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A Generalization of the Stroboscopic Method

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1. Introduction

In this note, we consider a real system of n nonlinear differential equations of the form as follows:

$$\frac{dx_i}{dt} = \varepsilon f_i(x, t, \varepsilon) \qquad (i=1, 2\cdots, n), \tag{1.1}$$

where $f_i(x, t, \varepsilon)$ $(i=1, 2,\cdots, n)$ are such that

1° *they are continuous with respect to* (x, t, ε) *in the domain*

$$D: \ |x| = \sum_{i=1}^{n} |x_i| < L, \ -\infty < t < +\infty, \ |\varepsilon| < \delta;$$

2° *they are periodic in* t *with the period* $T(>0)$;

3° *they are once continuously differentiable with respect to* x_i $(i=1, 2,\cdots, n)$ *and* ε.

For such a system, N. Minorsky proposed an interesting method [2]—— the so-called *stroboscopic method* —— to seek for periodic solutions and to decide their stability. After his proposal, his heuristic method was proved to be mathematically legal too by some writers [3, 4]. The stroboscopic method guaranteed mathematically advocates that, for sufficiently small $|\varepsilon|$,

1° *to each simple critical point of the system* —— *the so-called stroboscopic image of* (1.1) ——

$$\frac{dx_i}{dt} = \varepsilon F_i(x) \qquad (i=1, 2,\cdots, n), \tag{1: 2}$$

where

$$F_i(x) = \frac{1}{T}\int_0^T f_i(x, t, 0)dt \qquad (i=1, 2,\cdots, n), \tag{1. 3}$$

there corresponds one and only one periodic solution with period T *of the initial system* (1.1);

2° *the stability of the corresponding periodic solution of the initial system is the same as that of the corresponding critical point.*

Recently, R. Faure [1] considered the case where some of $F_i(x)$ $(i=1, 2, \cdots, n)$ vanish identically —— in this case, evidently there does not exist any simple critical point —— and he has found to seek for a periodic solution of the initial system making use of Haag's successive approximations. But his method seems to be insufficient owing to some assumptions which he laid for simplification.

In this note, we are concerned with the same case as that of R. Faure and we shall show that the problem can be solved more simply without any excessive assumption laid by R. Faure if we make use of the method which M. Urabe [4] used to prove mathematical legality for a stroboscopic method.

On the initial system (1.1), as R. Faure has done, we assume that

1° $f_i(x, t, \varepsilon)$ $(i=1, 2,\dots, n)$ *are continuous with respect to* (x, t, ε) *in the domain* D;

2° $f_i(x, t, \varepsilon)$ $(i=1, 2,\dots, n)$ *are periodic in* t *with the period* $T(>0)$;

3° $f_i(x, t, \varepsilon)$ $(i=1, 2,\dots, n)$ *are twice continuously differentiable with respect to* x_i $(i=1, 2,\dots, n)$ *and* ε;

4° $\displaystyle\int_0^T f_\alpha(x, t, 0)dt \equiv 0 \qquad (\alpha=1, 2,\dots, p,\ p\leqq n).$

2. Existence of a periodic solution

Let $x_i=\varphi_i(u, t, \varepsilon)$ $(i=1, 2,\dots, n)$ be the solution of (1.1) such that

$$\varphi_i(u, 0, \varepsilon)=u_i \qquad (i=1, 2,\dots, n). \tag{2.1}$$

Then, by the assumptions on $f_i(x, t, \varepsilon)$, in any finite interval of t, $\varphi_i(u, t, \varepsilon)$ can be expressed as follows:

$$\varphi_i(u, t, \varepsilon)=\varphi_i^{(0)}(u, t)+\varepsilon\varphi_i^{(1)}(u, t)+\varepsilon^2\varphi_i^{(2)}(u, t)+q_i(u, t, \varepsilon), \tag{2.2}$$

where $q_i(u, t, \varepsilon)=o(\varepsilon^2)$ as $\varepsilon\to 0$. Here, from the initial conditions (2.1), it is evident that

$$\begin{cases}\varphi_i^{(0)}(u, 0)=u_i,\\ \varphi_i^{(1)}(u, 0)=\varphi_i^{(2)}(u, 0)=q_i(u, 0, \varepsilon)=0,\end{cases} \tag{2.3}$$
$$(i=1, 2,\dots, n).$$

Now, if we substitute (2.2) into the initial equation (1.1) it follows that

$$\begin{aligned}\frac{d\varphi_i}{dt}(u, t, \varepsilon)&=\frac{d\varphi_i^{(0)}}{dt}(u, t)+\varepsilon\frac{d\varphi_i^{(1)}}{dt}(u, t)+\varepsilon^2\frac{d\varphi_i^{(2)}}{dt}(u, t)+o(\varepsilon^2)\\ &=\varepsilon[f_i(x, t, 0)+\varepsilon f'_i(x, t, 0)+o(\varepsilon)]\\ &=\varepsilon[f_i(\varphi^{(0)}, t, 0)+\varepsilon\sum_{k=1}^n f_{ik}(\varphi^{(0)}, t, 0)\varphi_k^{(1)}+\varepsilon f'_i(\varphi^{(0)}, t, 0)+o(\varepsilon)],\end{aligned}$$

where

$$f_{ik}(x, t, \varepsilon)=\frac{\partial f_i}{\partial x_k}(x, t, \varepsilon), \qquad f'_i(x, t, \varepsilon)=\frac{\partial f_i}{\partial \varepsilon}(x, t, \varepsilon).$$

Comparing the coefficients of powers of ε, there result the equations

$$\begin{cases} \dfrac{d\varphi_i^{(0)}}{dt}=0, \\ \dfrac{d\varphi_i^{(1)}}{dt}=f_i(\varphi^{(0)}, t, 0), \\ \dfrac{d\varphi_i^{(2)}}{dt}=\sum_{k=1}^{n} f_{ik}(\varphi^{(0)}, t, 0)\varphi_k^{(1)}+f'_i(\varphi^{(0)}, t, 0), \end{cases} \qquad (i=1, 2,\dots, n).$$

Therefore, solving these equations under the initial conditions (2.3), we obtain

$$(2.4)\qquad \begin{cases} \varphi_i^{(0)}(u, t)=u_i, \\ \varphi_i^{(1)}(u, t)=\displaystyle\int_0^t f_i(u, t', 0)dt', \\ \varphi_i^{(2)}(u, t)=\displaystyle\int_0^t [\sum_{k=1}^{n} f_{ik}(u, t', 0)\int_0^{t'} f_k(u, t'', 0)dt''+f'_i(u, t', 0)]dt', \end{cases} \qquad (i=1, 2,\dots, n).$$

Now, if the solution $x_i=\varphi_i(u, t, \varepsilon)$ $(i=1, 2,\dots, n)$ is periodic in t with period T, it must be that

$$(2.5)\qquad \varphi_i(u, T, \varepsilon)=u_i \qquad (i=1, 2,\dots, n).$$

This condition is written from (2.2) and (2.4) as follows:

$$(2.6)\qquad \varphi_i^{(1)}(u, T)+\varepsilon\varphi_i^{(2)}(u, T)+o(\varepsilon)=0.$$

Then, by the assumption 3° and 4°, this condition is written as follows:

$$(2.7)\qquad \begin{cases} \varphi_\alpha^{(2)}(u, T)+o(1)=0 & (\alpha=1, 2,\dots, p), \\ \varphi_\nu^{(1)}(u, T)+\varepsilon\varphi_\nu^{(2)}(u, T)+o(\varepsilon)=0 & (\nu=p+1,\dots, n), \end{cases}$$

since $\varphi_\alpha^{(1)}(u, T)=0$ $(\alpha=1, 2,\dots, p)$ by (2.4). Thus we see that a periodic solution of the initial system (1.1) is the solution of (1.1) such that their initial values u_i $(i=1, 2,\dots, n)$ $(|u_i|<L)$ may satisfy the above equations (2.7).

Therefore, if there exists no real solution u_i of the equations

$$(2.8)\qquad \begin{cases} \varphi_\alpha^{(2)}(u, T)=0 & (\alpha=1, 2,\dots, p), \\ \varphi_\nu^{(1)}(u, T)=0 & (\nu=p+1,\dots, n), \end{cases}$$

such that $|u_i|<L$ $(i=1, 2,\dots, n)$, there exists no periodic solution of the initial system (1.1) for a sufficiently small $|\varepsilon|$.

When there exists a real solution $u_i=c_i$ $(i=1, 2,\dots, n)$ of (2.8) such that $|c_i|<L$ $(i=1, 2,\dots, n)$, does there exist always a solution $u_i=u_i(\varepsilon)$ of (2.7) such that $u_i(0)=c_i$? As is readily seen from the assumption 3°, the left-hand sides of (2.7) are continuously differentiable with respect to u_i $(i=1, 2,\dots, n)$ and ε,

consequently $\varphi_\nu^{(1)}(u, T)$ and $\varphi_\nu^{(2)}(u, T)$ are evidently continuously differentiable with respect to u_i $(i=1, 2,\cdots, n)$.
Therefore, if the Jacobian

$$J=\det\begin{pmatrix} \dfrac{\partial\varphi_\alpha^{(2)}(c, T)}{\partial c_j} \\ \dfrac{\partial\varphi_\nu^{(1)}(c, T)}{\partial c_j} \end{pmatrix} \qquad \begin{pmatrix} \alpha=1, 2,\cdots, p, \\ \nu=p+1,\cdots, n, \\ j=1, 2,\cdots, n. \end{pmatrix}. \tag{2.9}$$

of the left-hand sides of (2.7) for $\varepsilon=0$ does not vanish, then, by the theorem on implicit functions, there exists a unique real solution $u_i=u_i(\varepsilon)$ of (2.7) such that $u_i(0)=c_i$, namely there exists a periodic solution of the initial equation (1.1) corresponding to the initial values $u_i=u_i(\varepsilon)$.

Now, if we put

$$\begin{cases} F_i(u, t)=\displaystyle\int_0^t f_i(u, t', 0)dt', \\ \Phi_i(u)=\displaystyle\int_0^T [\sum_{k=1}^n f_{ik}(u, t, 0)F_k(u, t)+f'_i(u, t, 0)]dt, \end{cases}$$

then, since $F_i(u)=\dfrac{1}{T}F_i(u, T)$, (2.8) can be written as

$$\begin{cases} \Phi_\alpha(u)=0 & (\alpha=1, 2,\cdots, p), \\ F_\nu(u)=0 & (\nu=p+1,\cdots, n), \end{cases} \tag{2.10}$$

therefore, the condition (2.9) can be written as

$$J_u(\Phi_\alpha, F_\nu)\Big|_{u=c}=\det\begin{pmatrix} \dfrac{\partial\Phi_\alpha(u)}{\partial u_j}\Big|_{u=c} \\ \dfrac{\partial F_\nu(u)}{\partial u_j}\Big|_{u=c} \end{pmatrix}\neq 0 \tag{2.11}$$

$$(\alpha=1, 2,\cdots, p,\ \nu=p+1,\cdots, n,\ j=1, 2,\cdots, n).$$

The above results are summarized as a theorem as follows:

Theorem 1. *Given a real system*

$$\frac{dx_i}{dt}=\varepsilon f_i(x, t, \varepsilon) \qquad (i=1, 2,\cdots, n), \tag{1.1}$$

where $f_i(x, t, \varepsilon)$ $(i=1, 2,\cdots, n)$ are the functions satisfying the conditions 1°, 2°, 3° *and* 4°. *Then, if there exists no real solution of* (2.10), *there exists no periodic solution of* (1.1), *and, when there exists a real solution $u_i=c_i$ $(i=1, 2,\cdots, n)$ of* (2.10), *there exists always one and only one periodic solution of* (1.1) *with period T corresponding to that solution $u_i=c_i$ of* (2.10) *provided $J_u(\Phi_\alpha, F_\nu)\neq 0$ for $u_i=c_i$.*

3. Stability of the periodic solution

The stability of a periodic solution of (1.1) whose existence is ascertained in Theorem 1, is decided in accordance with convergency of iteration of the transformation

$$r'_i=\varphi_i(\tilde{u}+r,\ T,\ \varepsilon)-\tilde{u}_i \qquad (i=1,\ 2,\cdots,\ n), \tag{3.1}$$

where $\tilde{u}_i$ is a solution of (2.7). Since $\tilde{u}_i$ is a solution of (2.5), the above transformation can be written as follows:

$$r'_i=\sum_{j=1}^{n}\frac{\partial\varphi_i(u,\ T,\ \varepsilon)}{\partial u_j}\bigg|_{u=\tilde{u}} r_j+o(r) \qquad (i=1,\ 2,\cdots,\ n), \tag{3.2}$$

where $r=\sum_{i=1}^{n}|r_i|$. Consequently the periodic solution corresponding to $u_i=\tilde{u}_i$ $(i=1,\ 2,\cdots,\ n)$ is stable when the characteristic roots of the matrix $A=\left(\frac{\partial\varphi_i(u,\ T,\ \varepsilon)}{\partial u_j}\bigg|_{u=\tilde{u}}\right)$ $(i,\ j=1,\ 2,\cdots,\ n)$ are all less than unity in absolute values.

Now, from (2.2) and (2.4), it follows that

$$\begin{cases}\varphi_\alpha(u,\ T,\ \varepsilon)=u_\alpha+\varepsilon^2\varphi^{(2)}_\alpha(u,\ T)+o(\varepsilon^2) & (\alpha=1,\ 2,\cdots,\ p),\\ \varphi_\nu(u,\ T,\ \varepsilon)=u_\nu+\varepsilon\varphi^{(1)}_\nu(u,\ T)+\varepsilon^2\varphi^{(2)}_\nu(u,\ T)+o(\varepsilon^2) & (\nu=p+1,\cdots,\ n),\end{cases}$$

consequently the matrix A is written as follows:

$$A=E+\varepsilon A_1+\varepsilon^2 A_2+o(\varepsilon^2), \tag{3.3}$$

where E is a unit matrix and

$$A_1=\begin{pmatrix} 0 \\ \dfrac{\partial\varphi^{(1)}_\nu(u,\ T)}{\partial u_j}\bigg|_{u=\tilde{u}}\end{pmatrix},\quad A_2=\begin{pmatrix}\dfrac{\partial\varphi^{(2)}_\alpha(u,\ T)}{\partial u_j}\bigg|_{u=\tilde{u}} \\ \dfrac{\partial\varphi^{(2)}_\nu(u,\ T)}{\partial u_j}\bigg|_{u=\tilde{u}}\end{pmatrix}, \tag{3.4}$$

$$(\alpha=1,\ 2,\cdots,\ p,\quad \nu=p+1,\cdots,\ n,\quad j=1,\ 2,\cdots,\ n).$$

The matrix A of the form (3.3) can be written in the exponential form as follows:

$$A=\exp(\varepsilon B), \tag{3.5}$$

and the matrix B satisfying the above relation is readily found as follows:

$$B=A_1+\varepsilon\left(A_2-\frac{1}{2}A_1^2\right)+o(\varepsilon). \tag{3.6}$$

From (3.5), it is evident that the characteristic roots of A are all less than unity in absolute values when and only when the characteristic roots of εB are are all negative in their real parts.

Let us express the matrix A_1 and A_2 as follows:

$$(3.7)\qquad A_1=\begin{pmatrix} 0 & 0 \\ A_{21}^{(1)} & A_{22}^{(1)} \end{pmatrix},\qquad A_2=\begin{pmatrix} A_{11}^{(2)} & A_{12}^{(2)} \\ A_{21}^{(2)} & A_{22}^{(2)} \end{pmatrix},$$

where $A_{11}^{(2)}$, $A_{12}^{(2)}$ are respectively $p\times p-$, $p\times(n-p)-$matrices and $A_{21}^{(i)}$, $A_{22}^{(i)}$ $(i=1, 2)$ are respectively $(n-p)\times p-$, $(n-p)\times(n-p)-$matrices. Then, from (3.7), the matrix B becomes

$$(3.8)\qquad B=\begin{pmatrix} \varepsilon A_{11}^{(2)}+o(\varepsilon) & \varepsilon A_{12}^{(2)}+o(\varepsilon) \\ A_{21}^{(1)}+o(1) & A_{22}^{(1)}+o(1) \end{pmatrix}.$$

Now, let us assume that

$$(3.9)\qquad \det A_{22}^{(1)}=\det\left(\frac{\partial\varphi_\nu^{(1)}(u, t)}{\partial u_\mu}\bigg|_{u=\tilde{u}}\right)\neq 0 \qquad (\nu, \mu=p+1,\cdots, n).$$

Then, by the lemma[1] due to M. Urabe [5], the characteristic rooots of the matrix B are the numbers of the forms

$$\varepsilon(\lambda+o(1)) \text{ and } \mu+o(1),$$

where λ and μ are respectively the characteristic roots of the matrices

$$A_{11}^{(2)}-A_{12}^{(2)}(A_{22}^{(1)})^{-1}A_{21}^{(1)} \text{ and } A_{22}^{(1)}.$$

Thus the real parts of the characteristic roots of the matrix B become all negative when $|\varepsilon|$ is sufficiently small and the real parts of the characteristic roots of the matrices

$$(3.10)\qquad \mathring{A}_{11}^{(2)}-\mathring{A}_{12}^{(2)}(\mathring{A}_{22}^{(1)})^{-1}\mathring{A}_{21}^{(1)} \text{ and } \mathring{A}_{22}^{(1)}$$

are all negative. But here $\mathring{A}_{21}^{(1)}$. $\mathring{A}_{22}^{(1)}$, $\mathring{A}_{11}^{(2)}$ and $\mathring{A}_{12}^{(2)}$ are the matrices as follows:

$$(3.11)\qquad \begin{cases} (\mathring{A}_{21}^{(1)}\ \mathring{A}_{22}^{(1)})=\left(\dfrac{\partial\varphi_\nu^{(1)}(u, T)}{\partial u_j}\bigg|_{u=c}\right)=\left(\dfrac{\partial F_\nu}{\partial u_j}\bigg|_{u=c}\right), \\ (\mathring{A}_{11}^{(2)}\ \mathring{A}_{12}^{(2)})=\left(\dfrac{\partial\varphi_\alpha^{(2)}(u, T)}{\partial u_i}\bigg|_{u=c}\right)=\left(\dfrac{\partial\Phi_\alpha}{\partial u_j}\bigg|_{u=c}\right), \end{cases}$$

$$(\alpha=1, 2,\cdots, p,\quad \nu=p+1,\cdots, n,\quad j=1, 2,\cdots, n).$$

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1) The lemma due to M. Urabe runs as follows:

When $\det B(0)\neq 0$ and $|\varepsilon|$ is sufficiently small, the characteristic roots of the continuous matrix of the form

$$\begin{bmatrix} A_1(\varepsilon) & A(\varepsilon) \\ B_1(\varepsilon) & A(\varepsilon) \end{bmatrix}$$

are the numbers of the forms

$$\varepsilon[\lambda+o(1)] \text{ and } \mu+o(1)$$

as $\varepsilon\to 0$, where λ and μ are respectively the characteristic roots of the matrices

$$A_1(0)-A(0)B^{-1}(0)B_1(0) \text{ and } B(0).$$

When $|\varepsilon|$ is sufficiently small, the condition (3.9) can also be written as follows:

$$\det\left(\frac{\partial F_\nu(u)}{\partial u_\mu}\bigg|_{u=c}\right)\neq 0. \tag{3.12}$$

Thus, on the stability, there is obtained

Theorem 2. *The periodic solution whose existence is ascertained in Theorem* 1, *is stable when real parts of the characteristic roots of the two matrices of* (3.10) *are all negative.*

In the sequel, let us investigate the meaning of the condition of stability in the above theorem.

In the present case, as is seen from (1.2), the stroboscopic image of the initial system is of the form

$$\begin{cases} \dfrac{dx_\alpha}{dt}=0 & (\alpha=1, 2,\cdots, p), \\[2ex] \dfrac{dx_\nu}{dt}=\varepsilon F_\nu(x) & (\nu=p+1,\cdots, n). \end{cases} \tag{3.13}$$

Therefore, in the neighborhood of the point $x_i=c_i$ $(i=1, 2,\cdots, n)$, the critical points of the above system make a p-dimensional manifold V^p:

$$F_\nu(x)=0 \qquad (\nu=p+1,\cdots, n), \tag{3.14}$$

because of (3.12). Then, the condition that the real parts of the characteristic roots of the matrix $\varepsilon \mathring{A}_{22}^{(1)}$ are all negative implies that the manifold V^p is stable in the sense that any point lying near V^p tends to V^p along the paths of (3.13) as $t\to+\infty$.

Due to the assumption (3.12), in the neighborhood of $x_i=c_i$ $(i=1, 2,\cdots, n)$, the equations (3.14) can be solved with respect to x_ν $(\nu=p+1,\cdots, n)$ as follows:

$$x_\nu=x_\nu(x_1,\cdots, x_p) \qquad (\nu=p+1,\cdots, n). \tag{3.15}$$

Now, from (2.2) follows

$$\varphi_\alpha(u, T, \varepsilon)=u_\alpha+\varepsilon^2\varphi_\alpha^{(2)}(u, T)+o(\varepsilon^2) \qquad (\alpha=1, 2,\cdots, p).$$

Then, if we consider the transformation $\boldsymbol{T}$ induced in V^p by the above transformation, as is readily seen, the point $x_\alpha=c_\alpha$ $(\alpha=1, 2,\cdots, p)$ of the V^p considered as a fixed point of the transformation $\boldsymbol{T}$ (except for the terms of higher than the second order in ε being neglected) is stable under the transformation $\boldsymbol{T}$ when the real part of the characteristic roots of the matrix

$$M=\left(\left[\frac{\partial\varphi_\alpha^{(2)}(x, T)}{\partial x_\beta}+\sum_{\nu=p+1}^{n}\frac{\partial\varphi_\alpha^{(2)}(x, T)}{\partial x_\nu}\frac{\partial x_\nu}{\partial x_\beta}\right]_{x=c}\right) \tag{3.16}$$

$$(\alpha, \beta=1, 2,\cdots, p)$$

are all negative.

Since (3.15) is a solution of (3.14), it is evident that

$$\left[\frac{\partial F_\nu}{\partial x_\alpha}+\sum_\mu \frac{\partial F_\nu}{\partial x_\mu}\frac{\partial x_\mu}{\partial x_\alpha}\right]_{x=c}=0.$$

This can be written in the matrix from using by the notations of (4.11) as follows:

$$\mathring{A}^{(1)}_{21}+\mathring{A}^{(1)}_{22}H=0$$

where $H=\left(\left.\frac{\partial x_\nu}{\partial x_\alpha}\right|_{x=c}\right)$ $(\alpha=1, 2,\cdots,p,\ \mu=p+1,\cdots, n)$. Since $\det\mathring{A}^{(1)}_{22}\neq 0$ from (3.12), the above relation can be solved with respect to H as follows:

$$H=-(\mathring{A}^{(1)}_{22})^{-1}\mathring{A}^{(1)}_{21}.$$

Then, by the latter of (3.11), the matrix M can be expressed in the matrix form as follows:

$$M=\mathring{A}^{(2)}_{11}-\mathring{A}^{(2)}_{12}(\mathring{A}^{(1)}_{22})^{-1}\mathring{A}^{(1)}_{21}.$$

This is nothing else but the matrix of the former of (3.10). Thus we see that the condition that the real parts of the characteristic roots of the former matrix of (3.10) are all negative implies the point $x_\alpha=c_\alpha$ $(\alpha=1, 2,\cdots,p)$ of V^p considered as the fixed point of the transformation $\boldsymbol{T}$ in the precceding sense is stable under the transformation $\boldsymbol{T}$ induced in V^p by α-ccomponent-paths $(\alpha=1, 2,\cdots, p)$ of the initial system, the terms of higher order than ε^2 being neglected.

The above remarks are summarized as

Theorem 3. *The former matrix of* (3.10) *is a Jacobian matrix of* $\Phi_1,\cdots,\Phi_p$ *with respect to* $x_1,\cdots, x_p$ *for* $x_i=c_i$ $(i=1, 2,\cdots, n)$, $x_{p+1},\cdots, x_n$ *being considered as the function of* $x_1,\cdots, x_p$ *satisfying* (3.14).

The condition that the real parts of the characteristic roots of this matrix are negative implies that the point $x_\alpha=c_\alpha$ $(\alpha=1, 2,\cdots, p)$ *of* V^p *considered as the fixed point of the transformation* $\boldsymbol{T}$ *is stable under the transformation induced in* V^p *by* α*-component-paths* $(\alpha=1, 2,\cdots, p)$ *of the initial system, the terms of higher order being neglected.*

The condition that the real parts of the characteristic roots of the latter matrix of (3.10) *are negative implies that* V^p *is stable as a manifold.*

In conclusion, the auther wishes to express his hearty gratitude to Prof. M. Urabe for his kind guidance and constant advice.

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