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On Relatively Semi-orthocomplemented Lattices

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A complete lattice L is called a Z-lattice if its center Z is a complete (Boolean) sublattice of L and $\bigvee(a_\alpha;\alpha\in I)\frown b=\bigvee(a_\alpha\frown b;\alpha\in I)$ holds when $a_\alpha\in Z$ for every $\alpha\in I$ or $b\in Z$ (F. Maeda [4], Definition 1.1). If a complete lattice L has a binary relation "$\perp$" which satisfies the six axioms $(1,\alpha)$—$(1,\zeta)$ introduced in my previous paper [5], then it is a Z-lattice (see Theorem 1.3 and Lemma 1.3 of [5]). In this paper, it will be proved that the axiom $(1,\varepsilon)$ is unnecessary for the proof that L is a Z-lattice. A lattice with 0, 1 (not necessarily complete) which has a binary relation "$\perp$" satisfying these five axioms except $(1,\varepsilon)$ will be called to be relatively semi-orthocomplemented. Then, the above statement means that a relatively semi-orthocomplemented complete lattice is a Z-lattice. This is the main theorem of this paper.

We shall show that relatively orthocomplemented lattices and complemented modular lattices are relatively semi-orthocomplemented lattices with some special properties. Moreover, we shall show that, in a ring A with unity, if the set $R_I(A)$ of all principal right ideals generated by idempotents of A forms a lattice by set-inclusion, then it is a relatively semi-orthocomplemented lattice; especially that if A is a Baer ring, then $R_I(A)$, equal to the set of all right-annihilators, is a relatively semi-orthocomplemented complete lattice.

Our main theorem includes the following theorems as special cases: Theorem 2 of Loomis [3], on a relatively orthocomplemented complete lattice (see [4], Remark 4.3); Theorem 5 of Kaplansky [1], on a complemented modular complete lattice; Theorem 5.3 of F. Maeda [4], on a lattice of the annihilators of a Baer ring.

1. Definitions and examples. We assume that, in a lattice L with 0, there is a binary relation "$\perp$" which satisfies the following axioms:

($\perp$1) $a\perp a$ *implies* $a=0$;
($\perp$2) $a\perp b$ *implies* $b\perp a$;
($\perp$3) $a\perp b,\ a_1\leqq a$ *imply* $a_1\perp b$;
($\perp$4) $a\perp b,\ a\smile b\perp c$ *imply* $a\perp b\smile c$.

These axioms coincide with $(1,\alpha)$, $(1,\beta)$, $(1,\gamma)$ and $(1,\delta)$ in [5, §1] respectively. It is obvious by ($\perp$1), ($\perp$2) and ($\perp$3) that $a\perp b$ implies $a\frown b=0$. Two elements $a,b\in L$ are called to be *semi-orthogonal* when $a\perp b$. L is called to be *semi-orthocomplemented* if it has 1 and every element $a\in L$ has a complement $a^\perp$ such that

$a \perp a^{\perp}$ ($a^{\perp}$ is called a semi-orthocomplement of a). A semi-orthocomplemented lattice L is called to be *relatively semi-orthocomplemented* if for every $a, b \in L$ with $b \leqq a$ there exists $c \in L$ such that $b \cup c = a$ and $b \perp c$ (c is called a relative semi-orthocomplement of b in a). The last condition coincides with $(1, \zeta)$ in [5, § 1].

Examples. (i) In an orthocomplented lattice L, let $a \perp b$ be difined by $a \leqq b^{\perp}$, where $b^{\perp}$ is the orthocomplement of b. Then since ($\perp$1)—($\perp$4) hold clearly, L is semi-orthocomplemented. Any relatively orthocomplemented lattice ([4], Definition 4.1) is relatively semi-orthocomplemented.

(ii) In a lattice with 0, let $a \perp b$ be defined by $a \cap b = 0$. Then ($\perp$1), ($\perp$2) and ($\perp$3) hold clearly, and ($\perp$4) holds if the lattice is modular. Hence, in a modular lattice with 0, semi-orthogonality can be defined by $a \cap b = 0$. Since any complemented modular lattice is relatively complemented, it is not only semi-orthocomplemented but relatively semi-orthocomplemented.

(iii) If every L_{α} is a lattice with semi-orthogonality, then so is the product $L = \prod(L_{\alpha};\ \alpha \in I)$, for, $(a_{\alpha})_{\alpha \in I} \perp (b_{\alpha})_{\alpha \in I}$ can be defined by $a_{\alpha} \perp b_{\alpha}$ in L_{α} for every $\alpha \in I$. If every L_{α} is semi-orthocomplemented (resp. relatively semi-orthocomplemented), then so is L.

2. Properties. Let L be a relatively semi-orthocomplemented lattice. $a \cup b$ will be denoted by $a \dot{\cup} b$ when $a \perp b$. $(a, b)M$ means that $(c \cup a) \cap b = c \cup (a \cap b)$ when $c \leqq b$.

LEMMA 1. (i) *$(a, b)M$ holds if $a \perp b$.*

(ii) *If c is a relative semi-orthocomplement of b in a ($b \leqq a$), there exists a semi-orthocomplement $b^{\perp}$ of b such that $c = b^{\perp} \cap a$.*

(iii) *L is relatively complemented.*

PROOF. (i) Let $c \leqq b$. Since $(c \cup a) \cap b \geqq c \cup (a \cap b) = c$, there is $d \in L$ with $c \dot{\cup} d = (c \cup a) \cap b$. Since $c \cup d \leqq b \perp a$, it follows from ($\perp$3) and ($\perp$4) that $d \perp c \cup a \geqq d$, which implies $d = 0$ by ($\perp$1). Therefore $(a, b)\,M$ holds.

(ii) Let $a^{\perp}$ be a semi-orthocomplement of a and $b^{\perp} = c \cup a^{\perp}$. Since $b \dot{\cup} c = a$, we have $b \perp b^{\perp}$ by ($\perp$4) and have $b \cup b^{\perp} = a \cup a^{\perp} = 1$. It follows from (i) that $b^{\perp} \cap a = (c \cup a^{\perp}) \cap a = c$.

(iii) Let $a \leqq c \leqq b$. There is $d \in L$ with $c \dot{\cup} d = b$. Then $(a \cup d) \cap c = a$ by (i) and $(a \cup d) \cup c = b$. Hence $a \cup d$ is a relative complement of c in the interval $[a, b]$.

THEOREM 1. *The following statements are equivalent.*

(α) *L is a relatively semi-orthocomplemented lattice where every element has a unique semi-orthocomplement.*

(β) *L is a relatively semi-orthocomplemented lattice where the semi-*

orthogonality satisfies the following axiom (stronger than ($\perp 4$)):

$$a \perp b,\ a \perp c \ \textit{ imply } \ a \perp b \cup c.$$

(γ) *L is a relatively orthocomplemented lattice.*

PROOF. (γ)$\Rightarrow$(β) is obvious (§1, Example (i)).

(β)$\Rightarrow$(α). Let b and c be semi-orthocomplements of a, and $b \dot{\cup} d = b \cup c$. Since $a \perp b \cup c$ by (β), we have $d \perp a \cup b = 1$, which implies $d=0$. Hence $b = b \cup c$ and similarly we have $c = b \cup c$.

(α)$\Rightarrow$(γ). Let $a^{\perp}$ be the unique semi-orthocomplement of a. It suffices to show that $a \to a^{\perp}$ is a dual automorphism of L with $a^{\perp\perp} = a$, $a \cap a^{\perp} = 0$ and $(a, a^{\perp})M$ ([4], Theorem 4.1). Since $a^{\perp\perp}$ and a are semi-orthocomplements of $a^{\perp}$, we have $a = a^{\perp\perp}$. If $a \leqq b$, then, putting $a \dot{\cup} c = b$, we have $a \perp b^{\perp} \cup c$. Then, since $b^{\perp} \cup c$ is a semi-orthocomplement of a, we have $a^{\perp} = b^{\perp} \cup c \geqq b^{\perp}$. Therefore $a \to a^{\perp}$ is a dual automorphism. $a \cap a^{\perp} = 0$ holds clearly, and $(a, a^{\perp})M$ holds by Lemma 1 (i).

THEOREM 2. *The following statements are equivalent.*

(α) *L is a relatively semi-orthocomplemented lattice where every complement of $a \in L$ is a semi-orthocomplement of a.*

(β) *L is a relatively semi-orthocomplemented lattice where $a \cap b = 0$ ($a, b \in L$) implies that a and b are semi-orthogonal.*

(γ) *L is a complemented modular lattice.*

PROOF. (γ)$\Rightarrow$(α) is obvious (§1, Example (ii)).

(α)$\Rightarrow$(β). Let $a \cap b = 0$, and c be a semi-orthocomplement of $a \cup b$. Since $(c, a \cup b)M$ by Lemma 1 (i), we have $(a \cup c) \cap b = (a \cup c) \cap (a \cup b) \cap b = a \cap b = 0$. Hence $a \cup c$ is a complement of b, and it follows from (α) that $b \perp a \cup c \geqq a$.

(β)$\Rightarrow$(γ). It suffices to show that $(c \cup a) \cap b = c \cup (a \cap b)$ when $c \leqq b$. Let $(c \cup a) \cap b = \{c \cup (a \cap b)\} \dot{\cup} d$ and $c \cup (a \cap b) = (a \cap b) \dot{\cup} c_1$. Since $d \perp c_1 \dot{\cup} (a \cap b)$ we have $d \dot{\cup} c_1 \perp a \cap b$, and since $d \cup c_1 \leqq b$ we have $(d \cup c_1) \cap a = (d \cup c_1) \cap a \cap b = 0$. Hence we have $d \dot{\cup} c_1 \perp a$ by (β), and then $d \perp a \cup c_1$. But, since $a \cup c_1 = a \cup (a \cap b) \cup c_1 = a \cup c \cup (a \cap b) = a \cup c \geqq d$, we have $d = 0$. This completes the proof.

REMARK. Let L be a lattice with semi-orthogonality. A finite subset F of L is called a *semi-orthogonal system* if $(a;\, a \in F_1) \perp (a;\, a \in F_2)$ holds for every pair of disjoint subsets F_1, F_2 of F. It is easy to prove the following properties.

(i) If F_i is a semi-orthogonal system for every $1 \leqq i \leqq n$ and $\{\bigcup(a;\, a \in F_i);\ 1 \leqq i \leqq n\}$ is also a semi-orthogonal system, then so is the union $\bigcup(F_i; 1 \leqq i \leqq n)$.

(ii) If $a_1 \cup \cdots \cup a_i \perp a_{i+1}$ for every $1 \leqq i \leqq n-1$, then $\{a_1, \cdots, a_n\}$ is a semi-orthogonal system.

(iii) If L is relatively semi-orthocomplemented and F is a semi-orthogonal system in L, then $\{\bigvee(a; a \in S); S \subset F\}$ form a sublattice of L isomorphic to the Boolean lattice of all subsets of F.

LEMMA 2. *Let L be a relatively semi-orthocomplemented lattice and Z be its center. An element of L is in Z if and only if it has a unique complement.*

PROOF. The "only if" part is trivial. To prove the converse, assuming that z has a unique complement z', it suffices to show that the correspondence $x \rightarrow [z \cap x, z' \cap x]$ is an isomorphism between L and the product of the sublattices $L(0, z) = \{x \in L: x \leq z\}$ and $L(0, z')$. By the assumption, z' is necessarily a semi-orthocomplement of z. Then, it folows from Lemma 1 (i) that if $a \leq z$, $b \leq z'$ then $(a \cup b) \cap z = a$ and $(a \cup b) \cap z' = b$. Hence this correspondence is onto. To show that it is one-to-one, it suffices to prove $x = (z \cap x) \cup (z' \cap x)$ for every $x \in L$. We can show that $z \cap a = 0$ $(a \in L)$ implies $a \leq z'$: Putting $(z \cup a) \dot{\cup} b = 1$, since $(a \cup b) \cap (z \cup a) = a$ by Lemma 1 (i), we have $z \cap (a \cup b) = z \cap a = 0$, and hence $a \cup b$ is a complement of z, which implies $z' = a \cup b \geq a$. Now, putting $x = (z \cap x) \dot{\cup} a$, since $z \cap a = z \cap x \cap a = 0$, we have $a \leq z'$, and hence $x = (z \cap x) \cup a \leq (z \cap x) \cup (z' \cap x) \leq x$. Since the correspondence is clearly order-preserving, it is an isomorphism.

LEMMA 3. *In a semi-orthocomplemented complete lattice L, let $a_\delta \uparrow a$ and $a_\delta \perp b$ for every δ. If $a_\delta \in Z$ for every δ or $b \in Z$, then $a \perp b$.*

PROOF. Let $b^\perp$ be a semi-orthocomplement of b. Since a_δ or $b \in Z$, we have $a_\delta = (a_\delta \cap b) \cup (a_\delta \cap b^\perp) = a_\delta \cap b^\perp \leq b^\perp$. Hence $a \leq b^\perp$, which implies $a \perp b$.

THEOREM 3. *Let L be a relatively semi-orthocomplemented complete lattice.*

(i) *The center Z of L is a complete Boolean sublattice of L.*

(ii) *Let $a_\delta \uparrow a$. If $a_\delta \in Z$ for every δ or $b \in Z$, then $a_\delta \cap b \uparrow a \cap b$.*

These two properties mean that L is a Z-lattice in the sense of F. Maeda [4].

PROOF. (i) If $z \in Z$, then since z has a unique complement (which is a semi-orthocomplement), we denote it by $1 - z$, which is obviously in Z. Let $z_\delta \uparrow a$, $z_\delta \in Z$ for every δ and a' be a complement of a. Since $a' \cap z_\delta = 0$ we have $a' \leq 1 - z_\delta$ for every δ, and we put $a' \dot{\cup} b = \bigwedge_\delta (1 - z_\delta)$. Since $\bigwedge_\delta (1 - z_\delta) \leq 1 - z_{\delta'}$, we have $z_{\delta'} \perp \bigwedge_\delta (1 - z_\delta)$ for every δ', and hence $a \perp \bigwedge_\delta (1 - z_\delta) = a' \dot{\cup} b$ by Lemma 3. Hence we have $b \perp a \cup a' = 1$, and then $b = 0$. Therefore we have $a' = \bigwedge_\delta (1 - z_\delta)$, which means that a has a unique complement, and it follows from Lemma 2 that $a \in Z$.

If $z_\delta \downarrow a$ and $z_\delta \in Z$, then $\{1 - z_\delta\}$ is an ascending set and it follows from the above result that $\bigvee_\delta (1 - z_\delta)$ has a unique complement $\bigwedge_\delta z_\delta$ and is in Z. Hence $a = \bigwedge_\delta z_\delta$ is also in Z. Therefore Z is a complete Boolean sublattice of L.

(ii) Let $b^\perp$ be a semi-orthocomplement of b. Then it follows from the assumption that $a_\delta=(a_\delta\cap b)\cup(a_\delta\cap b^\perp)\leqq\bigvee_\delta(a_\delta\cap b)\cup(a\cap b^\perp)\leqq a$. Hence $a=\bigvee_\delta(a_\delta\cap b)\cup(a\cap b^\perp)$. Since $a\cap b^\perp\perp b$ it follows from Lemma 1 (i) that $a\cap b=\bigvee_\delta(a_\delta\cap b)$.

3. Principal ideals generated by idempotents of a ring. In a ring A with unity, the set of all idempotents of A is denoted by $I(A)$, the principal right (resp. left) ideal generated by $e\in I(A)$ is denoted by $(e)_r$ (resp. $(e)_l$) and $R_I(A)=\{(e)_r; e\in I(A)\}$, $L_I(A)=\{(e)_l; e\in I(A)\}$. Each $R_I(A)$ and $L_I(A)$ is a partially ordered set with 0,1 by set-inclusion and there exists a dual-isomorphism between them by $(e)_r\leftrightarrow(1-e)_l$. Because $(e)_r\leqq(f)_r\Leftrightarrow fe=e\Leftrightarrow(1-f)(1-e)=(1-f)\Leftrightarrow(1-e)_l\geqq(1-f)_l$.

LEMMA 4. (i) *If $(e)_r\leqq(f)_r$ in $R_I(A)$, then there exists $e_0\in I(A)$ such that $(e_0)_r=(e)_r$, $e_0=e_0f=fe_0$ and exists $f_0\in I(A)$ such that $(f_0)_r=(f)_r$, $e=ef_0=f_0e$.*

(ii) *If $ef=fe$, $e,f\in I(A)$, then $(e)_r\cap(f)_r$ and $(e)_r\cup(f)_r$ exist and are equal to $(ef)_r$ and $(e+f-ef)_r$ respectively.*

Similar properties on $L_I(A)$ also hold.

PROOF. (i) Since $e=fe$, it is easy to prove that $e_0=ef$ and $f_0=e+f-ef$ have the desired properties.

(ii) It is easy to prove that ef and $e+f-ef$ are idempotents and that $(ef)_r\leqq(e)_r$ (or $(f)_r$) $\leqq(e+f-ef)_r$. If $g\in I(A)$ *and* $(g)_r\leqq(e)_r,(f)_r$ then $g=eg=fg$ and hence $(g)_r\leqq(ef)_r$, and if $(g)_r\geqq(e)_r,(f)_r$ then $e=ge$, $f=gf$ and hence $(g)_r\geqq(e+f-ef)_r$. This completes the proof.

LEMMA 5. *If for every $e,f\in I(A)$ the right annihilator of $\{e,f\}$ is of the form $(h)_l$, $h\in I(A)$,* then *$R_I(A)$ and $L_I(A)$ are lattices, where $(e)_r\cap(f)_r$ (resp. $(e)_l\cap(f)_l$) is the intersection of $(e)_r$ and $(f)_r$ (resp. $(e)_l$ and $(f)_l$).*

PROOF. Since the right annihilator of $\{e,f\}$ is equal to the intersection of $(1-e)_r$ and $(1-f)_r$, it follows from the assumption that $(g)_r=(1-e)_r\cap(1-f)_r$ in $R_I(A)$. Hence $(1-g)_l=(e)_l\cup(f)_l$ in $L_I(A)$. Similarly we have $(h)_l=(1-e)_l\cap(1-f)_l$ and $(1-h)_r=(e)_r\cup(f)_r$. This completes the proof.

Exemples. (i) A ring A with unity is called a *Baer ring* if the right annihilator of every subset of A is of the from $(e)_r$, $e\in I(A)$ (Kaplansky [2], Chap. I, Definition 1). Then the similar property of left annihilators also holds ([2], Chap. 1, Theorem 1). $R_I(A)$ (resp. $L_I(A)$) is equal to the set of the right (resp. left) annihilators and is a lattice by Lemma 5. Moreover, it is a complete lattice, because if the right annihilator of $\{e_\alpha\}$ ($e_\alpha\in I(A)$ for every α) is of the form $(g)_r$, $g\in I(A)$, then we have $(g)_r=\bigwedge_\alpha(1-e_\alpha)_r$ and $(1-g)_l=\bigvee_\alpha(e_\alpha)_l$.

(ii) A ring A with unity is called to be *regular* if for every $a\in A$ there exists $x\in A$ such that $a=axa$. Now, we assume that, in a ring A with unity, for every $e,f\in I(A)$ there exists $x\in A$ such that $ef=efxef$. Then, putting $g=$

$f-fxef$, it is easy to show that $g \in I(A)$ and that the right annihilator of $\{e, 1-f\}$ is equal to $(g)_r$. Similarly, putting $h=e-efxe$, we have $h \in I(A)$ and the left annihilator of $\{1-e, f\}$ is equal to $(h)_l$. Hence $R_I(A)$ and $L_I(A)$ are lattices by Lemma 5. Moreover, since $(1-g)_l=(e)_l \cup (1-f)_l$ and $1-g=1-f-fxe(1-f)+fxe$ belongs to the left ideal generated by e and $1-f$, $(e)_l \cup (1-f)_l$ is the left ideal generated by $(e)_l$ and $(1-f)_l$. Hence $L_I(A)$ is a sublattice of the modular lattice formed by all left ideals of A, whence $L_I(A)$ is also modular. Since $(1-e)_l$ is a complement of $(e)_l$, it is a complemented modular lattice. Similar properties of $R_I(A)$ also hold.

Now, we shall prove that if $R_I(A)$ is a lattice then it is relatively semi-orthocomplemented. To this end, we define a binary relation "$\perp$" in $R_I(A)$ as follows: $(e)_r \perp (f)_r$ if there are $e_0, f_0 \in I(A)$ with $(e_0)_r=(e)_r$, $(f_0)_r=(f)_r$ and $e_0f_0=f_0e_0=0$. We note that $ef=0$ $(e, f \in I(A))$ implies $(e)_r \perp (f)_r$; because, putting $f_0=f(1-e)$, it is easy to prove that $ef=0$ implies $f_0 \in I(A)$, $(f_0)_r=(f)_r$ and $ef_0=f_0e=0$.

THEOREM 4. *If the set $R_I(A)$ of the principal right ideals generated by idempotents of a ring A with unity forms a lattice by set-inclusion, then it is a relatively semi-orthocomplemented lattice.*

PROOF. Firstly, we shall show that the relation "$\perp$" defined as above satisfies the four axioms of semi-orthogonality. ($\perp 2$) is clearly satisfied. If $(e)_r \perp (e)_r$, then there are $e_1, e_2 \in I(A)$ with $(e_1)_r=(e_2)_r=(e)_r$, $e_1e_2=e_2e_1=0$. Hence $e_1=e_2e_1=0$, $(e_1)_r=0$, which means ($\perp 1$) is satisfied. If $(e)_r \leqq (f)_r$ and $(f)_r \perp (g)_r$ $(e, f, g \in I(A))$ then we may assume that $fg=gf=0$. It follows from Lemma 4 (i) that there is $e_0 \in I(A)$ with $(e_0)_r=(e)_r$, $e_0=e_0f=fe_0$. Then we have $e_0g=ge_0=0$, and hence ($\perp 3$) is satisfied. Let $(e)_r \perp (f)_r$ and $(e)_r \cup (f)_r \perp (g)_r$. We may assume that $ef=fe=0$ and that there is $h \in I(A)$ with $(h)_r=(e)_r \cup (f)_r$, $hg=gh=0$. Since $he=e$ we have $ge=ghe=0$ and similarly have $gf=0$. putting $f_0=f(1-g)$, we have $f_0 \in I(A)$, $(f_0)_r=(f)_r$ and $f_0g=gf_0=0$. It follows from Lemma 4 (ii) that $(f)_r \cup (g)_r=(f_0+g)_r$. But, since $(f_0+g)e=(f-fg+g)e=0$, we have $(f_0+g)_r \perp (e)_r$. Hence ($\perp 4$) is satisfied.

Next, we shall show that $R_I(A)$ is relatively semi-orthocomplemented. If $(e)_r \leqq (f)_r$, then we may assume that $e=ef=fe$ by Lemma 4 (i). Then we have $f-e \in I(A)$, $(e)_r \perp (f-e)_r$ and it follows from Lemma 4 (ii) that $(e)_r \cup (f-e)_r=(f)_r$. This completes the proof of the theorem.

$L_I(A)$ also has the same property.

COROLLARY. *The right (resp. left) annihilators of a Baer ring form a semi-orthocomplemented complete lattice and hence form a Z-lattice.*

REMARK. In the case of Example (ii), $R_I(A)$ is a complemented modular lattice, and we shall show that $(e)_r \cap (f)_r=0$ implies $(e)_r \perp (f)_r$. Let $(e)_r \cup (f)_r=$

$(g)_r$, $g \in I(A)$. Since $(g)_r$ is a right ideal generated by $(e)_r$ and $(f)_r$, there exist $e_0 \in (e)_r$ and $f_0 \in (f)_r$ with $g=e_0+f_0$. If $x \in (e)_r \leqq (g)_r$ then $x=gx=e_0x+f_0x$. Since $x-e_0x=f_0x \in (e)_r \cap (f)_r=0$, we have $e_0x=x$, $f_0x=0$. Especially we have $e_0^2=e_0$, $f_0e_0=0$ and $e_0e=e$. Similarly $f_0^2=f_0$, $e_0f_0=0$ and $f_0f=f$. Hence we have e_0, $f_0 \in I(A)$, $(e_0)_r=(e)_r$, $(f_0)_r=(f)_r$ and $e_0f_0=f_0e_0=0$, which means that $(e)_r \perp (f)_r$.

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