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On the Behavior of Paths of the Analytic Two-dimensional Autonomous System in a Neighborhood of an Isolated Critical Point (Continued)

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§ 1. Introduction

Given an analytic two-dimensional autonomous system

$$(1) \qquad \frac{dx}{dt}=X(x, y), \qquad \frac{dy}{dt}=Y(x, y),$$

for which the origin is an isolated critical point.

In the previous paper [1][1], to study the behavior of paths of (1) in a neighborhood of the origin, we considered the differential equation

$$(2) \qquad \frac{dy}{dx}=\frac{Y(x, y)}{X(x, y)}$$

and studied the classification of the integral curves of (2) tending to the origin in the fixed directions, namely we sought for all possible types of such integral curves tending to the origin that they may be represented by the equations of the form $y=y(x)$ or $x=x(y)$.

In this paper, making use of the results obtained there, for the case where there exists at least one integral curve tending to the origin in a fixed direction[2], we study the full configuration of the paths of (1) around the origin.

When $X(x, y)\equiv 0$ in a neighborhood of the origin, all the integral curves in that neighborhood are represented by the equation $x=const.$, so the configuration of the paths near the origin is completely known. Hence, in the sequel, we consider only the case where $X(x, y)\not\equiv 0$ in any small neighborhood of the origin.

At first, a sufficiently small circular neighborhood of the origin is divided into a finite number of sectors by the curve $X(x, y)=0$, and next, each sector is further divided into some subsectors by some kinds of integral curves. Then, after some considerations about the integral curves in each subsector, the full configuration of the paths around the origin can be known.

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1) The numbers in square brackets refer to the references listed at the end of this paper.

2) As is readily seen, whether or not there exists at least one integral curve tending to the origin in a fixed direction can be decided by a finite number of steps mentioned in the previous paper.

§ 2. Sectors

Let U be the region $x^2+y^2<\sigma^2$ $(\sigma>0)$ such that no critical points other than the origin lie in $\bar{U}$ (the closure of U), and let Γ_0 be the boundary of U.

When there exists no branch of $X=0$, we choose σ so small that X may never vanish in U except at the origin.

When there exists a branch of $X=0$, as was shown by Lefschetz [2], for sufficiently small σ, the curve $X=0$ has at most a finite number of branches in U and they are expressed in the form $y=\varphi(x)$ or $x=0$, where $\varphi(x)$ is a convergent fractional power series in x with real coefficients.

In the latter case, we choose σ so small that, in U,

(i) the branches of $X=0$ may not intersect one another except at the origin [2];

(ii) each branch may contain no arc connecting the points of Γ_0. (The validity of this condition is proved at the end of this paragraph.)

Then, by the branches of $X=0$, U is divided into a finite number of sectors (open sets).

In the former case, for convenience, let us regard $U-\{O\}$ as a sector.

Let S be any sector and Γ be its boundary. Then $X\neq 0$ in S, so, in S, by the existence theorem of solutions of differential equations, the arc of any integral curve is represented by the equation of the form $y=y(x)$ and it becomes either an arc connecting two points of $\Gamma-\{O\}$ or an arc connecting the origin with a point of $\Gamma-\{O\}$. In the sequel, for brevity, the arc of an integral curve lying in a sector and connecting the origin with a boundary point of that sector is called the integral arc tending to the origin. When $x=0$ is a branch of $X=0$, the segment $x=0$, $0<y<\sigma$, is also called the integral arc for convenience, because it is an arc of an integral curve though it does not lie in a sector but lies on a boundary of a sector.

When there exists a branch of $X=0$ of the form $y=\varphi(x)$, the open arc $\overparen{OA}$ of that branch cut by the boundary Γ_0 of U becomes an arc without contact, for

$$|\varphi'(x)|<\infty \qquad \text{and} \qquad \frac{dx}{dy}=\frac{X(x,\,\varphi(x))}{Y(x,\,\varphi(x))}=0$$

on $\overparen{OA}$ because there is no critical point on it. Hence, if there exists an integral arc $\overparen{OP}$ which connects the origin with a point P of $\overparen{OA}$ in a sector bounded partially by $\overparen{OA}$, then the domain bounded by the integral arc $\overparen{OP}$ and the arc $\overparen{OP}$ lying on $\overparen{OA}$ is filled with the integral arcs tending to the origin.

Now we show that we can choose σ so small that the condition (ii) may be satisfied. In fact, the branch $x=0$ or $y=0$, if any, satisfies this condition. If the branch $y=\varphi(x)$ $(\neq 0)$ contains an arc $\overparen{AB}$ connecting the points A and B of Γ_0 in U, then, by Rolle's theorem, there must be a point $(x_0,\ \varphi(x_0))$ on $\overparen{AB}$

for which $\varphi(x_0)\varphi'(x_0)+x_0=0$, because $x^2+\varphi(x)^2=\sigma^2$ at A and B. Now ,if

$$\varphi(x)=a_0x^{\mu}+a_1x^{\mu+\nu}+a_2x^{\mu+2\nu}+\cdots\ (a_0\neq 0;\ \mu,\ \nu>0),$$

then

$$\varphi(x)\varphi'(x)+x=\begin{cases} x^{2\mu-1}\{\mu a_0^2+o(1)\} & \text{when} \quad \mu<1,\\ x\{(1+a_0^2)+o(1)\} & \text{when} \quad \mu=1,\\ x\{1+o(1)\} & \text{when} \quad \mu>1.\end{cases}$$

So, if we choose $|x|\ (>0)$ sufficiently small, $\varphi(x)\varphi'(x)+x$ never vanishes. Then, since the number of the branches of $X=0$ is finite, it is seen that we can choose σ so small that all the branches of $X=0$ may satisfy the above-mentioned condition.

§ 3. Results of the previous paper

For the benefit of understanding, in this paragraph, we explain the terminology and the results of the previous paper [1].

Since, in this paper, we are concerned with the case where at least one integral curve tends to the origin in a fixed direction, any integral curve tending to the origin necessarily tends to the origin in a fixed direction,. Then, for instance, it is sufficient to consider only the integral curves tending to the origin on the right side of the y-axis. Because, the integral curves tending to the origin on the left side of the y-axis or tending to the origin in the direction of the y-axis staggering into both sides of the y-axis can be changed to the former ones by replacing x by $-x$ or by interchanging x with y respectively. Thus, in the sequel, we consider only the integral curves which tend to the origin as $x\downarrow 0$.

1°. Order and magnitude of an integral curve tending to the origin

The *order* of an integral curve tending to the origin of the form $y=y(x)$ is defined as follows:

the order of an integral curve $y=y(x)$ is μ in x, when, for any $\varepsilon>0$, there exists a positive number μ such that

$$y(x)/x^{\mu-\varepsilon}\to 0 \text{ and } |y(x)|/x^{\mu+\varepsilon}\to\infty \text{ as } x\to 0;$$

it is infinity in x, when, for any great $\mu>0$, $y(x)/x^{\mu}\to 0$ as $x\to 0$;

it is zero in x, when, for any $\varepsilon>0$, $|y(x)|/x^{\varepsilon}\to\infty$ as $x\to 0$.

The *magnitude* of an integral curve $y=y(x)$ of order μ $(0<\mu<\infty)$ in x is defined as follows:

it is ρ when $y(x)/x^{\mu}\to\rho$ as $x\to 0$;

it is infinity when $y(x)/x^{\mu}\to\infty$ *or* $-\infty$ *as* $x\to 0$.

Of course, when an integral curve is represented by the equation of the form $x=x(y)$, its order and magnitude in y are also defined in the same manner.

2°. Newton polygon

Since X and Y are analytic at the origin, we can expand them into power series in x and y as follows:

$$X(x, y)=\sum_{i,j} a_{ij}x^i y^j \qquad (a_{ij}\neq 0),$$

$$Y(x, y)=\sum_{k,l} b_{kl}x^k y^l \qquad (b_{kl}\neq 0).$$

Then, introducing the orthogonal coordinates (ξ, η) in a plane, we mark on this plane the points $X_{ij}(j+1, i-1)$ and $Y_{kl}(l, k)$ corresponding to the terms $a_{ij}x^i y^j$ and $b_{kl}x^k y^l$ respectively, and from these points we select the points V_p's $(p=1, 2, \cdots, s+1)$ so that

(i) the slope $-\mu_p$ of the segment V_pV_{p+1} may be negative for any $p=1, 2, \cdots, s$;

(ii) $0<\mu_1<\mu_2<\cdots\cdots<\mu_s<\infty$;

(iii) all of the points X_{ij} and Y_{kl} may lie in the closed domain bounded by V_1V_1', $V_1V_2, \cdots\cdots, V_sV_{s+1}$, $V_{s+1}V_{s+1}'$, where V_1V_1' and $V_{s+1}V_{s+1}'$ are the half lines respectively parallel to the positive directions of ξ- and η-axes. (In the sequel, for convenience, the slopes of V_1V_1' and $V_{s+1}V_{s+1}'$ are denoted by $-\mu_0$ and $-\mu_{s+1}$ respectively. Evidently the values of μ_0 and μ_{s+1} are respectively zero and infinity.)

We call the broken line $V_1V_2\cdots V_{s+1}$ the Newton polygon of (2) and the points V_p's $(p=1, 2, \cdots, s+1)$ the vertices of the polygon.

A vertex of the Newton polygon is a certain X_{ij} or a certain Y_{kl}. But it may happen that the vertex is a certain X_{ij} and is at the same time a certain Y_{kl}. In such a case, for any vertex V_q such that it is an $X_{i_qj_q}$ and is at the same time a $Y_{k_ql_q}$, we consider the quantity $\nu_q=b_{k_ql_q}/a_{i_qj_q}$, when $\mu_q>b_{k_ql_q}/a_{i_qj_q}>\mu_{q-1}$.

For any $\mu\ (\neq\infty)$ such that $\mu_p\geqq\mu>\mu_{p-1}$, let L_μ be the straight line passing through V_p with the slope $-\mu$ and let $X_{\alpha_m\beta_m}$ $(m=1, 2, \cdots, f)$ and $Y_{\gamma_n\delta_n}$ $(n=1, 2, \cdots, g)$ be respectively the points X_{ij} and Y_{kl} lying on L_μ, if any. Then, if we put

$$A_\mu(u)=\sum_{m=1}^{f} a_{\alpha_m\beta_m}u^{\beta_m},$$

$$B_\mu(u)=\sum_{n=1}^{g} b_{\gamma_n\delta_n}u^{\delta_n}-\mu(u)uA_\mu,$$

$$d(\mu)=\inf_{X_{ij},Y_{kl}}\{i-1+\mu(j+1),\ k+\mu l\},$$

and substitute $y=x^{\mu}y_{\mu}$ in (2), there is obtained

$$(E_{\mu})\qquad \frac{dy_{\mu}}{dx}=\frac{[Y(x,\ x^{\mu}y_{\mu})-\mu x^{\mu-1}y_{\mu}\,X(x,\ x^{\mu}y_{\mu})]x^{-d(\mu)}}{x^{\mu}X(x,\ x^{\mu}y_{\mu})x^{-d(\mu)}}$$

$$=\frac{B_{\mu}(y_{\mu})+B(x,\ y_{\mu})}{x\{A_{\mu}(y_{\mu})+A(x,\ y_{\mu})\}},$$

where

$$A(0,\ y_{\mu})=B(0,\ y_{\mu})\equiv 0.$$

3°. Possible types of integral curves and the conditions of existence to each type

Theorem 1. *The integral curve of* (2) *which tends to the origin as* $x\downarrow 0$ *must come under one of the following types:*

I : *that of order infinity in* x;
II : *that of order infinity in* y (*namely that of order zero in* x);
III: *that of order* ν_q *in* x *and of a finite non-zero magnitude*;
IV: *that of order* $\mu_p(1\leqq p\leqq s)$ *in* x *and of a definite magnitude.*

Theorem 2. (Theorem on integral curves of type I)

When $Y(x,\ 0)\not\equiv 0$, *there exists no integral curve of type I.*
When $Y(x,\ 0)\equiv 0$, *if* $Y_y(x,\ 0)$ *can be written as*

$$Y_y(x,\ 0)=b_{\beta 1}x^{\beta}+o(x^{\beta})$$

and it holds that

$$\beta<\alpha-1 \quad \textit{and} \quad a_{\alpha 0}b_{\beta 1}>0$$

for α *and* $a_{\alpha 0}$ *such that*

$$X(x,\ 0)=a_{\alpha 0}x^{\alpha}+o(x^{\alpha}),$$

then there exist really infinitely many integral curves of type I.

When $Y(x,\ 0)\equiv 0$, *if any one of the above three conditions is not satisfied, then, except for* $y=0$, *there exists no integral curve of type I.*

About the integral curves of type II, the analogous theorem is valid, x being interchanged with y.

Theorem 3. (Theorem on integral curves of type III)

When ν_q *exists*[1], *there exists always one and only one integral curve of* (2) *of type III* (*i.e. of order* ν_q *in* x) *with any given non-zero finite magnitude, and there exists no integral curve of type III with the magnitude zero or infinity.*

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1) This means that the vertex V_q is a certain $X_{i_q j_q}$ and is at the same time a certain $Y_{k_q l_q}$ and moreover that $\mu_q>b_{k_q l_q}/a_{i_q j_q}>\mu_{q-1}$.

Theorem 4. (Theorem on integral curves of type *IV* with the magnitude infinity) *The integral curve of type IV (of order μ_p in x) with the magnitude infinity can exist only when $\mu_p=b_{k_p l_p}/a_{i_p j_p}$ and $B_{\mu_p}(u)$ does not vanish identically.*

When these conditions are fulfilled, if we write $B_{\mu_p}(u)$ as

$$B_{\mu_p}(u)=c_0u^{\beta}+c_1u^{\beta-1}+\cdots\cdots+c_\beta \quad (c_0\neq 0),$$

then it is valid that,

(i) *when $\sigma=l_p-\beta$ is even, if $c_0a_{i_pj_p}<0$, there exist infinitely many integral curves of type IV with the magnitude infinity on both sides of the x-axis, and, if $c_0a_{i_pj_p}>0$, there exists no such integral curve on each side of the x-axis;*

(ii) *when $\sigma=l_p-\beta$ is odd, there exist infinitely many integral curves of type IV with the magnitude infinity on the upper or lower side of the x-axis according as $c_0a_{i_pj_p}<0$ or >0 and there exist none on the opposite side.*

About the integral curves of type *IV* with the magnitude zero, as is readily seen, the analogous theorem is valid, x being interchanged with y.

Theorem 5. (Theorem on integral curves of type *IV* with the finite non-zero magnitude) *The integral curve of type IV (of order μ_p in x) cannot have any non-zero finite magniude except for a value ρ such that $B_{\mu_p}(\rho)=0$.*

For any non-zero value ρ such that $B_{\mu_p}(\rho)=0$ but $A_{\mu_p}(\rho)\neq 0$,

(i) *when $B_{\mu_p}(u)\equiv 0$, there exists one and only one integral curve of type IV with magnitude ρ*[1]*;*

(ii) *when $B_{\mu_p}(u)\not\equiv 0$,*

(a) *if the multiplicity κ of ρ satisfying $B_{\mu_p}(\rho)=0$ is odd and $A_{\mu_p}(\rho)\, B_{\mu_p}^{(\kappa)}(\rho)<0$, there exists one and only one integral curve of type IV with magnitude ρ;*

(b) *if κ is odd and $A_{\mu_p}(\rho)B_{\mu_p}^{(\kappa)}(\rho)>0$, there exist infinitely many integral curves of type IV with magnitude ρ;*

(c) *if κ is even, there exist infinitely many integral curves of type IV with magnitude ρ; in this case, there exists one and only one integral curve Γ of the form*

$$(3) \qquad y=y(x)=(\rho+v_0(x))x^{\mu_p}$$

$$(\mu_p=q_1/p_1\ (p_1,\ q_1\text{: positive integers}) \quad \text{and} \quad v_0(x)=O(x^{1/p_1}) \text{ as } x\to 0)$$

and,

when $A_{\mu_p}(\rho)B_{\mu_p}^{(\kappa)}(\rho)<0$, the integral curves of type IV with magnitude ρ do not exist on the upper side of Γ and exist infinitely many on the lower side of Γ in the form

$$(4) \qquad y=y(x)=(\rho+v(x))x^{\mu_p} \qquad (v(x)\to 0 \text{ but } |v(x)|/x^{1/p_1}\to\infty \text{ as } x\to 0);$$

1) This says that, when $B_{\mu_p}(u)\equiv 0$, there exist one and only one integral curve of type *IV* with any given non-zero finite magnitude except for the value letting $A_{\mu_p}(u)$ vanish.

when $A_{\mu_p}(\rho)B_{\mu_p}^{(\kappa)}(\rho)>0$, *the distributions of the integral curves are interchanged.*

Lastly, let us study the integral curves of type *IV* (of order μ_p in x) with magnitude ρ ($\neq 0$) such that $B_{\mu_p}(\rho)=A_{\mu_p}(\rho)=0$.

Since μ_p is of the form q_1/p_1, for such value ρ, substituting

$$x=z_1^{p_1},\ y=(\rho+v_1)z_1^{q_1} \tag{5}$$

into (2), we consider the derived equation

$$\frac{dv_1}{dz_1}=\frac{p_1z_1^{p_1-1}\,Y(x,y)-q_1(\rho+v_1)z_1^{q_1-1}\,X(x,y)}{z_1^{q_1}\,X(x,y)} \tag{6}$$

$$=\frac{p_1B_{\mu_p}(\rho+v_1)+z_1g_\rho(z_1,v_1)}{z_1\{A_{\mu_p}(\rho+v_1)+z_1\,f_\rho(z_1,v_1)\}},$$

which is further reduced to

$$\frac{dv_1}{dz_1}=\frac{g_\rho(z_1,v_1)}{A_{\mu_p}(\rho+v_1)+z_1f_\rho(z_1,v_1)}, \tag{7}$$

when $B_{\mu_p}(u)\equiv 0$. Then, since any integral curve of type *IV* (of order μ_p in x) with magnitude ρ can be expressed by the equation of the form (5) with $v_1(z_1)$ such that $v_1(z_1)\to 0$ as $z_1\to 0$, it is sufficient to consider the integral curves $v_1=v_1(z_1)$ of (6) or (7) tending to the origin.

Now, for (7), the origin is not a critical point when $g_\rho(0,0)\neq 0$. In such a case, one and only one integral curve passes through the origin, therefore, we see that, when $B_{\mu_p}(u)\equiv 0$, for any value $\rho(\neq 0)$ such that $A_{\mu_p}(\rho)=0$ and $g_\rho(0,0)\neq 0$, there exists one and only one integral curve of type *IV* with magnitude ρ.

While, for (6) and also for (7) where $g_\rho(0,0)=0$, the origin is a critical point. So, for these equations, constructing again the Newton polygons, we apply the above-mentioned theorems and classify the integral curves.

Then the integral curves of type *IV* with the specified magnitude ρ are all classified except for those which correspond to such an integral curve $v_1=v_1(z_1)$ of (6) or (7′)[1] that it is of type *IV* and has the magnitude ρ_1 ($\neq 0$) for which both polynomials $A_{\mu_q}(u)$ and $B_{\mu_q}(u)$ of (6) or (7′) vanish. For unclassified integral curves, we repeat the above process and continue the process again and again. Then, as was shown in the previous paper, this process ends in a finite number of steps.

Thus, after a finite number of steps, the classification of the integral curves can be done completely.

Theorem 1 combined with Theorem 2 says that there is no integral curve such that it may tend to the origin in the direction of the x-axis staggering into the upper and lower sides of the x-axis. Then, since this is valid also for the y-axis, we see from the remarks at the beginning of this paragraph that

1) For simplicity, in the sequel, let us denote by (7′) the equation (7) such that $g_\rho(0,0)=0$.

any integral curve which tends to the origin is $x=0$ or one which tends to the origin as $x \downarrow 0$ or $x \uparrow 0$.

§ 4. Families of integral curves

Making use of the results of the preceding paragraph, for the equation of the form (2), we pick up the cases where the integral curves of the same category appear infinitely many in the first step, and let us collect the integral curves of each category into a separate family. Then there are turned out the following families:

(1) *the families* I^+ *and* I^-: the collections of all the integral curves of type I lying respectively on the upper side and on the lower side of the x-axis;

(2) *the families* II^+ *and* II^-: the collections of all the integral curves of type II lying respectively on the upper side and on the lower side of the x-axis exclusive of $x=0$;

(3) *the families* $III^+(\nu_q)$ *and* $III^-(\nu_q)$: the collections of all the integral curves of type III (order ν_q) respectively with positive magnitudes and with negative magnitudes;

(4.1) *the families* $IV_\infty^+(\mu_p)$ *and* $IV_\infty^-(\mu_p)$: the collections of all the integral curves of type IV(order μ_p) respectively with magnitude ∞ and with magnitude $-\infty$;

(4.2) *the families* $IV_0^+(\mu_p)$ *and* $IV_0^-(\mu_p)$: the collections of all the integral curves of type IV (order μ_p) with magnitude 0 lying respectively on the upper side and on the lower side of the x-axis;

(4.3) *the families* $IV'_\rho(\mu_p)$ *and* $IV''_\rho(\mu_p)$: the former is a collection of all the integral curves of type IV(order μ_p) with magnitude $\rho(\neq 0)$ which is a root of the odd multiplicity κ of the equation $B_{\mu_p}(u)=0$ such that $A_{\mu_p}(\rho)B_{\mu_p}{}^{(\kappa)}(\rho)>0$, and the latter is a collection of all the integral curves of the form (4) of type IV (order μ_p) with magnitude ρ $(\neq 0)$ which is a root of the even multiplicity of the equation $B_{\mu_p}(u)=0$ such that $A_{\mu_p}(\rho)\neq 0$;

(4.4) *the family* $IV_J(\mu_p)$: this collection can exist only when $B_{\mu_p}(u)\equiv 0$ and, in this case, this expresses the collection of all the integral curves of type IV (order μ_p) with magnitudes ρ's such that $\rho \in J$, where J is an arbitrary interval[1] from a real line cut off by the real roots of $A_{\mu_p}(u)=0$ and zero.

Evidently, for the equation of the form (2), the integral curves which tend to the origin as $x \downarrow 0$ and moreover do not belong to any of the above families in the first step are those of the following types:

$\hat{I}$: the integral curve $y=0$;

$\hat{II}^+$ and $\hat{II}^-$: the integral curves $(x=0, y>0)$ and $(x=0, y<0)$;

1) Here, by an interval, we mean an open interval and also take into account the intervals such that $(-\infty, a)$ and (b, ∞).

$\dot{IV}_\rho(\mu_p)$: the integral curve of type IV (order μ_p) with the magnitude $\rho\,(\neq 0)$ which is a root of the odd multiplicity κ of the equation $B_{\mu_p}(u)=0$ such that $A_{\mu_p}(\rho)B_{\mu_p}{}^{(\kappa)}(\rho)<0$;

$\ddot{IV}_\rho\,(\mu_p)$: the itntegral curve of the form (3) of type IV (order μ_p) with the magnitude $\rho(\neq 0)$ which is a root of the even multiplicity of the equation $B_{\mu_p}(u)=0$ such that $A_{\mu_p}(\rho)\neq 0$;

$\hat{IV}_\rho(\mu_p)$: this type can appear only when $B_{\mu_p}(u)\equiv 0$ and, in that case, the integral curves of this type are those of type IV (order μ_p) with magnitude ρ $(\neq 0)$ such that $A_{\mu_p}(\rho)=0$ and $g_\rho(0,0)=0$ (*cf.* (7)).

In the sequel, let us call the integral curves of the above types *the exceptional integral curves.*

Now, as is shown in the preceding paragraph, after each step of the transformation of the form (5), there is again turned out the equation of the form (2), except for the case where the integral curves of type $\hat{IV}_\rho(\mu_p)$ are turned out. Therefore, after each step of the transformation, there are turned out a finite number of the families $I^{\pm}$, $II^{\pm}$, $III^{\pm}(\nu_q)$, $IV^{\pm}_{\infty}(\mu_p)$, $IV^{\mp}_{0}(\mu_p)$, $IV'_\rho(\mu_p)$, $IV''_\rho(\mu_p)$, $IV_J(\mu_p)$ of the integral curves and a finite number of the exceptional integral curves corresponding to the equation of the form (2) obtained after that step, and the remaining integral curves which tend to the origin as $x\downarrow 0$ become those corresponding to the similar integral curves of the equation transformed once more by the transformation of the form (5). But, as is shown in the preceding paragraph, these steps end, after a finite number of repetitions, in the equation such that all its integral curves which tend to the origin as $x\downarrow 0$ are the exceptional integral curves of those belonging to any one of the families $I^{\pm}$, $II^{\pm},\cdots$, $IV_J(\mu_p)$ of the integral curves.

Thus it is seen that all the integral curves of the equation (2) which tends to the origin as $x\downarrow 0$ consist of a finite number of exceptional integral curves or those belonging to a finite number of the families

$$I^{\pm},\ II^{\pm},\ III^{\pm}(\nu_q),\ IV^{\pm}_{\infty}(\mu_p),\ IV^{\mp}_{0}(\mu_p),\ IV'_\rho(\mu_p),\ IV''_\rho(\mu_p),\ IV_J(\mu_p)$$

corresponding to the equations of the form (2) obtained after a finite number of repetitions of the transformations of the form (5).

§ 5. Properties of families

In this paragraph, we prove some lemmas about the properties of the families of the integral curves introduced in the preceding paragraph.

In the sequel, we consider only the arcs of the integral curves bounded by the origin and the boundary curves of the sectors and, for brevity, we call the arcs of the integral curves belonging to a certain family simply the integral arc belonging to that family and the arcs of the exceptional integral curves simply the exceptional integral arcs.

Lemma 1. *The integral arcs of each family lie all in a sector.*

Proof. When the curve $X=0$ has no branch through the origin or has no branch besides $x=0$, the lemma evidently holds, because all the integral arcs belonging to any family tend to the origin on the right side of the y-axis. Hence we have only to consider the case where the curse $X=0$ has branches different from $x=0$.

Now, by the preceding paragraph, the integral arcs belonging to any family are represented by the equations of the form

(8) $$y=y(x)=\psi(z)+z^q(z), \qquad x=z^p,$$

where

$$\psi(z)=a_m z^m+a_{m+1}z^{m+1}+\cdots\cdots+a_q z^q \qquad (a_m,\ a_q\neq 0,\ m\leqq q)$$

and $v=v(z)$ is an integral arc belonging to one of the families

(9) $$I^{\pm},\ II^{\pm},\ III^{\pm}(\nu_q),\ IV^{\pm}_{\infty}(\mu_p),\ IV^{\pm}_{0}(\mu_p),\ IV'_{\rho}(\mu_p),\ IV''_{\rho}(\mu_p),\ IV_J(\mu_p)$$

of the equation of the form

(10) $$\frac{dv}{dz}=\frac{V(z,\ v)}{Z(z,\ v)},$$

which is of the same form as (2). While, as is mentioned in § **2**, except for $x=0$, any branch of the curve $X(x,\ y)=0$ is represented by the equation of the form

(11) $$y=\varphi(x)=c_r z^r+c_{r+1}z^{r+1}+\cdots\cdots+c_t z^t+\cdots\cdots, \qquad x=z^s.$$

Then, so long as $a_m x^{m/p}+a_{m+1}x^{(m+1)/p}+\cdots+a_q x^{q/p}$ does not coincide identically with $c_r x^{r/s}+c_{r+1}x^{(r+1)/s}+\cdots\cdots+c_t x^{t/s}$, the integral curves represented by (8) lie all, in a sufficiently small neighborhood of the origin, either on the upper or on the lower side of the branch of the curve $X(x,\ y)=0$ represented by (11).

But, as is seen from the process by which the function $\psi(z)$ is turned out, there may certainly be such a branch of the curve $X(x,\ y)=0$ that

(12) $$y=\psi(z)+z^q v_1(z),$$

where $v=v_1(z)$ is a branch of the curve $Z(z,\ v)=0$ connected with (10). In this case, as is well known, $v=v_1(z)$ has a definite non-zero order in z and also a definite finite non-zero magnitude, and they are determined from the Newton polygon constructed in connection with the function $Z(z,\ v)$. So, the order cannot be 0, ∞, or ν_q. And further, when the order is μ_p, the magnitude must be a root of $A_{\mu_p}(u)=0$ as is seen from the relation

$$Z(z,\ z^{\mu}v)=z^{d(\mu)+1-\mu}[A_{\mu}(v)+A(z,\ v)] \qquad (A(0,\ v)\equiv 0)\ (cf.\ (E_{\mu})).$$

Then it follows thst, either all the integral curves of (10) belonging to any one of the families (9) have the greater or the small order in z than the branch $v=v_1(z)$, or otherwise, namely when they have the same order as the branch $v=v_1(z)$, they have all the greater or the smaller magnitude than the branch $v=v_1(z)$. This implies that the integral curves belonging to any one of the families (9) lie all, in a sufficiently small neighborhood of the origin, either on the upper or on the lower side of the branch $v=v_1(z)$. This expresses that *the integral curves of the initial equation* (2) *belonging to any one of the families lie all, in a sufficiently small neighborhood of the origin, either on the upper or on the lower side of the branch* (11) *of the curve* $X(x, y)=0$.

Thus, combined with the first case, we see that the above statement is always valid for any branch of the curve $X(x, y)=0$ exclusive of $x=0$.

Then, since the number of the branches of the curve $X(x, y)=0$ is finite, it is readily seen that, if σ (*cf.* §2) is chosen sufficiently small, the integral arcs belonging to any one of the families mentioned in the preceding paragraph lie all in a sector.

Q.E.D.

Lemma 2. *Let* $y=y_i(x)$ $(0<x<\delta_i)$ $(i=1, 2)$ *be any two integral arcs belonging to the same family and D be the region bounded by these two arcs and the line* $x=\delta=\min(\delta_1, \delta_2)$. *Then D is filled with the arcs of the integral arcs belonging to that family.*

Proof. Let us suppose that $y_1(x)>y_2(x)$ in the interval (O, δ).

By lemma 1, D is contained in a sector—say S, so any integral curve passing through any point of D is expressed as $y=y(x)$ and it tends to the origin as $x\downarrow 0$ and reaches the line $x=\delta$ as x is increased. Then, in the interval (O, δ), it holds that

$$y_1(x)>y(x)>y_2(x). \tag{13}$$

Now, as is mentioned in the proof of the preceding lemma, $y=y_i(x)$ $(i=1, 2)$ are of the form (8), so let us write them as follows:

$$y_i(x)=\psi(z)+z^q v_i(z) \qquad (x=z^p) \quad (i=1, 2).$$

Then, by (13), $y(x)$ can be written as follows:

$$y(x)=\psi(z)+z^q v(z),$$

where $v(z)$ is a function such that

$$v_1(z)>v(z)>v_2(z) \tag{14}$$

in the interval (O, δ).

Then, since the integral curves $v=v_i(z)$ $(i=1, 2)$ both belong to the same one of the families (9) of the equation (10), it is readily seen that $v=v(z)$ also belongs to that family. This implies that the integral curve $y=y(x)$ also belongs to the same family as that to which the integral curves $y=y_i(x)$ $(i=1, 2)$ belong. Since $y=y(x)$ is an integral curve passing through an arbitrary point of D, this proves the lemma.

Let $\mathfrak{F}$ be any family and S be the sector containing all the integral arcs belonging to $\mathfrak{F}$. Let Γ be the boundary of S and, on $\Gamma-\{O\}$, let us consider the set M consisting of such a point that, through that point, there exists an integral arc belonging to $\mathfrak{F}$.

Since we are, at present, concerned with the integral arcs tending to the origin as $x \downarrow 0$, the set M lies on the right side of the y-axis, consequently it lies on an open arc. Then, mapping this open arc homeomorphically onto a bounded open interval of the real line, we can consider the end points p and q of the set M which correspond to the least upper bound and the greatest lower bound of the image of M.

Suppose the point p is not the origin. Then one of the half-paths of (1) passing through the point p must completely lie in $\bar{S}$. In fact, if not so, the point p cannot be a point of M and moreover both half-paths passing through the point p must contain the points outside $\bar{S}$. Then, by the continuity of paths, both half-paths passing through any point sufficienly near the point p contain the points outside $\bar{S}$. This implies that there exists no points of M sufficiently near the point p. While, since the point p is not a point of M, it must be a point of accumulation of M. This is a contradiction. Thus we see that, if the point p is not the origin, there exists an integral arc passing through the point p which tends to the origin in $\bar{S}$. In the sequel, let us call such an integral arc the boundary integral arc of the family $\mathfrak{F}$.

For the point q which is not the origin, the matter is the same, and the integral arc passing through the point q which tends to the origin in $\bar{S}$ is also called the boundary integral arc of $\mathfrak{F}$.

Of these boundary integral arcs, the one above which there is no integral arc belonging to $\mathfrak{F}$ is called the upper boundary integral arc of $\mathfrak{F}$ and the other, namely the one below which there is no integral arc belonging to $\mathfrak{F}$, is called the lower boundary integral arc of $\mathfrak{F}$.

About the boundary integral arcs, there is proved

Lemma 3. *The boundary integral arcs of any family do not belong to that family.*

Proof. From the definition, all the families are made of the families

$$(15) \qquad I^{\pm},\ II^{\pm},\ III^{\pm}(\nu_q),\ IV^{\pm}_{\infty}(\mu_p),\ IV^{\pm}_{0}(\mu_p),\ IV'_{\rho}(\mu_p),\ IV''_{\rho}(\mu_p),\ IV_J(\mu_p)$$

of the equations obtained by a finite number of repetitions of the transformations of the form (5), therefore, top rove the present lemma, it is enough to prove the lemma about the families (15) of the initial equation (2), because the equations obtained after the transformation of the form (5) are always of the same form as the initial equation (2).

First, let us prove the lemma about the family I^+. As has been shown in the previous paper ([1], pp. 187–188), the integral curves belonging to the family I^+ are those yielded by the integral curves of the equation (E_μ) tending to the origin in the first quadrant of the (x, y_μ)-plane. Here μ is an arbitrary positive number greater than μ_s such that it is further greater than ν_{s+1} if this exists. When the family I^+ exists, as is seen from the previous paper ([1], pp. 187–188), any integral curve of (E_μ) passing through any point which lies near the origin and on the right side of the y_μ-axis tends to the origin in the direction of the x-axis. Then it follows that any integral arc of the family I^+ cannot be a boundary integral arc of I^+, because, in the first quadrant of the (x, y_μ)-plane, there exists always an integral curve tending to the origin on each side of any such integral curve. This proves that any one of the boundary integral curves of the family I^+ cannot belong to that family.

About the family I^-, the proof is carried on in quite a similar way.

If x is interchanged with y in the above proof, the proof about the families $II^\pm$ is got.

About the families $III^\pm(\nu_q)$ and $IV_J(\mu_p)$, the validity of the lemma is readily seen from Theorem 3 and Theorem 5 respectively.

About the families $IV_0^\pm(\mu_p)$, let us consider the transformation $y=x^{\mu_p}y_{\mu_p}$. Then, by this transformation, the equation (2) is transformed to the equation (E_{μ_p}) and the integral curves belonging to the families $IV_0^\pm(\mu_p)$ and their boundary integral arcs are transformed to the integral curves belonging to the families $II^\pm$ and their boundary integral arcs respectively. Then, if the initial equation has the family $IV_0^+(\mu_p)$ or $IV_0^-(\mu_p)$, the transformed equation (E_{μ_p}) has the family II^+ or II^- respectively and so the equation (E_{μ_p}) becomes that of the same form as the initial equation. Then, since any boundary integral arc of the families $II^\pm$ cannot belong to the respective families as is proved above, it is seen that the lemma is also valid for the families $IV_0^\pm(\mu_p)$.

About the families $IV_\infty^\pm(\mu_p)$, the lemma is derived by interchanging x with y in the above result.

Lastly, about the families $IV'_\rho(\mu_p)$ and $IV''_\rho(\mu_p)$, the lemma is proved if we transform y to v by the transformation

$$y=(\rho+v)x^{\mu_p}$$

and apply to the transformed equation the analogous reasonings as those in the proof about the family $I^\pm$ (*cf.* [1], p. 191).

Thus the proof is completed.

From this lemma, we have

Lemma 4. *The boundary integral arcs of any family do not belong to any family, or, in other words, any boundary integral arc of any family is an* *exceptional integral arc.*

Proof. First we show that, on each side of any integral arc C belonging to any family $\mathfrak{F}$, there exists always an integral arc belonging to that family $\mathfrak{F}$. In fact, if not so, then all the integral arcs belonging to $\mathfrak{F}$ must lie on the same side of C, consequently, by the definition of the boundary integral arc, C must be a boundary integral arc of $\mathfrak{F}$. This contradicts lemma 3, because C belongs to $\mathfrak{F}$.

Suppose that the boundary integral arc C of any family $\mathfrak{F}$ belongs to a certain family $\mathfrak{F}'$. Then, by the above notice, on each side of C, there exist respectively the integral arc C_1 and C_2 belonging to $\mathfrak{F}'$, consequently, by lemma 2, in a sufficiently small neighborhood of the origin, the sector bounded by C_1 and C_2 is filled with the integral curves belonging to $\mathfrak{F}'$. Now, as is seen from the definition, all the families are mutually disjoint. Consequently, provided $\mathfrak{F}'$ does not coincide with $\mathfrak{F}$, in a sufficiently small neighborhood of the origin, there exists no integral curve belonging to the family $\mathfrak{F}$ in the sector bounded by C_1 and C_2. Then, since C does not belong to the family $\mathfrak{F}$ by lemma 3, by the definition of the boundary integral arc, C cannot be a boundary integral arc of $\mathfrak{F}$. This contradicts the assumption upon C. While, by lemma 3, it is evident that $\mathfrak{F}'$ cannot coincide with $\mathfrak{F}$.

Thus we see that the boundary integral arc of any family does not belong to any family. This proves the lemma.

§ 6. Conclusions—configuration of the paths around the origin

Since the number of the exceptional integral arcs is finite, we can choose σ of §2 so small that all the exceptional integral arcs may be cut off by Γ_0. Then every sector is further divided into a finite number of subsectors (open sets) by the exceptional integral arcs.

If such a subsector S contains an integral arc tending to the origin, then that integral arc must belong to a certain family—say $\mathfrak{F}$—because S cannot contain any exceptional integral arc. Then, by lemma 1, all the integral arcs belonging to $\mathfrak{F}$ tend to the origin inside this subsector S and moreover the boundary integral arcs of $\mathfrak{F}$, if any, become the integral arcs bounding the subsector S. Thus we see that the distribution of the paths in a subsector can be classified into six types as is shown in *Fig.* 1.

In *Fig.* 1, $\widehat{OA}$ and $\widehat{OA'}$ show the branches of $X=0$ exclusive of $x=0$ and, $\widehat{OB}$ and $\widehat{OB'}$ show the exceptional integral arcs.

From this result, in order to obtain the full configuration of the paths of (1) around the origin, it needs only to connect the adjacent subsectors successively.

The results obtained in this paper are summarized and, as the conclusion of this paper, they are stated in the form of the theorem as follows:

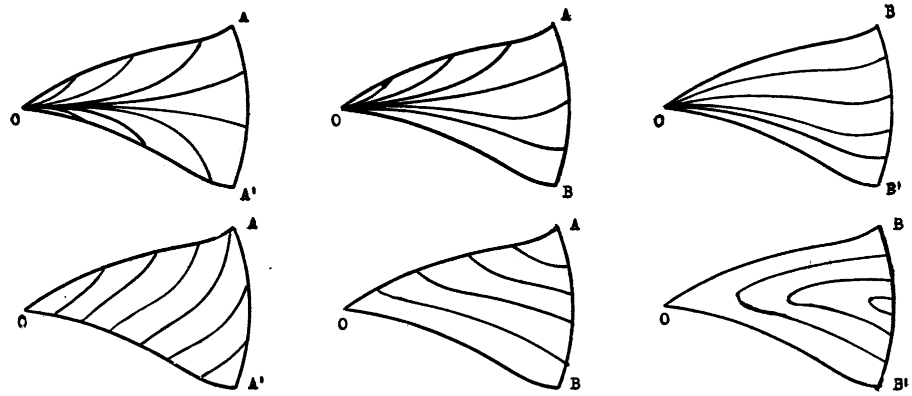

Fig. 1.

Theorem 6. *When there exists an integral curve tending to the origin in a fixed direction, a sufficiently small circular neighborhood of the origin can be divided into a finite number of sectors by the branches of the curve* $X(x, y)=0$ *and the exceptional integral arcs, and, in each sector, the paths behave as is shown in Fig.* 1. *Then full configuration of the paths around the origin can be got by connecting these sectors successively.*

This theorem gives an answer to the problem in the ambiguous case[1].

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