|^\natural$, and in particular $|(A \cdot T)^\natural| \leq \|A\| \, |T|^\natural$ for every $A \in \mathbf{M}$;*

(2) *If $S, T \in \mathfrak{S}$, then $|S+T|^\natural \leq |S|^\natural + |T|^\natural$;*

(3) *If $S, T \in \mathfrak{S}^{\frac{1}{2}}$, such that $S \cdot T^* = 0$, then $(|S+T|^2)^\natural = (|S|^2)^\natural + (|T|^2)^\natural$;*

(4) *If $T \in \mathfrak{S}$, then $T \geq 0$ if and only if $(A \cdot T)^\natural \geq 0$ for every $A \in \mathbf{M}^+$;*

(5) *If $A \in \mathbf{M}$, then $A \geq 0$ if and only if $(A \cdot T)^\natural \geq 0$ for every $T \in \mathfrak{S}^+$;*

(6) *If $S \in \mathfrak{S}^{\frac{1}{2}}$, then $S \geq 0$ if and only if $(S \cdot T)^\natural \geq 0$ for every $T \in \mathfrak{S}^{\frac{1}{2}+}$;*

(7) *If $S, T \in \mathfrak{S}^{\frac{1}{2}}$ such that $|S| \leq |T|$, then $(|S|^2)^\natural \leq (|S| \cdot |T|)^\natural \leq (|T|^2)^\natural$;*

(8) *If S and T are self-adjoint elements of $\mathfrak{S}^{\frac{1}{2}}$ such that $(S^2)^\natural \leq (T^2)^\natural$, then $(S \cdot T)^\natural \leq (T^2)^\natural$;*

(9) *If $T \in \mathfrak{S}^{\frac{1}{2}}$ and $U \in \mathbf{M}_U$, then $(|T|^2)^\natural = (|UTU^*|^2)^\natural$;*

(10) *If $S, T \in \mathfrak{S}^{\frac{1}{2}}$, then $|(S \cdot T)^\natural|^2 \leq (|T| \cdot |S^*|)^\natural (|S| \cdot |T^*|)^\natural \leq |T \cdot S|^\natural \, |S \cdot T|^\natural$;*

(11) *If $S, T \in \mathfrak{S}^{\frac{1}{2}}$, then $|(S \cdot T)^\natural|^2 \leq (|S \cdot T|^\natural)^2 \leq (S^* S)^\natural (T^* T)^\natural$* (Schwarz's Inequality), *and $((S^* S)^\natural)^{\frac{1}{2}} = \underset{(T^*T)^\natural \leq 1}{\text{l.u.b.}} |(S \cdot T)^\natural|$.*

PROOF. First we shall prove a part of (11): $|(S \cdot T)^\natural|^2 \leq (S^* S)^\natural (T^* T)^\natural$. For any complex numbers α and β,

$$|\alpha|^2 (SS^*)^\natural + 2\Re\, \bar{\alpha}\beta (S \cdot T)^\natural + |\beta|^2 (T^* T)^\natural = ((\alpha S^* + \beta T)^* \cdot (\alpha S^* + \beta T))^\natural \geq 0$$

By means of this inequality, we do the trick in the usual canonical fashion.

Ad (1): Let $T = U|T|$ be the polar decomposition of T and $\|A\| \leq 1$. Then

$$|(A \cdot T)^\natural|^2 = |(A \cdot U|T|)^\natural|^2 = |(AU \cdot |T|^{\frac{1}{2}} \cdot |T|^{\frac{1}{2}})^\natural|^2 \leq (|T|^{\frac{1}{2}} \cdot U^* A^* A U \cdot |T|^{\frac{1}{2}})^\natural |T|^\natural$$

by Schwarz's Inequality just proved. But as is easily verified, $|T|^{\frac{1}{2}} \cdot U^* A^* A U \cdot |T|^{\frac{1}{2}} \leq |T|$. Hence $(|T|^{\frac{1}{2}} \cdot U^* A^* A U \cdot |T|^{\frac{1}{2}})^\natural \leq |T|^\natural$. Thus we have $|(A \cdot T)^\natural| \leq |T|^\natural$ for every $A \in \mathbf{M}$, $\|A\| \leq 1$. $|T| = U^* T$ shows that $|T|^\natural$ is the least upper bound really attainable by an $A = U^*$.

Ad (2): Let $S+T = U|S+T|$ be the polar decomposition of $S+T$. Then by using (1) we obtain

$$|S+T|^\natural = (U^*(S+T))^\natural = (U^* \cdot S)^\natural + (U^* \cdot T)^\natural \leq |S|^\natural + |T|^\natural.$$

Ad (3): From the assumption, we have $(S \cdot T^*)^\natural = 0$. Hence

$$(|S+T|^2)^\natural = ((S^*+T^*)(S+T))^\natural = (S^*S)^\natural + (T^* \cdot S)^\natural + (T \cdot S^*)^\natural + (T^*T)^\natural$$
$$= (|S|^2)^\natural + (|T|^2)^\natural + (S \cdot T^*)^\natural + \overline{(S \cdot T^*)^\natural} = (|S|^2)^\natural + (|T|^2)^\natural.$$

Ad (4): By Lemma 7, it is sufficient to prove the "if" part. If $T = T_1 + iT_2$ with $T_1 = T_1^*$ and $T_2 = T_2^*$, then $(A \cdot T_2)^\natural = 0$ for every $A \in \mathbf{M}^+$. Let $T_2 = \int_{-\infty}^{\infty} \lambda dF_\lambda$ be the spectral resolution. Then for any $\lambda < 0$, $F_\lambda T_2 \leq 0$. But, as $F_\lambda \in \mathbf{M}^+$, we have $(F_\lambda T_2)^\natural = 0$. Hence $F_\lambda T_2 = 0$ since the mapping $\natural$ is faithful. This shows us that $F_\lambda = 0$ for every $\lambda < 0$. In the same way, we can prove that for any $\lambda > 0$, $F_\lambda^\perp = 0$. Thus we have $T_2 = 0$. Let $T_1 = \int_{-\infty}^{\infty} \lambda dE_\lambda$ be the srectral resolution. Then for any $\lambda < 0$, $E_\lambda T_1 \leq 0$ and $(E_\lambda T_1)^\natural = (E_\lambda T)^\natural \geq 0$ since $E_\lambda \in \mathbf{M}^+$. This shows $(E_\lambda T_1)^\natural = 0$ so that $E_\lambda T_1 = 0$, and hence $E_\lambda = 0$ for every $\lambda < 0$. Thus $T = T_1 = \int_0^\infty \lambda dE_\lambda \geq 0$. This proves (4).

Ad (5): By Lemma 7, it is sufficient to prove the "if" part. If $A = A_1 + iA_2$ with $A_1 = A_1^*$ and $A_2 = A_2^*$, then $(A_2 \cdot T)^\natural = 0$ for every $T \in \mathfrak{S}^+$. Hence $(A_2 \cdot T)^\natural = 0$ for every $T \in \mathfrak{S}$, so that $(|A_2| \cdot T)^\natural = 0$ for every $T \in \mathfrak{S}$. Thus $T^{\frac{1}{2}}|A_2|T^{\frac{1}{2}} = 0$ for every $T \in \mathfrak{S}^+$. Let $|A_2| = \int_0^\infty \lambda dF_\lambda$ be the spectral resolution. If $F_{\lambda_0}^{\frac{1}{2}} \neq 0$ for some $\lambda_0 > 0$, then there is a non-zero projection $Q \in \mathfrak{S}$ such that $Q \leq F_{\lambda_0}^\perp$. For every $x \in \mathfrak{H}$ we have

$$0 = \langle Q|A_2|Qx, x\rangle = \int_0^\infty \lambda d\,\|F_\lambda Qx\|^2 \geq \int_{\lambda_0}^\infty \lambda d\,\|F_\lambda Qx\|^2.$$

Hence $0 = F_{\lambda_0}^\perp Q = Q$. This is a contradiction. Therefore $F_\lambda^\perp = 0$ for every $\lambda > 0$. That is $A_2 = 0$. Let $A_1 = \int_{-\infty}^{\infty} \lambda dE_\lambda$ be the spectral resolution. If $E_{\lambda_0} \neq 0$ for some $\lambda_0 > 0$, then there exists a non-zero projection $P \in \mathfrak{S}$ such that $P \leq E_{\lambda_0}$. As $PA_1P \leq 0$ and $0 \leq (PAP)^\natural = (PA_1P)^\natural$, we see that $(PA_1P)^\natural = 0$ and hence $PA_1P = 0$. From this we can prove in the same manner as above $0 = E_{\lambda_0}P = P$. This is a contradiction. Thus we have $A = A_1 = \int_0^\infty \lambda dE_\lambda$. The proof is complete.

Ad (6): The "only if" part is evident by Lemma 8. The proof of the "if" part is nearly the same as that of (4). Hence details are omitted.

Ad (7): $|T| - |S| \geq 0$. Hence $(|S| \cdot (|T| - |S|))^\natural \geq 0$. This leads to the first inequality $(|S|^2)^\natural \leq (|T| \cdot |S|)^\natural$. The second is similarly proved.

Ad (8): $0\leq((T-S)^2)^\natural=(T^2)^\natural-2(T\cdot S)^\natural+(S^2)^\natural\leq 2((T^2)^\natural-(S\cdot T)^\natural)$. Hence $(T\cdot S)^\natural\leq(T^2)^\natural$.

Ad (9): $|UTU^*|^2=UT^*U^*UTU^*=U|T|^2U^*$. Hence $(|UTU^*|^2)^\natural=(U|T|^2U^*)^\natural=(|T|^2)^\natural$.

Ad (10): Let $S=U|S|$ and $T=V|T|$ be the polar decompositions of S and T respectively. Then $|S^*|=U|S|U^*=SU^*$ and $|T^*|=V|T|V^*=TV^*$. Hence

$$\begin{aligned}|(S\cdot T)^\natural|^2&=|(U|S|\cdot V|T|)^\natural|^2=|((|T|^{\frac12}\cdot U\cdot|S|^{\frac12})\cdot(|S|^{\frac12}\cdot V\cdot|T|^{\frac12}))^\natural|^2\\&\leq(|S|^{\frac12}\cdot U^*\cdot|T|\cdot U\cdot|S|^{\frac12})^\natural(|T|^{\frac12}\cdot V^*|S|\cdot V\cdot|T|^{\frac12})^\natural\\&=(U^*\cdot|T|\cdot U\cdot|S|)^\natural(V^*\cdot|S|\cdot V\cdot|T|)^\natural=(|T|\cdot U|S|U^*)^\natural(|S|\cdot V|T|V^*)^\natural\\&=(|T|\cdot|S^*|)^\natural(|S|\cdot|T^*|)^\natural=(|T|\cdot SU^*)^\natural(|S|\cdot TV^*)^\natural\\&=(V^*\cdot T\cdot SU^*)^\natural(U^*\cdot S\cdot TV^*)\leq|T\cdot S|^\natural|S\cdot T|^\natural\end{aligned}$$

Ad (11): Consider the polar decomposition $W|S\cdot T|$ of $S\cdot T$, where W is a partially isometric operator. Then

$$\begin{aligned}&|(S\cdot T)^\natural|^2=|(W|S\cdot T|)^\natural|^2\leq\|W\|^2(|S\cdot T|^\natural)^2\leq(|S\cdot T|^\natural)^2=((W^*\cdot S)\cdot T)^\natural)^2\\&\leq(S^*\cdot WW^*\cdot S)^\natural(T^*T)^\natural=(WW^*SS^*)^\natural(T^*T)^\natural\leq(SS^*)^\natural(T^*T)^\natural\end{aligned}$$

This proves Schwarz's Inequality. The proof of the last statement goes as follows. Put $g=\underset{(T^*T)^\natural\leq 1}{\mathrm{l.u.b.}}|(S\cdot T)^\natural|$. Then, by Schwarz's Inequality just proved, it follows that $g\leq((S^*S)^\natural)^{\frac12}$. Let $S=U|S|$ be the polar decomposition and P_n be the central projection corresponding to the open-closed set $\overline{\{\omega;\ ((S^*S)^\natural)^{\frac12}(\omega)>1/n\}}$. Then $\dfrac{P_n}{((S^*S)^\natural)^{\frac12}}\in C(\Omega)$ and hence we may regard $\dfrac{P_n}{((S^*S)^\natural)^{\frac12}}$ as an operator $\in\mathbb{M}^\natural$. Thus $T_n=\dfrac{P_n}{((S^*S)^\natural)^{\frac12}}|S|U^*\in\mathfrak{S}^{\frac12}$, $(T_n^*T_n)^\natural\leq 1$ and

$$|(S\cdot T_n)^\natural|=\frac{P_n}{((S^*S)^\natural)^{\frac12}}|(U|S|^2U^*)^\natural|=\frac{P_n}{((S^*S)^\natural)^{\frac12}}(SS^*)^\natural=P_n((S^*S)^\natural)^{\frac12}.$$

Therefore $g\geq P_n((S^*S)^\natural)^{\frac12}$ for every n, and hence $g\geq((S^*S)^\natural)^{\frac12}$, completing the proof. The theorem is thus completely proved.

In the rest of this §, we consider, as an example, the canonical ♮-application of an H-system (= Ambrose space [14]). Let $\boldsymbol{H}$ be an H-system, and $\boldsymbol{B}$, $\mathbb{L}$ and $\mathbb{R}$ be its bounded algebra, left ring and right ring respectively. The partial applications $y\to xy$ and $y\to yx$ are denoted by L_x and R_x respectively. An element $x\in\boldsymbol{H}$ is called *central* if $xb=bx$ for every $b\in\boldsymbol{B}$, that is, $L_x\eta\mathbb{L}^\natural=\mathbb{R}^\natural$. The set of all central elements forms a closed linear subspace $\boldsymbol{H}^{\natural}\eta\mathbb{L}\cup\mathbb{R}$. Let $x\to x^{\natural}$ be the projection of x on $\boldsymbol{H}^{\natural}$. It is known that $\boldsymbol{B}^{\natural}\subset\boldsymbol{B}$ and $x^{\natural}\geq 0$ for every $x\geq 0$. Put $L_b^{\natural}=L_{b^{\natural}}$ for $b\in\boldsymbol{B}$. Then $L_b\to L_b^{\natural}$ is an application of the ideal $\mathbb{L}_{\boldsymbol{B}}=\{L_b\,;\,b\in\boldsymbol{B}\}$ of $\mathbb{L}$ into the center $\mathbb{L}^{\natural}$ of $\mathbb{L}$ with the following properties:

1. If $B\in\mathbb{L}_{\boldsymbol{B}}\cap\mathbb{L}^{\natural}$, then $B^{\natural}=B$;
2. $B\to B^{\natural}$ is a positive, linear and normal mapping;
3. $(AB)^{\natural}=(BA)^{\natural}$ for every $A\in\mathbb{L}$ and $B\in\mathbb{L}_{\boldsymbol{B}}$;
4. $(AB)^{\natural}=AB^{\natural}$ for every $A\in\mathbb{L}^{\natural}$ and $B\in\mathbb{L}_{\boldsymbol{B}}$;
5. $\|B^{\natural}\|\leq\|B\|$ for every $B\in\mathbb{L}_{\boldsymbol{B}}$.

Thus $B\to B^{\natural}$ is a normal and essential $\natural$-application defined on $\mathbb{L}_{\boldsymbol{B}}$. Owing to the property (5), $B\to B^{\natural}$ is uniquely extended to a normal and essential $\natural$-application defined on $\mathbb{L}$. We have called this extended application the *canonical $\natural$-application* of $\boldsymbol{H}$ [13]. The pseudo-$\natural$-application, obtained by restricting it to $\mathbb{L}^{+}$, can be extended by means of ($\natural$) to an extended pseudo-$\natural$-application defined on the set of all positive operators T on $\boldsymbol{H}$:

$$T^{\natural}=\underset{\mathbb{L}^{+}\ni A\leq T}{\text{l. u. b.}}\,A^{\natural}.$$

As remarked earlier, every element of $\boldsymbol{Z}'$ is identified with an operator $\eta\mathbb{L}^{\natural}$ and *vice versa*. With this identification we obtain the following

THEOREM 11. *$L_x^{\natural}=L_{x^{\natural}}$ for every $x\in\boldsymbol{H}$.*

PROOF. We need only to consider the case $x\geq 0$. Let $L_x=\int_0^{\infty}\lambda dE_{\lambda}$ be the spectral resolution. Then $\underset{\lambda}{\text{l.u.b.}}\,L_{E_{\lambda}x}=L_x$. Thus by the normality of the extended pseudo-$\natural$-application (Remark 5), we have

$$\underset{\lambda}{\text{l.u.b.}}\,L_{(E_{\lambda}x)^{\natural}}=\underset{\lambda}{\text{l.u.b.}}\,L_{E_{\lambda}x}^{\natural}=L_x^{\natural}.$$

As $\{E_{\lambda}x\}$ is an increasing set with an upper bound x, $\{L_{(E_{\lambda}x)^{\natural}}\}$ is a commutative and increasing set with an upper bound $L_{x^{\natural}}$. Hence $\{L^2_{(E_{\lambda}x)^{\natural}}\}$ is an increasing set of positive operators with an upper bound $L^2_{x^{\natural}}$. It follows that, by Theorem 1, $\text{l.u.b.}\,L_{(E_{\lambda}x)^{\natural}}=T_0\leq L_{x^{\natural}}$, where l.u.b. is taken in the sense of the ordering of the positive operators $\eta\mathbb{L}$. T_0 is a measurable operator $\eta\mathbb{L}$ with $\mathfrak{D}_{T_0}\supset\mathfrak{D}_{L_{x^{\natural}}}\supset\boldsymbol{B}$ and $\lim\langle(E_{\lambda}x)^{\natural}b,\,b\rangle=\langle T_0b,\,b\rangle$ for every $b\in\boldsymbol{B}$. On the other hand, as $\|(E_{\lambda}x)^{\natural}-x^{\natural}\|\to 0$ for $\lambda\to\infty$, we have $\lim\langle(E_{\lambda}x)^{\natural}b,\,b\rangle=\langle x^{\natural}b,\,b\rangle$ for every $b\in\boldsymbol{B}$. Hence T_0 and $L_{x^{\natural}}$ are identical on the dense set $\boldsymbol{B}$. Measurability of T_0 and $L_{x^{\natural}}$ assures that $T_0=L_{x^{\natural}}$ [13]. Thus $\text{l.u.b.}\,L_{(E_{\lambda}x)^{\natural}}=L_{x^{\natural}}$ in the sense of the ordering of the positive operators on $\boldsymbol{H}$, and *a fortiori* in the sense of the ordering of the real elements of $\boldsymbol{Z}'$. Thus we have $L_x^\natural = L_{x^\natural}$, completing the proof.

§ 3. Application to the theory of integration

In this § some applications of the previous results to the theory of non-commutative integrations will be considered. In contrast to our previous paper [13], we assume the classical theory of integrations over an abstract measure space.

Let m be a normal, faithful and essential pseudo-trace defined on $\mathbb{M}^+$. Then there exists a unique normal, faithful and essential pseudo-measure φ on Ω such that $m(A) = \varphi(A^\natural)$ holds for every $A \in \mathbb{M}^+$ [4]. Put

$$m(T) = \underset{\mathbb{M}^+ \ni A \leq T}{\text{l. u. b.}}\, m(A)$$

for every positive operator $T\eta\mathbb{M}$. Then by Theorem 2, $T^\natural = \text{l.u.b.}\, T_n^\natural$, where $T = \int_0^\infty \lambda dE_\lambda$ is the spectral resolution and $T_n = \int_0^n \lambda dE_\lambda$. Hence on account of the normality of φ we obtain

$$m(T) = \underset{\mathbb{M}^+ \ni A \leq T}{\text{l.u.b.}}\, \varphi(A^\natural) = \underset{n}{\text{l.u.b.}}\, \varphi(T_n^\natural) = \varphi(T^\natural).$$

LEMMA 10. *If T is a positive operator $\eta\mathbb{M}$ with $m(T) < +\infty$, then $T \in \mathfrak{S}^+$ and the support of $T^\natural$ is of* countable genre, *that is every family of disjoint non-void open-closed sets contained in this support is at most countable* [3].

PROOF. Essentiality of the pseudo-measure φ shows us that $\varphi(T^\natural) = m(T) < +\infty$ implies $T^\natural(\omega) < +\infty$ except on a nowhere dense set, that is $T \in \mathfrak{S}^+$. If the support of $T^\natural$ is not of countable genre, it is not difficult to see that $m(T) = +\infty$, a contradiction.

LEMMA 11. *Let $T \in \mathfrak{S}^+$. Then following statements are equivalent:*

(1) *There is a normal, faithful and essential pseudo-trace m such that $m(T) < +\infty$;*

(2) *The support of $T^\natural$ is of countable genre.*

PROOF. The lemma is evident from the classical theory of integration. So the proof is omitted.

A positive operator $T\eta\mathbb{M}$ is integrable only if $T \in \mathfrak{S}^+$. The converse does not hold in general. For this we have

LEMMA 12. *The following statements are equivalent:*

(1) *For every $T \in \mathfrak{S}^+$, there is a normal, faithful and essential pseudo-trace m such that $m(T) < +\infty$;*

(2) *Ω is of countable genre;*

(3) $\mathbf{M}^{\natural}$ *is countably decomposable.*

PROOF. Ad (1)→(2): Let f be an arbitrary element of $\boldsymbol{Z}$ such that $0<f(\omega)<+\infty$ except on a nowhere dense set. Then there exists a positive operator $T\eta\mathbf{M}$ with $T^{\natural}=f$ by Theorem 5. Hence by assumption, a normal, faithful and essential pseudo-trace m on $\mathbf{M}^+$, and hence the corresponding normal, faithful and essential pseudo-measure φ on Ω exist, such that $\varphi(f)=\varphi(T^{\natural})=m(T)<+\infty$. Put $\varphi_f(g)=\varphi(fg)$. Then φ_f is also a normal faithful and essential pseudo-measure on Ω. $\varphi_f(1)=\varphi(f)<+\infty$ shows us that φ_f is a measure with support Ω. Hence Ω is of countable genre [3].

Ad (2)→(1): If Ω is of countable genre, there exists a bounded normal measure [3]. Hence for every $f\in\boldsymbol{Z}$, there exists a normal, faithful and essential pseudo-measure φ such that $\varphi(f)<+\infty$. This shows (2)→(1). Equivalence of (2) and (3) is obvious. The proof is thus complete.

In the sequel, m is a fixed normal, faithful and essential pseudo-trace defined on $\mathbf{M}^+$, and φ is the corresponding pseudo-measure on Ω.

DEFINITION 3. An operator $T\eta\mathbf{M}$ is called *integrable* if $m(|T|)<+\infty$. T is called *square-integrable* if $m(T^*T)<+\infty$. The set of all integrable operators is denoted by $\boldsymbol{L}_1$ and that of all square-integrable operators by $\boldsymbol{L}_2$.

$\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are invariant linear systems satisfying $(\ll)_1$ and $(\ll)_2$. $\boldsymbol{L}_2=\boldsymbol{L}_1^{\frac{1}{2}}$, $\boldsymbol{L}_1\subset\mathfrak{S}$ and $\boldsymbol{L}_2\subset\mathfrak{S}^{\frac{1}{2}}$. The proof is not difficult and the details are omitted. By a canonical fashion $m(T)$ is uniquely extended as a linear form on $\boldsymbol{L}_1$. Then we have

$$m(T)=\varphi(T^{\natural})$$

for every $T\in\boldsymbol{L}_1$. $m(T)$ is called the *integral* of T.

As an immediate consequence of Theorem 3 we have

THEOREM 12. *The integral $m(T)$, $T\in\boldsymbol{L}_1$ has the following properties:*

(1) *If $T\in\boldsymbol{L}_1$ and $T_1\in\boldsymbol{L}_1$, and α, α_1 are complex numbers, then $m(\alpha T+\alpha_1T_1)=\alpha m(T)+\alpha_1m(T_1)$;*

(2) *If $T\in\boldsymbol{L}_1$ and $A\in\mathbf{M}$, then $m(A\cdot T)=m(TA)$;*

(3) *If $T\in\boldsymbol{L}_1^+$, then $m(T)\geq 0$;*

(4) *$m(T^*)=\overline{m(T)}$ for every $T\in\boldsymbol{L}_1$;*

(5) *If $SS^*\in\boldsymbol{L}_1$ for an operator S, then $S^*S\in\boldsymbol{L}_1$ and $m(SS^*)=m(S^*S)$.*

REMARK 8. The statements in Theorem 10 may be transferred to the relations in terms of integrals. For instance: $|m(S\cdot T)|^2\leq m(|T|\cdot|S^*|)m(|S|\cdot|T^*|)$ for every $S\in\boldsymbol{L}_2$ and $T\in\boldsymbol{L}_2$ (10); $|m(S\cdot T)|^2\leq m(|S\cdot T|)^2\leq m(S^*S)\,m(T^*T)$ for every $S\in \boldsymbol{L}_2$ and $T\in \boldsymbol{L}_2$ (Schwarz's Inequality). Details are omitted.

As in our previous paper [13], we denote $\|T\|_1=m(|T|)$ for $T\in \boldsymbol{L}_1$ and $\|T\|_2=m(T^*T)^{\frac{1}{2}}$ for $T\in \boldsymbol{L}_2$. Then it is clear that $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are normed spaces with norms $\|T\|_1$ and $\|T\|_2$ respectively (Theorem 10). First we show

THEOREM 13. (Monotone Convergence Theorem). *Let $\{T_n\}$ be a monotone increasing sequence of positive operators $\in \boldsymbol{L}_1$. Then there exists a $T\in \boldsymbol{L}_1$ such that* $\text{l.u.b.}\, T_n=T$, *if and only if $\{\|T_n\|_1\}$ is bounded. In this case* $\lim\|T-T_n\|_1=0$, $T^\natural=\text{l.u.b.}\, T_n^\natural$, *and $\{T_n\}$ converges n. e. to T in the star sense.*

PROOF. If $\{\|T_n\|_1\}$ is not bounded, no such T exists. Assume that $\{\|T_n\|_1\}$ is bounded. By taking a subsequence, if necessary, we may assume that $\|T_{n+1}-T_n\|_1<1/4^n$ $(n=1, 2, 3, \cdots)$. Let $T_{n+1}-T_n=\int_0^\infty \lambda dE_\lambda^{(n)}$ be the spectral resolution of $T_{n+1}-T_n\geq 0$. Then

$$(1/2^n)\, m(E_{1/2^n}^{(n)\perp})=-\int_{1/2^n}^\infty (1/2^n)\, dm(E_\lambda^{(n)\perp})\leq -\int_{1/2^n}^\infty \lambda dm(E_\lambda^{(n)\perp})$$

$$\leq -\int_0^\infty \lambda dm(E_\lambda^{(n)\perp})=\|T_{n+1}-T_n\|_1<1/4^n$$

Hence $m(E_{1/2^n}^{(n)\perp})<1/2^n$. Put $P_n=\bigcap_{k=n}^\infty E_{1/2^k}^{(k)}$. Then $m(P_n^\perp)<1/2^{n-1}$. Thus we have $P_n^\perp\downarrow 0$ and $P_n^\perp$ is finite. Since $\|(T_{n+1}-T_n)P_n\|\leq 1/2^n$ and $\{P_n\}$ is increasing, we have $\|(T_m-T_n)P_n\|\leq 1/2^{n-1}$ for every $m>n$. Let $\mathfrak{D}$ be the intersection of all $\mathfrak{D}_{T_n}$ $(n=1, 2, 3, \cdots)$ and the set-theoretic sum of all $P_n\mathfrak{H}$ $(n=1, 2, 3, \cdots)$. Then $\mathfrak{D}$ is strongly dense [13]. Now, for every $x\in\mathfrak{D}$, $\{T_nx\}$ is a Cauchy sequence of elements of $\mathfrak{H}$. Hence $\lim_{n\to\infty} T_nx$ exists which we will denote by Sx. Clearly S is a linear not necessarily closed operator with strongly dense domain $\mathfrak{D}$, and has the adjoint $S^*\supset S$. Therefore S has its own closure T. Evidently $T\geq 0$. For every $x\in\mathfrak{D}$, $\text{l.u.b.}\langle T_nx, x\rangle=\langle Tx, x\rangle$. Hence by Theorem 1, $\text{l.u.b.}\, T_n=T$, and by normality of $\natural$, $\text{l.u.b.}\, T_n^\natural=T^\natural$. Thus $\|T\|_1-\|T_n\|_1=\|T-T_n\|_1=\varphi(T^\natural-T_n^\natural)\to 0$. This proves the theorem.

COROLLARY 1. *Let $\{T_n\}$ be a monotone increasing sequence of positive operators $\eta\mathbf{M}$. If* $\text{l.u.b.}\, T_n^\natural=g\in \boldsymbol{Z}'$ *and the support of g is of countable genre, then* $\text{l.u.b.}\, T_n=T\eta\mathbf{M}$ *exists with $T^\natural=g$. And $\{T_n\}$ converges n. e. to T in the star sense.*

PROOF. Since the support of g is of countable genre, there is a normal, faithful and essential pseudo-measure φ' such that $\varphi'(g)<+\infty$. Let m' be the corresponding normal, faithful and essential pseudo-trace. Then the norm $\|T_n\|_1'$ $=m'(T_n)\leq\varphi'(g)$, that is, $\{\|T_n\|_1'\}$ is bounded. To complete the proof we have only to apply the preceding theorem.

COROLLARY 2. *Let $\{T_n\}$ be a monotone increasing sequence of positive operators* $\eta\mathbb{M}$. *If* $\text{l.u.b.}\, T_n^\natural = g \in \boldsymbol{Z}'$, *then* $\text{l.u.b.}\, T_n = T \eta \mathbb{M}$ *exists and* $T^\natural = g$.

PROOF. As $\mathbb{M}$ is a central direct sum of countably decomposable centers, the proof follows from the preceding corollary.

THEOREM 14. *$\boldsymbol{L}_1$ is a Banach space.*

PROOF. The only point to be proved here is the completeness of $\boldsymbol{L}_1$ with respect to the norm $\| \ \|_1$. Let $\{T_n\}$ be a Cauchy sequence, that is, $\|T_m - T_n\|_1 \to 0$ $(m, n \to \infty)$. We have to prove the existence of $T \in \boldsymbol{L}_1$ such that $\|T - T_n\|_1 \to 0$ $(n \to \infty)$. With no loss of generality, we may assume that (1) : $T_n = T_n^*$ for every n and (2) : $\|T_{n+1} - T_n\| < 1/2^n$ for every n. Put

$$S_n = |T_1 - T_2| + |T_2 - T_3| + \cdots + |T_n - T_{n+1}|.$$

Then $\{S_n\}$ is an increasing sequence and,

$$\|S_n\|_1 = \|T_1 - T_2\|_1 + \|T_2 - T_3\|_1 + \cdots + \|T_n - T_{n+1}\|_1 \leq \sum 1/2^n = 1$$

for every n. Hence by Theorem 13, there is an $S \in \boldsymbol{L}_1$ such that $\|S - S_n\| \to 0$ and $\text{l.u.b.}\, S_n = S$. Put $T_n' = T_n - T_1 + S_{n-1}$ for $n = 2, 3, \cdots$ and $T_1' = 0$. Then $T_{n+1}' - T_n' = T_{n+1} - T_n + |T_n - T_{n+1}| \geq 0$ and $\|T_n'\|_1 \leq \|T_n - T_1\|_1 + \|S_{n-1}\|_1 \leq c$ for some constant c. Again Theorem 13 is applicable to the sequence $\{T_n'\}$, and there exists a $T' \in \boldsymbol{L}_1$ such that $\text{l.u.b.}\, T_n' = T'$ and $\|T' - T_n'\|_1 \to 0$ $(n \to \infty)$. $T = T' + T_1 - S$ is the desired limit. In fact

$$T - T_n = T' + T_1 - S - T_n = (T' - T_n') + (S_{n-1} - S)$$

and $\|T' - T_n'\|_1 \to 0$, $\|S_{n-1} - S\|_1 \to 0$. This completes the proof.

From this proof we have

COROLLARY. *If $T_n \to T$ in $\boldsymbol{L}_1$, then $T_n^\natural \to T^\natural$ in the star sense and $T_n \to T$ n. e. in the star sense.*

PROOF. $T^\natural - T_n^\natural = T'^\natural - T_n'^\natural + S_{n-1}^\natural - S^\natural$ and $T_n'^\natural \to T'^\natural$, $S_{n-1}^\natural - S^\natural$. Hence the first assertion holds. By using $T - T_n = T' - T_n' + S_{n-1} - S$, the second assertion may be similarly proved.

As for $\boldsymbol{L}_2$ we have the next analogue to Theorem 13.

THEOREM 15. *Let $\{T_n\}$ be a monotone increasing sequence of positive operators $\in \boldsymbol{L}_2$. Then there exists a $T \in \boldsymbol{L}_2$ such that* $\text{l.u.b.}\, T_n = T$, *if and only if $\{\|T_n\|_2\}$ is bounded. In this case* $\lim \|T - T_n\|_2 = 0$, $T^\natural = \text{l.u.b.}\, T_n^\natural$, $(T^2)^\natural = \text{l.u.b.}\, (T_n^2)^\natural$, *and $\{T_n\}$ converges n.e. to T in the star sense.*

PROOF. *If* $T=\text{l.u.b.}\, T_n$ exists in $\boldsymbol{L}_2$, then $(T^2)^\natural \geq (T_n^2)^\natural$ (Theorem 10, (7)) implies that $\{\|T_n\|_2\}$ is bounded. Assume the converse. If $m>n$, then

$$((T_m - T_n)^2)^\natural = (T_m^2 - T_m \cdot T_n - T_n \cdot T_m + T_n^2)^\natural \leq (T_m^2)^\natural - (T_n^2)^\natural$$

Hence $\|T_m - T_n\|_2^2 \leq \|T_m\|_2^2 - \|T_n\|_2^2$ for $m>n$. Thus by taking a subsequence, if necessary, we may assume that $\|T_{n+1} - T_n\|_2 < 1/4^n$ ($n=1, 2, 3, \cdots$). As in the proof of Theorem 13, we can construct a $T\eta\mathbf{M}$ such that $\{T_n\}$ converges n. e. to T and $\text{l.u.b.}\, T_n = T$. Hence $\text{l.u.b.}\, T_n^\natural = T^\natural$. We are now to show that $T \in \boldsymbol{L}_2$ and $\lim \|T - T_n\|_2 = 0$. Since $\{T_n^2\}$ is a Cauchy sequence in $\boldsymbol{L}_1$, there is an $S \in \boldsymbol{L}_1$ such that $\|T_n^2 - S\|_1 \to 0$. Hence by the preceding corollary $T_n^2 \to S$ n. e. in the star sense. On the other hand, as $T_n \to T$ n. e. in the star sense, $T_n^2 \to T^2$ n. e. in the star sense [13]. Hence $S=T^2$. But $((T-T_n)^2)^\natural = (T^2)^\natural - 2(T \cdot T_n)^\natural + (T_n^2)^\natural$ and $T_n \leq T$. This shows us that $((T-T_n)^2)^\natural \leq (T^2)^\natural - (T_n^2)^\natural = S^\natural - (T_n^2)^\natural$. Hence $\|T-T_n\|_2 \to 0$. Thus $\|T\|_2 = \text{l.u.b.}\, \|T_n\|_2$ or $\varphi((T^2)^\natural) = \text{l.u.b.}\, \varphi((T_n^2)^\natural)$ which implies $(T^2)^\natural = \text{l.u.b.}\, (T_n^2)^\natural$. This completes the proof.

THEOREM 16. *$\boldsymbol{L}_2$ is a Hilbert space with an inner product $\langle S, T\rangle = m(S \cdot T^*)$.*

PROOF. The proof of the completeness of $\boldsymbol{L}_2$ is the same as that of $\boldsymbol{L}_1$, except that $\|\ \|_1$ is replaced by $\|\ \|_2$, and that Theorem 15 is used in place of Theorem 13. Details are omitted.

To each $A \in \mathbf{M}$ corresponds a mapping $\theta(A)$ of $\boldsymbol{L}_2$ into itself, defined by the relation $\theta(A)T = A \cdot T$ for every $T \in \boldsymbol{L}_2$. It is easy to see that θ is a normal $*$-isomorphism, so that $\theta(\mathbf{M})$ is a ring of operators on $\boldsymbol{L}_2$ [6]. We can also show that $\boldsymbol{L}_2$ is an H-system whose left ring is $\theta(\mathbf{M})$. But this will not be used in the sequel, so the proof is omitted.

THEOREM 17. (Radon-Nikodym's Theorem). *For every $T \in \boldsymbol{L}_1$, $\Phi_T(A) = m(A \cdot T)$ is a linear form on $\mathbf{M}$ continuous in the ultraweak topology on $\mathbf{M}$. Conversely, every such linear form on $\mathbf{M}$ is a Φ_T, $T \in \boldsymbol{L}_1$, and $\|\Phi_T\| = \|T\|_1$. $\mathbf{M}$ is the conjugate space of $\boldsymbol{L}_1$.*

PROOF. First we prove that Φ_T is continuous in the ultraweak topology on $\mathbf{M}$. Since $T \in \boldsymbol{L}_1$ is a linear combination of positive operators $\in \boldsymbol{L}_1$, we may assume that $T \geq 0$. We note that a positive linear form on $\mathbf{M}$ is normal if and only if it is continuous in the ultraweak topology on $\mathbf{M}$ [6]. Hence the problem is reduced to prove that $\Phi_T(A) = m(A \cdot T)$ is normal for $T \geq 0$. But we have shown that $A \to (A \cdot T)^\natural$ is a normal mapping (Theorem 9). Hence the normality of Φ_T follows directly from that of φ. Conversely, let Φ be a linear form continuous in the ultraweak topology. We may assume that Φ is positive. Then Φ is normal. Define $\tilde{\Phi}(\theta(A)) = \Phi(A)$. $\tilde{\Phi}$ is a normal linear form on $\theta(\mathbf{M})$, so that we may write

$$\Phi(A)=\tilde{\Phi}(\theta(A))=\sum_{n=1}^{\infty}\langle A\cdot S_n, S_n\rangle=\sum_{n=1}^{\infty}m(A\cdot S_n^2),$$

where $S_n\in \boldsymbol{L}_2^+$ and $\sum\|S_n\|_2^2<+\infty$ [6]. Let $T_n=\sum_{i=1}^{n}S_i^2$. Then $\|T_n\|_1=\sum_{i=1}^{n}\|S_i\|_2^2$. Theorem 13 shows us that $T=$ l.u.b. T_n exists and $\|T-T_n\|_1\to 0$. Thus $\Phi(A)=\lim m(A\cdot T_n)=m(A\cdot T)$, or $\Phi=\Phi_T$. $\|\Phi_T\|=\underset{A\in\mathbb{M},\,\|A\|\leq 1}{\text{l. u. b.}}|m(A\cdot T)|=\|T\|_1$ is obvious from Theorem 10, (1).

It remains to prove the last statement. For each $A\in\mathbb{M}$, $\Psi_A(T)=m(A\cdot T)$ is a bounded linear form on $\boldsymbol{L}_1$. That $\|\Psi_A\|=\|A\|$ may be proved in the following way. Since $\|A\|=\||A|\|$ and $\underset{T\in\boldsymbol{L}_1,\|T\|_1\leq 1}{\text{l. u. b.}}|m(A\cdot T)|=\underset{T\in\boldsymbol{L}_1,\|T\|_1\leq 1}{\text{l. u. b.}}|m(|A|\cdot T)|$ we may assume that $A\in\mathbb{M}^+$. Clearly $\|\Psi_A\|\leq\|A\|$ by Theorem 10, (1). If $0\leq a<\|A\|$ for some a, then $aE_a^{\perp}\leq AE_a^{\perp}$ where $A=\int_0^{\|A\|}\lambda dE_\lambda$ is the spectral resolution of A. As $E_a^{\perp}\neq 0$, there exists a projection $P\leq E_a^{\perp}$ such that $0<m(P)<+\infty$. Put $T=\frac{1}{m(P)}P$. Then $\|T\|_1=1$ and $aT\leq PA\cdot T$. Hence $a=am(T)\leq m(PA\cdot T)=m(A\cdot T)$. Thus $\|A\|\leq\underset{T\in\boldsymbol{L}_1,\|T\|_1\leq 1}{\text{l. u. b.}}|m(A\cdot T)|=\|\Psi_A\|$. That is $\|A\|=\|\Psi_A\|$. That every bounded linear form on $\boldsymbol{L}_1$ is of the form Ψ_A with $A\in\mathbb{M}$ is obvious from Dixmier's Theorem ([6], Theorem 1), since we have already shown that $\boldsymbol{L}_1$ may be regarded as the set $\mathbb{M}_*$ of all ultraweakly continuous linear forms on $\mathbb{M}$. Thus the theorem is completely proved.

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