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On the Behavior of Paths of the Analytic Two-dimensional Autonomous System in a Neighborhood of an Isolated Critical Point

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§ 1. Introduction

The object of this paper is to study the behavior of paths of an analytic two-dimensional autonomous system in a neighborhood of an isolated critical point.

First I. Bendixson [1][1)] treated this problem systematically. He showed that, by a finite number of quadratic transformations, one might reduce the study of such a problem to that of the case where the critical point is simple or of the so-called Bendixson type.

Later in 1928, M. Frommer [2] introduced the notions of the orders and magnitudes of curvature of the paths tending to a critical point and, making use of these notions and of the exceptional directions, he gave a systematic method to determine the behavior of paths in a neighborhood of a critical point. But, as is noticed by V. V. Nemytzkii and V. V. Stepanov [3], his method seems to be not sufficient.

Recently S.Lefschetz [4] gave a step-by-step process to reduce the study of the behavior of paths in a neighborhood of a critical point to the study of the paths in the case where the critical point is simple or of the Bendixson type. At this juncture, he made use of the coordinate transformations so often that the behavior of the paths of the initial system is not always clear.

In this paper, making use of a Newton polygon, exceptional directions and Keil's theorem, we give a simple step-by-step process to obtain a local phase portrait of the paths near the critical point without using any coordinate transformation except for a simple one used only once.

§ 2. Preliminaries.

2.1 Exceptional directions and Keil's theorem.

Given a continuous autonomous system

(1) $$\frac{dx}{dt}=P(x, y), \quad \frac{dy}{dt}=Q(x, y),$$

1) The numbers in square brackets refer to the references listed at the end of this paper.

for which the origin is an isolated critical point. We assume that $P(x, y)$ and $Q(x, y)$ can be written in the forms as follows:

$$P(x, y)=P_m(x, y)+o(r^m),$$
$$Q(x, y)=Q_n(x, y)+o(r^n),$$

as $r=\sqrt{x^2+y^2}\to 0$, where $P_m(x, y)$ and $Q_n(x, y)$ are homogeneous polynomials of degree m and n respectively. Corresponding to (1), we consider a differential equation

$$\frac{dy}{dx}=\frac{Q(x, y)}{P(x, y)}. \tag{2}$$

Then evidently the paths of (1) are nothing but the integral curves of (2). Therefore, in the investigation of the behavior of paths of (1), we often use (2) in place of (1). In the sequel, for simplicity, let us call the critical point of (1) also that of (2).

For the equation of the form (2), Frommer considered the polynomial $F(x, y)$ as follows:

$$F(x, y)=xQ_n(x, y)-yP_m(x, y) \qquad \text{when } m=n;$$
$$F(x, y)=xQ_n(x, y) \qquad \text{when } n<m;$$
$$F(x, y)=-yP_m(x, y) \qquad \text{when } m<n;$$

and, for any real linear factor l of $F(x, y)$, he called the direction determined by $l=0$[1] the *exceptional direction* of (2) at the origin. Then it is well known that the integral curve of (2) tending to the origin must tend to the origin either spiraling to the origin or in a certain exceptional direction. Now, when the origin is not a critical point, it is evident that the integral curve of (2) tending to the origin tends to the origin in the unique direction $l=0$. Therefore we may also call the direction $l=0$ the exceptional direction of (2) even when the origin is not a critical point. Thus, for convenience, in the present note, we always call the direction $l=0$ the exceptional direction of (2) at the origin either when the origin is a critical point or not.

In this note, we frequently use the following

Keil's theorem. *Given a differential equation*

$$\frac{dy}{dx}=\frac{g(x, y)}{x+f(x, y)}, \tag{3}$$

where $f(x, y)$ and $g(x, y)$ are continuously differentiable with respect to x and y and $f(x, y)$, $g(x, y)=o(r)$ as $r\to 0$. Suppose the origin is an isolated critical point of (3). *Then there exists one and only one integral curve of* (3) *tending to the origin in either direction (positive and negative) of the x-axis.*

Further, if we denote the curve made of these two integral curves by L and put $G(y)=g(x(y), y)$ for $x=x(y)$ such that $x(y)+f(x(y), y)=0$, then, for the integral curves lying above L, it holds that

(i) *when $G(y)<0$ for sufficiently small $y>0$, one and only one integral*

1) In the sequel, we call the direction determined by $l=0$ simply the *direction* $l=0$.

curve tends to the origin in the positive direction of the y-axis and any other integral curve does not tend to the origin;

(ii) *when $G(y)>0$ for sufficiently small $y>0$, all integral curves tend to the origin in the positive direction of the y-axis.*

For the integral curves lying below L, the same conclusion holds if the inequality signs are inverted.

For the proof of this theorem, refer to his original paper [5].

2.2 Order and magnitude of an integral curve tending to the origin.

In this note, we are concerned with an analyitc two-dimensional system

$$\frac{dx}{dt}=X(x, y), \quad \frac{dy}{dt}=Y(x, y), \tag{4}$$

for which the origin is an isolated critical point. As is remarked at the beginning of **2.1**, the investigation of the behavior of paths of (4) in a neighborhood of the origin is equivalent to that of the behavior of integral curves of the differential equation

$$\frac{dy}{dx}=\frac{Y(x, y)}{X(x, y)} \tag{5}$$

in a neighborhood of the origin. For the equation (5), without loss of generality, we may suppose that X and Y have no common factor vanishing at the origin.

After Frommer and Lefschetz, we define the *order* and *magnitude* of an integral curve $y=y(x)$ of (5) tending to the origin as follows:
About the order, we say that

an integral curve $y=y(x)$ is of order μ in x, when, for any $\varepsilon>0$, there exists a positive number μ such that

$$|y(x)|/|x|^{\mu-\varepsilon} \to 0 \text{ and } |y(x)|/|x|^{\mu+\varepsilon} \to \infty \quad \text{as } x \to +0 \text{ or } -0;$$

it is of order infinity in x, when, for any great $\mu>0$, $|y(x)|/|x|^{\mu} \to 0$ as $x \to +0$ or -0;

it is of order zero in x, when, for any $\varepsilon>0$, $|y(x)|/|x|^{\varepsilon} \to \infty$ as $x \to +0$ or -0.

About the magnitude, we say that, *when an integral curve $y=y(x)$ is of order μ $(0<\mu<\infty)$ in x, it is of magnitude ρ or infinity according as $y(x)/|x|^{\mu} \to \rho$ or ∞* (or $-\infty$) *as $x \to +0$ or -0.*

When an integral curve is expressed as $x=x(y)$, its order and magnitude are defined in the same manner with respect to y.

2.3 Newton polygon.

Since X and Y are analytic at the origin, we can expand them in power series in x and y in the domain $|x|, |y| \leq \triangle$ as follows:

$$(6) \qquad \begin{aligned} X(x, y) &= \sum_{i,j} a_{ij} x^i y^j \qquad (a_{ij} \neq 0), \\ Y(x, y) &= \sum_{k,l} b_{kl} x^k y^l \qquad (b_{kl} \neq 0). \end{aligned}$$

These expansions do not contain constant terms, since $X(0, 0) = Y(0, 0) = 0$ by the assumption.

As, in the sequel, we want to make use of the orders of the integral curves of (5) tending to the origin, for the differential equation (5), after Briot and Bouquet, we construct a polygon like the Newton polygon in the theory of algebraic functions. Indeed, introducing the orthogonal coordinates (ξ, η) in a plane, we mark, in this plane, the points X_{ij} $(j+1, i-1)$ and Y_{kl} (l, k) corresponding to the terms $a_{ij}x^iy^j$ and $b_{kl}x^ky^l$ respectively, and out of these points we choose the points V_p's $(p=1, 2, \cdots, s+1)$ so that they may satisfy the conditions as follows:

(i) the slopes of the segments V_pV_{p+1}'s $(p=1, 2, \cdots, s)$ are all negative. (In the sequel, the slope of V_pV_{p+1} is denoted by $-\mu_p$.)

(ii) $0<\mu_1<\mu_2<\cdots<\mu_s<\infty$;

(iii) all of the points X_{ij} and Y_{kl} lie in the closed domain bounded by $V_1'V_1$, V_1V_2, $\cdots$, V_sV_{s+1} and $V_{s+1}V_{s+1}'$, where $V_1'V_1$ and $V_{s+1}V_{s+1}'$ are the half lines respectively parallel to the positive directions of ξ- and η-axes. (If necessary, the slopes of V_1V_1' and $V_{s+1}V_{s+1}'$ are denoted by $-\mu_0$ and $-\mu_{s+1}$ respectively. It is evident that the values of μ_0 and μ_{s+1} are zero and infinity respectively.)

Evidently, as in the case of algebraic functions, such choice of points V_p's $(p=1, 2, \cdots, s+1)$ is possible. In this note, let us call the broken line $V_1V_2\cdots V_{s+1}$ the *Nowton polygon* of (5) and call the points V_p's $(p=1, 2, \cdots, s+1)$ and the segments V_pV_{p+1}'s $(p=1, 2, \cdots, s)$ the *vertices* and the *sides* of the polygon respectively. Needles to say, the Newton polygon of (5) is determined uniquely, because X and Y are supposed to have no common factor vanishing at the origin.

A vertex of the Newton polygon is a certain X_{ij} or a certain Y_{kl}. But it may happen that the vertex is a certain X_{ij} and at the same time a certain Y_{kl}. In such a case, for any vertex V_p such that it is an $X_{i_qj_q}$ and at the same time a $Y_{k_ql_q}$, as will be seen later, it is convenient to introduce also the quantity $v_q=b_{k_ql_q}/a_{i_qj_q}$, if $\mu_q>b_{k_ql_q}/a_{i_qj_q}>\mu_{q-1}$.

On a transformation of variables turning the Newton polygon to that of simple characters, let us prove the following

Lemma 1. *When a positive integer λ_0 is chosen suitably, by applying the transformation $w=x^{1/\lambda_0}$ to (5), we can make the Newton polygon of the transformed differential equation satisfy the conditions as follows:*

(A_1) μ_p's $(p=1, 2, \cdots, s)$ *are integers and* $\mu_p \geqq \mu_{p-1}+2$;

(A_2) *for any* q $(1\leqq q\leqq s+1)$ *for which* v_q *is introduced,*

(i) $\mu_q-1>v_q\geqq \mu_{q-1}+2$;

(ii) $i-1+v_q(j+1) \geqq i_q-1+v_q(j_q+1)+2,$
$k+v_q l \geqq k_q+v_q l_q+2$

for any points X_{ij} and Y_{kl} other than the vertex V_q for which v_q is defined.

Proof. By the transformation $w=x^{1/\lambda_0}$, the differential equation (5) is transformed to the equation as follows:

$$\frac{dy}{dw}=\frac{\lambda_0 w^{\lambda_0-1+\sigma}Y(w^{\lambda_0}, y)}{w^{\sigma}X(w^{\lambda_0}, y)}\ , \tag{7}$$

where σ is $1-\lambda_0$ or 0 according as $X(x, y)$ has a factor x or not. Here $w^{\sigma}X(w^{\lambda_0}, y)$ and $w^{\lambda_0-1+\sigma}X(w^{\lambda_0}, y)$ cannot have any common factor vanishing at the origin. For, since $X(w^{\lambda_0}, y)$ and $Y(w^{\lambda_0}, y)$ have no such common factor from the assumption, $X(w^{\lambda_0}, y)$ and $w^{\lambda_0-1}Y(w^{\lambda_0}, y)$ evidently have no such common factor, provided $X(x,y)$ has not a factor x. When $X(x, y)$ has a factor x, if we write $X(x, y)$ as $x\widetilde{X}(x, y)$, then evidently $Y(w^{\lambda_0}, y)$ and $w^{1-\lambda_0}X(w^{\lambda_0}, y)=w\widetilde{X}(w^{\lambda_0}, y)$ have no common factor vanishing at the origin. Thus $w^{\sigma}X(w^{\lambda_0}, y)$ and $\lambda_0 w^{\lambda_0-1+\sigma}\ Y(w^{\lambda_0}, y)$ cannot have any common factor vanishing at the origin either when $X(x, y)$ has a factor x or not. Then the Newton polygon of (7) is constructed uniquely corresponding to the terms of $w^{\sigma}X(w^{\lambda_0}, y)$ and $\lambda_0 w^{\lambda_0-1+\sigma}\ Y(w^{\lambda_0}, y)$ themselves. As is readily seen, by the transformation $w=x^{1/\lambda_0}$, the points $X_{ij}(j+1, i-1)$ and $Y_{kl}(l, k)$ are shifted to the points $(j+1, \lambda_0(i-1)+\lambda_0-1+\sigma)$ and $(l, \lambda_0 k+\lambda_0-1+\sigma)$ respectively and the coefficients b_{kl}'s in (6) are multiplied by λ_0. Hence it follows that, by the transformation $w=x^{1/\lambda_0}$, the slopes of the Newton polygon and the quantities v_q's are multiplied by λ_0.

Then we can choose λ_0 sufficiently large so that $\lambda_0\mu_p$'s $(p=1, 2, \cdots, s)$ may all become integers and $\lambda_0(\mu_p-\mu_{p-1}) \geqq 2\ (p=1, 2, \cdots, s)$ may hold. (This is evidently possible because μ_p's $(p=1, 2, \cdots, s)$ are all rational numbers.) This means that the condition (A_1) is valid for the Newton polygon of the transformed equation (7).

For q for which v_q is introduced, let us consider the straight lines of the slope $-v_q$ passing through the points X_{ij} and Y_{kl}. Then, as is readily seen, the distances from the origin to these lines become

$$\frac{i-1+v_q(j+1)}{\sqrt{1+v_q^2}} \quad \text{and} \quad \frac{k+v_q l}{\sqrt{1+v_q^2}}\ .$$

Now, since $\mu_q>v_q>\mu_{q-1}$, by the property of the Newton polygon, the above parallel lines cannot lie below the line

$$\eta+v_q\xi=i_q-1+v_q(j_q+1)=k_q+v_q l_q$$

passing through V_q. Therefore there exists a positive number δ such that

$$i-1+v_q(j+1) \geqq i_q-1+v_q(j_q+1)+\delta,$$
$$k+v_p l \geqq k_q+v_q l_q+\delta$$

for any points X_{ij} and Y_{kl} other than V_q, because the set of marked points is discrete. Then we can choose the preceding λ_0 still larger so that, for any points X_{ij} and Y_{kl} other than V_q, it may hold that

$$\lambda_0(\mu_q - v_q) > 1, \quad \lambda_0(v_q - \mu_{q-1}) \geqq 2,$$
$$\lambda_0(i-1) + \lambda_0 v_q(j+1) \geqq \lambda_0(i_q - 1) + \lambda_0 v_q(j_q + 1) + 2,$$
$$\lambda_0 k + \lambda_0 v_q\, l \geqq \lambda_0 k_q + \lambda_0 v_q\, l_q + 2.$$

This says that the condition (A_2) is also valid for the Newton polygon of the transformed equation (7). Thus lemma 1 has been proved.

In this note, we consider the case where $x \geqq 0$, because the case where $x \leqq 0$ is reduced to the former case by replacing x by $-x$. Then, since the transformation $w = x^{1/\lambda_0}$ becomes a real transformation, by the present lemma, without loss of generality, we may assume that the Newton polygon of the given equation (5) satisfy the conditions (A_1) and (A_2).

2.4 Differential equations $(\mathbf{E}_\mu)$ and $(\mathbf{F}_\mu)$.

Let (E_μ) be the differential equation obtained from (5) by the substitution $y = x^\mu y_\mu$, where μ is a positive number such that. $\mu_p \geqq \mu \geqq m_{p-1} = \max(\mu_{p-1}, \varepsilon)$ and ε is any small positive number less than 1/2. This substitution $y = x^\mu y_\mu$ is really possible when $(x, y_\mu) \in \Omega_\mu : 0 \leqq x \leqq \Delta_1, |y_\mu| \leqq \Delta_2$, provided $\Delta_1 \leqq \Delta$ and $\Delta_1^\mu \Delta_2 \leqq \Delta$. In the sequel, assuming $(x, y_\mu) \in \Omega_\mu$, we investigate the characters of (E_μ).

Substitution of $y = x^\mu y_\mu$ into (5) entails

$$\frac{dy_\mu}{dx} = \frac{Y(x, x^\mu y_\mu) - \mu x^{\mu-1} y_\mu X(x, x^\mu y_\mu)}{x^\mu X(x, x^\mu y_\mu)}, \tag{8}$$

which is an (E_μ). But this equation can be reduced to that of a simpler form. In fact, from (6) $x^{\mu-1}X(x, x^\mu y_\mu)$ and $Y(x, x^\mu y_\mu)$ are written as follows:

$$\begin{aligned} x^{\mu-1}X(x, x^\mu y_\mu) &= \sum_{i,j} a_{ij} x^{i-1+\mu(j+1)} y_\mu^j, \\ Y(x, x^\mu y_\mu) &= \sum_{k,l} b_{kl} x^{k+\mu l} y_\mu^l \end{aligned} \tag{9}$$

Then, if we draw the straight line L_μ of the slope $-\mu$ through V_p and consider the marked points X_{ij} and Y_{kl} lying on L_μ (these points are, if any, denoted by $X_{\alpha_m \beta_m}$ $(m = 1, 2, \cdots, e)$ and $Y_{\gamma_n \delta_n}$ $(n = 1, 2, \cdots, f)$ respectively), then, by the same reasonings as in the proof of lemma 1, we see that the quantities

$$\alpha_m - 1 + \mu(\beta_m + 1) \quad \text{and} \quad \gamma_n + \mu\delta_n$$

corresponding to $X_{\alpha_m \beta_m}$ and $Y_{\gamma_n \delta_n}$ are all equal to

$$d(\mu) = \inf_{X_{ij},\, Y_{kl}} \{i - 1 + \mu(j+1), k + \mu\, l\}, \tag{10}$$

and that

$$\begin{aligned} i - 1 + \mu(j+1) &\geqq d(\mu) + \varepsilon_1, \\ k + \mu\, l &\geqq d(\mu) + \varepsilon_1 \end{aligned} \tag{11}$$

for any points X_{ij} and Y_{kl} lying outside L_μ. Here ε_1 is a positive number depending on μ such that its value is $\geqq 1$ or $\geqq 2$ according as μ is an integer or v_p (by lemma 1). Then the right hand sides of (9) are expressed as follows:

$$(12)\qquad \begin{aligned} x^{\mu-1}X(x, x^{\mu}y_{\mu}) &= x^{d(\mu)}\{\sum_{m=1}^{e} a_{\alpha_m\beta_m}y_{\mu}^{\beta_m} + A(x, y_{\mu})\}, \\ Y(x, x^{\mu}y_{\mu}) &= x^{d(\mu)}\{\sum_{n=1}^{f} b_{\gamma_n\delta_n}y_{\mu}^{\delta_n} + B'(x, y_{\mu})\}, \end{aligned}$$

where

$$A(x, y_{\mu}),\ B'(x, y_{\mu}) = O(x^{\varepsilon_1})\ ^{1)}.$$

Consequently, if we put

$$A_{\mu}(y_{\mu}) = \sum_{m=1}^{e} a_{\alpha_m\beta_m}y_{\mu}^{\delta_m},$$

the numerator of the right hand side of (8) is written as follows:

$$(13)\qquad Y(x, x^{\mu}y_{\mu}) - \mu x^{\mu-1}y_{\mu}X(x, x^{\mu}y_{\mu}) = x^{d(\mu)}\{B_{\mu}(y_{\mu}) + B(x, y_{\mu})\},$$

where

$$(14)\qquad B_{\mu}(y_{\mu}) = \sum_{n=1}^{f} b_{\gamma_n\delta_n}y_{\mu}^{\delta_n} - \mu y_{\mu}A_{\mu}(y_{\mu}),$$

and

$$B(x, y_{\mu}) = O(x^{\varepsilon_1}).$$

Now, as is readily seen, $B_{\mu}(y_{\mu})$ can vanish identically only when V_p is an $X_{i_pj_p}$ and at the same time a $Y_{k_pl_p}$ and moreover $\mu = b_{k_pl_p}/a_{i_pj_p}$. In this case, μ is an integer or v_p, consequently the value of ε_1 in (11) is $\geqq 1$ or $\geqq 2$, therefore, when $B_{\mu}(y_{\mu}) \equiv 0$, we have

$$Y(x, x^{\mu}y_{\mu}) - \mu x^{\mu-1}y_{\mu}X(x, x^{\mu}y_{\mu}) = x^{d(\mu)+1}C(x, y_{\mu}),$$

where

$$C(x, y_{\mu}) = O(x^{\varepsilon_1-1}).$$

Thus, according as $B_{\mu}(y_{\mu}) \not\equiv 0$ or $\equiv 0$, dividing the denominator and the numerator of the right hand side of (8) by $x^{d(\mu)}$ or $x^{d(\mu)+1}$, we have

$$(15)\qquad \frac{dy_{\mu}}{dx} = \frac{B_{\mu}(y_{\mu}) + B(x, y_{\mu})}{x\{A_{\mu}(y_{\mu}) + A(x, y_{\mu})\}} \qquad \text{when } B_{\mu}(y_{\mu}) \not\equiv 0;$$

$$(16)\qquad \frac{dy_{\mu}}{dx} = \frac{C(x, y_{\mu})}{A_{\mu}(y_{\mu}) + A(x, y_{\mu})} \qquad \text{when } B_{\mu}(y_{\mu}) \equiv 0.$$

In the sequel, we call these equations (E_{μ}). Evidently the critical points of these equations lying on the y_{μ}-axis are, if any, isolated.

So far we are concerned with the order with respect to x of the terms contained in the denominator and the numerator of the right hand side of (8). Now let us turn our attention to the orders with respect to x and y_{μ} and let us consider the case where $\mu \leqq \mu_p - 1$. In this case, we consider the line $L_{\mu+1}$ and denote the points X_{ij} and Y_{kl} lying on $L_{\mu+1}$, if any, by $X_{\alpha'_m\beta'_m}$ $(m=1, 2, \cdots, g)$ and $Y_{\gamma'_n\delta'_n}$ $(n=1, 2, \cdots, h)$ respectively. Then, as before, it follows that the quantities

$$\alpha'_m - 1 + (\mu+1)(\beta'_m + 1) \qquad \text{and} \qquad \gamma'_n + (\mu+1)\delta'_n$$

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1) In this note, the symbol $f(u,v) = O(u^{\lambda})$ means that, as $u \to 0$, $\left|\frac{f(u, v)}{u^{\lambda}}\right|$ is bounded, provided $|v|$ is bounded.

corresponding to $X_{\alpha'_m\beta'_m}$ and $Y_{\gamma'_n\delta'_n}$ are all equal to $d(\mu+1)$ and that

$$\begin{aligned} &i-1+(\mu+1)(j+1) \geqq d(\mu+1)+\varepsilon_2, \\ &k+(\mu+1)l \geqq d(\mu+1)+\varepsilon_2 \end{aligned} \tag{17}$$

for any points X_{ij} and Y_{kl} lying outside $L_{\mu+1}$. Here ε_2 is a positive number depending on μ such that its value is $\geqq 1$ or $\geqq 2$ according as μ is an integer or υ_p-1 (by lemma 1).

Since V_p lies on $L_{\mu+1}$ and L_μ, if we write the coordinates of V_p as (ζ, κ), it is evident that $d(\mu)=\kappa+\mu\zeta$ and $d(\mu+1)=\kappa+(\mu+1)\zeta$, consequently also that $d(\mu+1)-d(\mu)=\zeta$.

Now, since the orders with respect to x and y_μ of the terms contained in the right hand sides of (9) are respectively

$$i-1+\mu(j+1)+j=[i-1+(\mu+1)(j+1)]-1$$

and

$$k+\mu l+l=k+(\mu+1)l,$$

the terms of the lowest orders among these terms are those corresponding to $X_{\alpha'_m\beta'_m}$ and $Y_{\gamma'_n\delta'_n}$, consequently their orders are respectively

$$d(\mu+1)-1=d(\mu)+(\zeta-1) \quad \text{and} \quad d(\mu+1)=d(\mu)+\zeta.$$

By the way, by (17), the orders of the terms other than the above ones are higher than the above least orders by at least ε_2. Hence, taking (10) into consideration, if we put

$$\begin{aligned} H_\mu(x, y_\mu) &= \sum_{m=1}^{g} a_{\alpha'_m\beta'_m} x^{\zeta-1-\beta'_m} y_\mu^{\beta'_m}, \\ K_\mu(x, y_\mu) &= \sum_{n=1}^{h} b_{\gamma'_n\delta'_n} x^{\zeta-\delta'_n} y_\mu^{\delta'_n}, \end{aligned} \tag{18}$$

we have

$$\begin{aligned} x^{\mu-1}X(x, x^\mu y_\mu) &= x^{d(\mu)}\{H_\mu(x, y_\mu)+H(x, y_\mu)\}, \\ Y(x, x^\mu y_\mu) &= x^{d(\mu)}\{K_\mu(x, y_\mu)+K(x, y_\mu)\}, \end{aligned} \tag{19}$$

where

$$H(x, y_\mu),\ K(x, y_\mu)=O(r_\mu^{\zeta+\varepsilon_2})$$

as $r_\mu=\sqrt{x^2+y_\mu^2} \to 0$. Then, from (12) and (13), it follows that

$$\begin{aligned} A_\mu(y_\mu)+A(x, y_\mu) &= H_\mu(x, y_\mu)+H(x, y_\mu), \\ B_\mu(y_\mu)+B(x, y_\mu) &= [K_\mu(x, y_\mu)-\mu y_\mu H_\mu(x, y_\mu)]+L(x, y_\mu), \end{aligned} \tag{20}$$

where

$$L(x, y_\mu)=O(r_\mu^{\zeta+\varepsilon_2}).$$

Therefore, when $\mu \leqq \mu_p-1$, the equations (15) and (16), namely the equations (E_μ), can be written in the forms as follows:

$$\frac{dy_\mu}{dx}=\frac{[K_\mu(x, y_\mu)-\mu y_\mu H_\mu(x, y_\mu)]+L(x, y_\mu)}{x\{H_\mu(x, y_\mu)+H(x, y_\mu)\}}; \tag{21}$$

$$\frac{dy_\mu}{dx}=\frac{[K_\mu(x, y_\mu)-\mu y_\mu H_\mu(x, y_\mu)]x^{-1}+L(x, y_\mu)x^{-1}}{H_\mu(x, y_\mu)+H(x, y_\mu)}. \tag{22}$$

As is seen from the way of formation, these equations are of the same form as that of (2) and, as we have seen, the origin of the (x, y_μ)- plane cannot be a critical point without being an isoloated one. Therefore we can consider the exceptional directions of (E_μ) at the origin. Indeed, for (E_μ) $(\mu \leqq \mu_p - 1)$, the polynomial defined in **2.1** becomes as follows:

$$(23)\qquad \begin{aligned} &F_\mu(x, y_\mu) = x\{K_\mu(x, y_\mu) - (\mu+1)y_\mu H_\mu(x, y_\mu)\} &&\text{when } B_\mu(y_\mu) \not\equiv 0;\\ &F_\mu(x, y_\mu) = K_\mu(x, y_\mu) - (\mu+1)y_\mu H_\mu(x, y_\mu) &&\text{when } B_\mu(y_\mu) \equiv 0. \end{aligned}$$

In the sequel, let us consider the case where $\mu \leqq \mu_p - 1$ and $B_\mu(y_\mu) \not\equiv 0$. In this case, evidently the direction $x=0$ is an exceptional direction of (E_μ) at the origin. So, in the following, let us study the integral curves tending to the origin in the direction of the y_μ-axis. For this purpose, putting $x = zy_\mu$ in (E_μ), let us make differential equations as follows:

$$(24)\qquad \frac{dz}{dy_\mu} = \frac{z\{y_\mu A_\mu(y_\mu) - B_\mu(y_\mu)\}y_\mu^{-\zeta} + z\cdot a(y_\mu, z)}{y_\mu\{B_\mu(y_\mu)y_\mu^{-\zeta} + b(y_\mu, z)\},},$$

$$(25)\qquad \frac{dz}{dy_\mu} = \frac{-z\{D_\mu(z) + d(y_\mu, z)\}}{y_\mu\{C_\mu(z) + c(y_\mu, z)\}},$$

where

$$\begin{aligned} &a(y_\mu, z),\ b(y_\mu, z) = O(z^{\varepsilon_1}),\\ &c(y_\mu, z) = c_1(y_\mu) + c_2(y_\mu, z),\quad d(y_\mu, z) = d_1(y_\mu) + d_2(y_\mu, z),\\ &c_1(y_\mu),\ d_1(y_\mu) = O(y_\mu^{\varepsilon_2}),\quad c_2(y_\mu, z),\ d_2(y_\mu, z) = z^{\varepsilon_1}\cdot O(y_\mu^{\varepsilon_2}), \end{aligned}$$

and

$$(26)\qquad \begin{aligned} &C_\mu(z) = K_\mu(z, 1) - \mu H_\mu(z, 1),\\ &D_\mu(z) = K_\mu(z, 1) - (\mu+1)H_\mu(z, 1). \end{aligned}$$

In the sequel, let us denote the differential equations (24) and (25) by (F_μ). Evidently the constant terms of $H_\mu(z, 1)$ and $K_\mu(z, 1)$ are, if any, the coefficients of the terms in (6) corresponding to V_p, because

$$(27)\qquad \begin{aligned} &H_\mu(z, 1) = \sum_{m=1}^{g} a_{\alpha'_m \beta'_m} z^{\zeta-1-\beta'_m},\\ &K_\mu(z, 1) = \sum_{n=1}^{h} b_{\gamma'_n \delta'_n} z^{\zeta-\delta'_n}. \end{aligned}$$

From (25), a lemma can be proved about integral curves of (E_μ) tending to the origin in the direction of the y_μ-axis as follows:

Lemma 2. *When $C_\mu(0)D_\mu(0) > 0$ or $D_\mu(z) \equiv 0$, the integral curve of (E_μ) tending to the origin in the direction of the y_μ-axis is only $x=0$.*

Proof. If we put $y_\mu = u^\lambda$ and $z = v^\lambda$ for any odd positive integer λ such that $\lambda\varepsilon_1, \lambda\varepsilon_2 \geqq 1$, then the equation (25) becomes equivalent to $u=0$, $v=0$ and

$$(28)\qquad \frac{dv}{du} = \frac{-v\{D_\mu(v^\lambda) + d(u^\lambda, v^\lambda)\}}{u\{C_\mu(v^\lambda) + c(u^\lambda, v^\lambda)\}}.$$

But $u=0$ and $v=0$ are evidently integral curves of (28), consequently (25) becomes equivalent to (28) itself.

In general, μ is not an integer, consequently $c(u^{\lambda}, v^{\lambda})$ and $d(u^{\lambda}, v^{\lambda})$ are not always defined in the 2nd and 4th quadrants of the (u, v)-plane. But, from the manner of choosing λ, it is evident that the denominator and the numerator of the right hand side of (28) are continuously differentiable with respect to u and v in the part $uv \geqq 0$ of a neighborhood of the origin.

Hence, when $C_{\mu}(0)D_{\mu}(0)>0$, since $u=0$ and $v=0$ are integral curves, by the same reasonings as in the case of a saddle-point, it is readily seen that the integral curves of (28) tending to the origin are only $u=0$ and $v=0$. Then it follows that the integral curves of (25) tending to the origin are only $z=0$ and $y_{\mu}=0$.

When $D_{\mu}(z)\equiv 0$, as is readily seen, V_p is an $X_{i_p j_p}$ and at the same time a $Y_{k_p l_p}$ and moreover $b_{k_p l_p}-(\mu+1)a_{i_p j_p}=0$. In this case, the value of ε_2 in (17) is $\geqq 1$ or $\geqq 2$, because μ is an integer or v_p-1. Hence, dividing the denominator and the numerator of the right hand of (25) by y_{μ}, we can write (25) as follows:

$$\frac{dz}{dy_{\mu}}=z\cdot f(y_{\mu}, z), \tag{29}$$

where

$$f(y_{\mu}, z)=\frac{d(y_{\mu}, z)y_{\mu}^{-1}}{C_{\mu}(z)+c(y_{\mu}, z)},$$

and

$$C_{\mu}(0)=b_{k_p l_p}-\mu a_{i_p j_p}=a_{i_p j_p}\neq 0.$$

Then, since $zf(y_{\mu}, z)$ is continuously differentiable with respect to z in the part $y_{\mu}z\geqq 0$ of a neighborhood of the origin, one and only one integral curve of (29) passes through the origin. But from the form of (29), this unique integral curve must be $z=0$. Consequently it follows that the integral curve tending to the origin is only $z=0$.

Thus, in either case, returning to the (x, y_{μ})-plane, we see that the integral curve of (E_{μ}) tending to the origin in the direction of the y_{μ}-axis is only $x=0$. This proves the lemma.

§ 3. Order and magnitude of a certain integral curve tending to the origin.

In this paragraph, corresponding to the vertices V_p's $(p=1, 2, \cdots, s+1)$ of the Newton polygon of (5), let us consider the integral curve $y=y(x)$ of (5) such that $y(x)=o(x^{m_{p-1}})$ as $x\to+0$, and let us study its behavior near the origin, making use of the subsidiary equation (E_{μ}) for μ such that $m_{p-1}\leqq\mu\leqq\mu_p-1$.

For the vertex V_p, there arise three cases as follows:

Case I. V_p is a certain Y_{kl} (this point is denoted by $Y_{k_p l_p}$) and not any X_{ij};

Case II. V_p is a certain X_{ij} (this point is denoted by $X_{i_p j_p}$.) and not any Y_{kl};
Case III. V_p is a certain X_{ij} and at the same a certain Y_{kl}.

Case I. V_p is a $Y_{k_p l_p}$ and not any X_{ij}.

At first, let us consider the vertex V_p for p such that $p \leqq s$. Then evidently $l_p \geqq 1$.

Now, for any $\mu < \mu_p - 1$, there is no marked point on $L_{\mu+1}$ other than V_p. Therefore, for (E_μ) with such μ, from (23) and (26), we have

$$F_\mu(x, y_\mu) = b_{k_p l_p} x y_\mu^{l_p}, \quad C_\mu(z) = D_\mu(z) = b_{k_p l_p}, \tag{30}$$

consequently it is evident that $C_\mu(0)D_\mu(0) = (b_{k_p l_p})^2 > 0$. Then the exceptional directions of (E_μ) at the origin are only $x=0$ and $y_\mu=0$ and, by virtue of lemma 2, the integral curve tending to the origin in the direction of the y_μ-axis is only $x=0$. Therefore, if an integral curve of (E_μ) other than $x=0$ tends to the origin, it must do so in the direction of the x-axis. In other words, if the integral curve $y=y(x)$ of (5) is such that $y(x)=o(x^\mu)$[1], then it must be that $y(x)=o(x^{\mu+1})$.

Now, we choose a positive integer n so that $\mu_p - 1 \leqq m_{p-1} + n < \mu_p$. Then, if $y=y(x)$ is an integral curve of (5) such that $y(x)=o(x^{m_{p-1}})$, it follows from the preceding result that $y(x)=o(x^{m_{p-1}+1}) = o(x^{m_{p-1}+2}) = \dots = o(x^{m_{p-1}+n})$, consequently that $y(x)=o(x^{\mu_p - 1})$. This says that the integral curve $y=y(x)$ such that $y(x)=o(x^{m_{p-1}})$ must be the one such that $y(x)=o(x^{\mu_p-1})$.

Now, for (E_μ) with $\mu=\mu_p-1$, the side V_pV_{p+1} of the Newton polygon lies on $L_{\mu+1}$, therefore, from (26) we have $C_\mu(0) = D_\mu(0) = b_{k_p l_p}$, consequently $C_\mu(0)D_\mu(0) = (b_{k_p l_p})^2 > 0$. Then, as before, the integral curve of (E_μ) tending to the origin in the direction of the y_μ-axis is only $x=0$. But, in this case, as is seen from (18) and (23), there may exist exceptional directions of (E_μ) at the origin other than $x=0$, namely it may happen that an integral curve tends to the origin in a certain exceptional direction $y_\mu = \rho x$. Thus we see that the integral curve $y=y(x)$ such that $y(x)=o(x^{\mu_p-1})$ must be the one such that $y(x) = \rho x^{\mu_p} + o(x^{\mu_p})$.

Then, combined with the preceding result, it is concluded that the integral curve $y=y(x)$ of (5) such that $y(x)=o(x^{m_{p-1}})$ $(p \leqq s)$ is either the one of order μ_p in x and of a finite non-zero magnitude or the one such that $y(x)=o(x^{\mu_p})$.

Next, let us consider the vertex V_{s+1}. For this vertex, it may happen that $l_{s+1}=0$, but, when $l_{s+1} \neq 0$, it must be that $l_{s+1}=1$, because otherwise $j \geqq 1$, which implies the existence of a common factor y between $X(x, y)$ and $Y(x, y)$.

When $l_{s+1}=1$, by the same reasonings as for the vertex V_p $(p \leqq s)$, it follows that the integral curve $y=y(x)$ such that $y(x)=o(x^{m_{p-1}})$ must be the one of order infinity in x.

When $l_{s+1}=0$, from (30), for $\mu=m_{p-1}$, we see that the exceptional

1) In the sequel, unless otherwise stated, $y(x)=o(x^\mu)$ implies $y(x)=o(x^\mu)$ as $x \to +0$.

direction of $(E_{m_{p-1}})$ is only $x=0$, consequently, by the same reasonings as for V_p $(p\leqq s)$, we see that there exists no integral curve $y=y(x)$ such that $y(x)=o(x^{m_{p-1}})$.

Case II. V_p **is an** $X_{i_pj_p}$ **and not any** Y_{kl}.

As in the preceding case, for (E_μ) with $\mu\leqq\mu_p-1$, we have $C_\mu(0)=-\mu a_{i_pj_p}$, $D_\mu(0)=-(\mu+1)a_{i_pj_p}$, consequently $C_\mu(0)D_\mu(0)=\mu(\mu+1)(a_{i_pj_p})^2>0$. While, in particular, for (E_μ) with $\mu<\mu_p-1$, it is valid that

$$F_\mu(x,y_\mu)=-(\mu+1)a_{i_pj_p}xy_\mu^{j_p+1}.$$

Therefore, by the same reasonings as for V_p $(p\leqq s)$ in the preceding case, there is obtained the same conclusion as that in **Case I** except for the case where $l_{s+1}=0$.

Case III. V_p **is an** $X_{i_pj_p}$ **and at the same time a** $Y_{k_pl_p}$.

In this case $l_p\geqq 1$, because $l_p=j_p+1$.

Now, for (E_μ) with $\mu\leqq\mu_p-1$, we have

$$C_\mu(0)=b_{k_pl_p}-\mu a_{i_pj_p},\quad D_\mu(0)=b_{k_pl_p}-(\mu+1)a_{i_pj_p};$$

$$B_\mu(y_\mu)=\begin{cases}(b_{k_pl_p}-\mu a_{i_pj_p})y_\mu^{l_p} & (\mu>\mu_{p-1})\\ (b_{k_pl_p}-\mu a_{i_pj_p})y_\mu^{l_p}+O(y_\mu^{l_p+1}) & (\mu=\mu_{p-1}>0)\end{cases}\tag{31}$$

In particular, for (E_μ) with $\mu<\mu_p-1$, it is valid that

$$F_\mu(x,y_\mu)=\{b_{k_pl_p}-(\mu+1)a_{i_pj_p}\}xy_\mu^{l_p}\qquad\text{when } B_\mu(y_\mu)\not\equiv 0,\tag{32}$$

$$F_\mu(x,y_\mu)=-a_{i_pj_p}y_\mu^{l_p}\qquad\text{when } B_\mu(y_\mu)\equiv 0.\tag{33}$$

From these formulas, it is evident, for (E_μ) with μ such that $\mu<\mu_p-1$ and $\mu\neq b_{k_pl_p}/a_{i_pj_p}-1$, $\neq b_{k_pl_p}/a_{i_pj_p}$, that $B_\mu(y_\mu)\not\equiv 0$ and that the exceptional directions at the origin are only $x=0$ and $y_\mu=0$.

In the sequel, for (E_μ) with $\mu\leqq\mu_p-1$, we shall attack the problem separately according to the magnitude of the quantity $b_{k_pl_p}/a_{i_pj_p}$.

1°. Case where $b_{k_pl_p}/a_{i_pj_p}>\mu_p$ **or** $<\mu_{p-1}$. In this case, evidently $b_{k_pl_p}/a_{i_pj_p}-1$ and $b_{k_pl_p}/a_{i_pj_p}$ are at the same time greater than or less than μ, consequently it is valid that $C_\mu(0)D_\mu(0)>0$. Then, by the same reasonings as for V_p $(p\leqq s)$, there are obtained the same conclusions as those of **Case I** except for the case where $l_{s+1}=0$.

2° Case where $\mu_p>b_{k_pl_p}/a_{i_pj_p}>\mu_{p-1}$. In this case, by definition $b_{k_pl_p}/a_{i_pj_p}=\nu_p$ and so, by lemma 1, it holds that

$$\mu_p-1>\nu_p\geqq\mu_{p-1}+2>m_{p-1}+1.$$

For (E_μ) with $\mu<\nu_p-1$, it is evident that $C_\mu(0)D_\mu(0)>0$ and that the exceptional directions at the origin are only $x=0$ and $y_\mu=0$. On the other hand, for (E_μ) with $\mu=\upsilon_p-1$, evidently $F_\mu(x, y_\mu)\equiv D_\mu(z)\equiv 0$. Consequently, for such an equation, all directions are exceptional directions at the origin and moreover, by lemma 2, the integral curve tending to the origin in the direction of the y_μ-axis is only $x=0$. Then, by the same reasonings as for V_p $(p\leqq s)$ in **Case I**, it is seen that the integral curve $y=y(x)$ of (5) such that $y(x)=o(x^{m_{p-1}})$ must be either the one of order υ_p in x and of a finite non-zero magnitude or the one such that $y(x)=o(x^{\nu_p})$. From this, it is evident that there does not exist any integral curve of (5) of order υ_p in x and of the magnitude infinity.

Now the integral curve of (5) of order υ_p in x and of magnitude ρ corresponds to the integral curve of (E_{υ_p}) tending to the point $(0, \rho)$ on the y_{υ_p}-axis. But, since $A_{\upsilon_p}(y_{\upsilon_p})=a_{i_pj_p}y_{\upsilon_p}^{j_p}$ and $B_{\upsilon_p}(y_{\upsilon_p})\equiv 0$, by (16), (E_{υ_p}) is written as follows.

$$\frac{dy_{\upsilon_p}}{dx}=\frac{C(x, y_{\upsilon_p})}{a_{i_pj_p}y_{\upsilon_p}^{j_p}+A(x, y_{\upsilon_p})}\,.$$

Evidently $A(x, y_{\upsilon_p})$ and $C(x, y_{\upsilon_p})$ are continuously differentiable with respect to x and y_{υ_p} in a neighborhood of any point $(0, \rho)$ $(\rho\neq 0)$, because $\varepsilon_1\geqq 2$ from the definition. Then, since the critical point lying on the y_{υ_p}-axis is only the origin, we see that, through any point $(0, \rho)$ $(\rho\neq 0)$, there passes one and only one integral curve, which evidently differs from $x=0$ as is seen from the form of (E_{υ_p}), This says that, for any number ρ $(\neq 0)$, there exists always one and only one integral curve of (5) of order υ_p in x and of magnitude ρ.

Lastly, let us consider the integral curve $y=y(x)$ such that $y(x)=o(x^{\nu_p})$. By (33), for (E_{υ_p}), the exceptional direction at the origin is only $y_{\nu_p}=0$. But, for (E_μ) with $\mu>\upsilon_p$, by (32), the exceptional directions at the origin consist of $x=0$ and $y_\mu=0$. Then, since $C_\mu(0)D_\mu(0)>0$ for (E_μ) with $\mu>\upsilon_p$, by the same reasonings as in **Case I**, it is seen that, when $p\leqq s$, the integral curve $y=y(x)$ of (5) such that $y(x)=o(x^{\nu_p})$ must be either the one of order μ_p in x and of a finite non-zero magnitude or the one such that $y(x)=o(x^{\mu_p})$ and that the integral curve $y=y(x)$ such that $y(x)=o(x^{\nu_{s+1}})$ must be the one of order infinity in x. From this, it is evident that there does not exist any integral curve of (5) of order υ_p in x and of the magnitude zero.

Thus, summarizing the above results, we have the conclusion that the integral curve $y=y(x)$ such that $y(x)=o(x^{m_{p-1}})$ must be the one of order υ_p in x and of a finite non-zero magnitude or otherwise, when $p\leqq s$, it must be either the one of order μ_p in x and of a finite non-zero magnitude or the one such that $y(x)=o(x^{\mu_p})$ and, when $p=s+1$, it must be the one of order infinity in x.

In addition, about the integral curves of order υ_p in x, there holds

Theorem 1. *When υ_p exists, for any number ρ $(\neq 0)$, there exists always one and only one integral curve of (5) of order υ_p in x and of magnitude ρ, and there does not exist any integral curve of order υ_p in x and of the magnitude either zero or infinity.*

3°. Case where $b_{k_pl_p}/a_{i_pj_p}=\mu_p$. In this case, evidently it cannot be that

$p=s+1$, because $\mu_{s+1}=\infty$.

Now, for (E_μ) with $\mu<\mu_p-1$, evidently $b_{k_pl_p}/a_{i_pj_p}$ and $b_{k_pl_p}/a_{i_pj_p}-1$ are both greater than μ, consequently $C_\mu(0)D_\mu(0)>0$. Then, as for V_p $(p\leqq s)$ in **Case I**, it follows that, for an integral curve $y=y(x)$ of (5), $y(x)=o(x^{\mu})$ implies $y(x)=o(x^{\mu+1})$, consequently that, for an integral curve $y=y(x)$, $y(x)=o(x^{m_{p-1}})$ implies $y(x)=o(x^{\mu_p-1})$. Then, if $y=y(x)$ is an integral curve such that $y(x)=o(x^{m_{p-1}})$, for any v such that $\mu_p>v\geqq\mu_p-1$, it is valid that $y(x)=o(x^{\mu_p-1})=o(x^{v-1})=o(x^v)$. This says that, for an integral curve $y=y(x)$, $y(x)=o(x^{m_{p-1}})$ implies $y(x)=o(x^v)$ for any $v<\mu_p$.

Now, for $\mu=\mu_p-1$, as for $V_p(p\leqq s)$ in **Case I**, an integral curve of (E_μ) tending to the origin in the exceptional direction different from the y_μ-axis yields an integral curve $y=y(x)$ of (5) such that $y(x)$ is of the form $\rho x^{\mu_p}+o(x^{\mu_p})$. While, an integral curve of (E_μ) tending to the origin in the direction of the y_μ-axis yields an integral curve of (5) of order μ_p in x and of the magnitude infinity, provided the former does not coincide with the y_μ-axis. Because, for such an integral curve $y=y(x)$ of (5), as $x\to+0$, it holds that $|y(x)|/x^{\mu_p}\to\infty$ and, as is proved at first, also that $y(x)/x^v\to0$ for any $v<\mu_p$.

Thus we have the conclusion that the integral curve $y=y(x)$ of (5) such that $y(x)=o(x^{m_{p-1}})$ must be the one such that $y(x)=o(x^{\mu_p})$ or otherwise must be either the one of order μ_p in x and of a finite non-zero magnitude or the one of order μ_p in x and of the magnitude infinity.

Remark. When $D_{\mu_p-1}(z)\equiv0$, there does not exist any integral curve of order μ_p in x and of the magnitude infinity, for, as is seen from lemma 2, except for $x=0$, there does not exist any integral curve of (E_{μ_p-1}) tending to the origin in the direction of the y_{μ_p-1}-axis.

4°. **Case where** $b_{k_pl_p}/a_{i_pj_p}=\mu_{p-1}$. In this case, as a matter of course, $p\geqq2$, because $b_{k_pl_p}/a_{i_pj_p}\neq0$ and $\mu_0=0$. Therefore it is evident that $m_{p-1}=\mu_{p-1}$.

Since $b_{k_pl_p}-(\mu_{p-1}+1)a_{i_pj_p}\neq0$ from the assumption, for $(E_{\mu_{p-1}})$, by (32) and (33), the exceptional directions at the origin are either $x=0$ and $y_{\mu_{p-1}}=0$ or only $y_{\mu_{p-1}}=0$ according as $B_{\mu_{p-1}}(y_{\mu_{p-1}})\not\equiv0$ or $\equiv0$.

Now, for (E_μ) with $\mu>\mu_{p-1}$, evidently $b_{k_pl_p}/a_{i_pj_p}-1$ and $b_{k_pl_p}/a_{i_pj_p}$ are both less than μ, consequently $C_\mu(0)D_\mu(0)>0$. Therefore, by the same reasonings as in **Case I**, it is seen that, when $p\leqq s$, the integral curve $y=y(x)$ of (5) corresponding to the integral curve of $(E_{\mu_{p-1}})$ which tends to the origin in the direction of the x-axis must be either the one such that $y(x)=o(x^{\mu_p})$ or the one of order μ_p in x and of a finite non-zero magnitude, and that, when $p=s+1$, such an integral curve must be the one of order infinity in x.

On the contrary, if any, the integral curve l of $(E_{\mu_{p-1}})$ which differes from $x=0$ and tends to the origin in the positive direction of the $y_{\mu_{p-1}}$-axis yields an integral curve $y=y(x)$ such that $y(x)/x^{\mu_{p-1}+1}\to\infty$ as $x\to+0$.

Now, if we mark the points $(k+1,\ l-1)$ and $(i,\ j)$ in the $(\xi,\ \eta)$-plane corresponding to the terms $b_{kl}y^lx^k$ and $a_{ij}y^jx^i$, and consider the Newton polygon of the differential equation

$$\frac{dx}{dy}=\frac{X(x,y)}{Y(x,y)}, \tag{34}$$

considering x as a function of y, then it is readily seen that this polygon and the Newton polygon of (5) are symmetric with respect to the line $\eta=\xi-1$. Therefore evidently the slopes of the sides of the Newton polygon of (34) can be expressed by $\mu_j'=1/\mu_j$ $(j=1, 2, \cdots, s)$. For convenience, let us put $\mu_0'=\infty$, $\mu_{s+1}'=0$ and also $m_{p-1}'=\max(\mu_{p-1}',\delta)$, where δ is a sufficiently small positive number less than $1/(\mu_{p-1}+1)$.

Although μ_p''s are not integers, the relations of order between x and y obtained in the preceding cases are also valid for these μ_p''s without any altering, for such relations are invariant for the transformation of the variable used in the proof of lemma 1 in order to let the inclinations have integral values. Then, by the results in case $3°$, it is seen that, for an integral curve $x=x(y)$ of (34), $x(y)=o(y^{m_p'})$ implies $x(y)=o(y^{v})$ for any $v<\mu_{p-1}'$. But, since $1/(\mu_{p-1}+1)>\delta=m_p'$ when $p=s+1$ and $1/(\mu_{p-1}+1)>1/\mu_p=\mu_p'=m_p'$ when $p\leqq s$, it holds that $1/(\mu_{p-1}+1)>m_p'$. Therefore it is evident that, for an integral curve $x=x(y)$ of (34) such that $x(y)=o(x^{1/(\mu_{p-1}+1)})$, there holds $x(y)=o(y^{v})$ for any $v<\mu_{p-1}'$. This says that, for an integral curve $y=y(x)$ of (5) such that $y(x)/x^{\mu_{p-1}+1}\to\infty$ as $x\to+0$, it holds that $y(x)/x^{\mu}\to\infty$ for any $\mu>\mu_{p-1}$ as $x\to+0$. Now, for the integral curve l, it evidently holds that $y(x)/x^{\mu_{p-1}}\to 0$ as $x\to+0$. Therefore it is seen that the integral curve l yields an integral curve of (5) of order μ_{p-1} in x and of the magnitude zero. If we replace y by $-y$ in (5), the same result is also obtained for the integral curve of $(E_{\mu_{p-1}})$ which differs from $x=0$ and tends to the origin in the negative direction of the $y_{\mu_{p-1}}$-axis.

Thus we see that the integral curve $y=y(x)$ of (5) such that $y(x)=o(x^{\mu_{p-1}})$ $(p\neq 1)$ must be the one of order μ_{p-1} in x and of the magnitude zero or otherwise, when $p\leqq s$, it must be either the one of order μ_p in x and of a finite non-zero magnitude or the one such that $y(x)=o(x^{\mu_p})$ and, when $p=s+1$, it must be the one of order infinity in x.

Remark. When $B_{\mu_{p-1}}(y_{\mu_{p-1}})\equiv 0$, there exists no integral curve of (5) of order μ_{p-1} in x and of the magnitude zero, because the direction $x=0$ is not an exceptional direction of $(E_{\mu_{p-1}})$ at the origin.

The preceding results of this paragraph are summarized as follows:

The integral curve $y=y(x)$ such that $y(x)=o(x^{m_{p-1}})$ must, if any, be one of the following types:

1. *the integral curve of order μ_p in x and of a finite non-zero magnitude* $(p\leqq s)$,
2. 〃 *of order μ_p in x and of the magnitude infinity* $(p\leqq s)$,
3. 〃 *of order v_p in x and of a finite non-zero magnitude* $(p\leqq s+1)$,
4. 〃 *of order μ_{p-1} in x and of the magnitude zero* $(p\geqq 2)$,

5. 〃 *of order infinity in* x $(p=s+1)$,

6. 〃 $y=y(x)$ *such that* $y=o(x^{\mu_p})$ $(p\leqq s)$.

The integral curve of the 2nd type may appear only when $\mu_p=b_{k_p l_p}/a_{i_p j_p}$ *and* $D_{\mu_p-1}(z)\neq 0$.

The integral curve of the 3rd type always appears when v_p *exists and there always exists such an integral curve of the arbitrary magnitude* ρ $(\neq 0)$.

The integral curve of the 4th type may appear only when $\mu_{p-1}=b_{k_p l_p}/a_{i_p j_p}$ *and* $B_{\mu_{p-1}}(y_{\mu_{p-1}})\neq 0$.

From this result, taking $p=1,2,\cdots,s+1$ successively, we have a conclusion: the integral curve $y=y(x)$ of (5) such that $y(x)=o(x^{\varepsilon})$ for any small $\varepsilon>0$ must, if any, be either of order μ_p $(p\leqq s)$ in x or of order v_q $(q\leqq s+1)$ in x or otherwise be of order infinity in x.

Now, if x is replaced by $-x$ in (5), there is obtained a differential equation

$$\frac{dy}{dx}=\frac{Y(-x,y)}{-X(-x,y)}\,. \tag{35}$$

This equation is (5) itself but with a_{ij} and b_{kl} replaced by $\bar{a}_{ij}=(-1)^{i+1}a_{ij}$ and $\bar{b}_{kl}=(-1)^k b_{kl}$ respectively. Consequently the Newton polygon of (35) coincides with that of (5) and moreover, for any vertex V_q which is an $X_{i_q j_q}$ and at the same time a $Y_{k_q l_q}$, it holds that

$$\bar{b}_{k_q l_q}/\bar{a}_{i_q j_q}=(-1)^{k_q}b_{k_q l_q}/(-1)^{i_q+1}a_{i_q j_q}=b_{k_q l_q}/a_{i_q j_q},$$

because $k_q=i_q-1$. Therefore the values of μ_p's and v_q's for (34) determined by its Newton polygon are same as those for (5). Hence the same conclusion is valid for the integral curve $y=y(x)$ of (35) such that $y(x)=o(x^{\varepsilon})$ as $x\to+0$, in other words, is valid for the integral curve $y=y(x)$ of (5) such that $y(x)=o(|x|^{\varepsilon})$ as $x\to-0$.

Now, as is remarked in **2.1**, the integral curve of (5) tending to the origin tends to the origin either spiraling to the origin or in a certian exceptional direction. While, the integral curve of (5) tending to the origin in the direction $y=\rho x$ $(x\geqq 0)$ is expressed as $y=y(x)=\rho x+o(x)$ (as $x\to+0$) and that tending to the origin in the direction $x=0$ is expressed as $x=x(y)=o(y)$ (as $y\to+0$ or -0). By the preceding conclusion, the integral curve $y=y(x)$ such that $y(x)=\rho x+o(x)$ must be either of order μ_p $(p\leqq s)$ in x or of order v_q $(q\leqq s+1)$ in x or otherwise be of order infinity in x. For the integral curve $x=x(y)$ such that $x(y)=o(y)$, if we interchange the roles of x and y, from the preceding discussions on (34), it follows that this integral curve must be either of order $\mu'_p=1/\mu_p$ $(p\leqq s)$ or of order $v'_q=1/v_q$ $(q\leqq s+1)$ in y or otherwise be of order infinity in y. This implies that the integral curve $x=x(y)$ such that $x(y)=o(y)$ must be the integral curve $y=y(x)$ either of order μ_p $(p\leqq s)$ in x or of order v_q $(q\leqq s+1)$ in x or otherwise be the integral curve $x=x(y)$ of order infinity in y.

Thus, summarizing these, we have

Thorem 2. *The integral curve of (5) tending to the origin in a fixed direction must, if any, be one of the following types:*

1. *that of order* μ_p $(p \leqq s)$ *in* x *and of a certain magnitude,*
2. *that of order* v_q $(q \leqq s+1)$ *in* x *and of a finite non-zero magnitude,*
3. *that of order infinity in* x,
4. *that of order infinity in* y.

§ 4. The integral curve of order infinity.

First, let us consider an integral curve of (5) of order infinity in x. By the results of the preceding paragraph, we can suppose that such an integral curve is the one yielded by an integral curve of (E_μ) tending to the origin in the direction of the x-axis for μ which is an integer greater than m_s and further greater than v_{s+1} if this exists. Consequently, taking the vertex V_{s+1} of the Newton polygon of (5), let us seek for an integral curve of (E_μ) tending to the origin in the direction of the x-axis for the above-mentioned value of μ.

When V_{s+1} is a vertex of **Case I**, as we have already seen, it must be that $l_{s+1}=0$ or $l_{s+1}=1$.

When $l_{s+1}=0$, namely when $Y(x, 0) \neq 0$, there exists no integral curve of (5) of order infinity in x.

When $l_{s+1}=1$, it must be that $X(x, 0) \neq 0$, for otherwise $X(x, y)$ and $Y(x, y)$ would have a common factor y. Then $X(x, 0)$ and $Y_y(x, 0)$ can be written as follows:

$$X(x, 0)=a_{\alpha 0}x+o(x^\alpha),$$
$$Y_y(x, 0)=b_{\beta 1}x+o(x^\beta),$$

where $\beta=k_{s+1}$ and $\alpha>\beta+1$. Then, by the definition,

$$A_\mu(y_\mu)\equiv 0, \quad B_\mu(y_\mu)=b_{\beta 1}y_\mu,$$
$$A(x, y_\mu)=a_{\alpha 0}x^N+o(x^N)+O(y_\mu), \quad B(x, y_\mu)=y_\mu\cdot O(x),$$

where $N=\alpha-\beta-1 \geqq 1$.

Now, since μ is an integer, $A(x, y_\mu)$ and $B(x, y_\mu)$ are analytic in x and y_μ at the origin. Hence, applying Keil's theorem, we see that, when $a_{\alpha 0}b_{\beta 1}<0$, the integral curve of (E_μ) tending to the origin in the positive direction of the x-axis is only $y_\mu=0$, and, when $a_{\alpha 0}b_{\beta 1}>0$, all the integral curves lying on the right side of the y_μ-axis tend to the origin in the direction of the x-axis.

This says that, when $a_{\alpha 0}b_{\beta 1}<0$, the integral curve of (5) of order infinity in x is only $y=0$ and that, when $a_{\alpha 0}b_{\beta 1}>0$, infinitely many such integral curves really exist.

When V_{s+1} is a vertex of **Case II**, it must be that $j_{s+1}=0$, because otherwise $l \geqq 2$, which implies the existence of the common factor between $X(x, y)$ and $Y(x, y)$. Further it must be that $Y(x, 0)\equiv 0$, because $l \geqq 1$. Thus we see that

$$A_\mu(y_\mu)=a_{i_p j_p}, \quad B_\mu(y_\mu)=-\mu a_{i_p j_p}y_\mu, \quad B(x, y)=y_\mu\cdot O(x) \quad (p=s+1).$$

Then, since μ is an integer, $A(x, y_\mu)$ and $B(x, y_\mu)$ become analytic in x and y_μ at the origin, consequently, for the equation (15), the origin of the (x, y_μ)-plane becomes a saddle-point and only the integral curves $x=0$ and $y_\mu=0$ tend to the origin. This says that the integral curve of (5) of order infinity in x is only $y=0$.

When V_{s+1} is a vertex of **Case III**, as above, it must be that $j_{s+1}=0$, consequently also that $l_{s+1}=1$. Then we see that

$$A_\mu(y_\mu)=a_{i_pj_p},\quad B_\mu(y_\mu)=(b_{k_pl_p}-\mu a_{i_pj_p})y_\mu,\quad B(x, y_\mu)=y_\mu\cdot O(x)\quad (p=s+1),$$

because $Y(x, 0)\equiv 0$.

Now, $(b_{k_pl_p}-\mu a_{i_pj_p})a_{i_pj_p}<0$, as is seen from the way of choosing μ. Therefore, as in the former case, for the equation (15), the origin of the (x, y_μ)-plane becomes a saddle-point and only the integral curves $x=0$ and $y_\mu=0$ can tend to the origin. This says that the integral curve of (5) of order infinity in x is only $y=0$.

Thus there is obtained

Theorem 3. *When $Y(x, 0)\not\equiv 0$, there exists no integral curve of* (5) *of order infinity in x. When $Y(x, 0)\equiv 0$, if $X(x, 0)$ and $Y_y(x, 0)$ are written as*

$$X(x, 0)=a_{\alpha 0}x+o(x^\alpha),\quad Y_y(x, 0)=b_{\beta 1}x^\beta+o(x^\beta)$$

and it holds that

$$\alpha>\beta+1 \text{ and } a_{\alpha 0}b_{\beta 1}>0,$$

then there exist really infinitely many integral curves of (5) *of order infinity in x. But, when $Y(x, 0)\equiv 0$, if any one of the above conditions is not satisfied, then $y=0$ is a unique integral curve of* (5) *of order infinity in x.*

Since the variables x and y are mutually interchangeable, the analogous theorem is evidently valid for the integral curves of (5) of order infinity in y, but it is omitted here for brevity.

§ 5. The integral curves of order μ_p in x.

1°. The integral curve of order μ_p in x and of the magnitude infinity.

By the result of § **3**, the integral curves of (5) of order μ_p in x and of the magnitude infinity can appear only as the one yielded by an integral curve of (E_{μ_p-1}) tending to the origin in the direction of the y_{μ_p-1}-axis for the vertex V_p $(p\geqq s)$ of **Case III, 3°** such that $D_{\mu_p-1}(z)\not\equiv 0$. But, by the reasonings of § **2**, such an integral curve of (E_{μ_p-1}) is yielded by the one of (F_{μ_p-1}) tending to the origin. So, in this section, let us seek for such an integral curve of (F_{μ_p-1}).

For brevity, put $\mu_p-1=\mu$. Then, comparing (14) with (26), we see that

$$D_\mu(z)=z^{l_p}B_{\mu_p}(1/z), \tag{36}$$

from which follows that $B_{\mu_p}(y_{\mu_p})\not\equiv 0$ because $D_\mu(z)\not\equiv 0$. Put

$$B_{\mu_p}(y_{\mu_p})=c_0y_{\mu_p}^\beta+c_1y_{\mu_p}^{\beta-1}+\cdots+c_\beta\qquad (c_0\neq 0),$$

then, from

$$\mu=b_{k_pl_p}/a_{i_pj_p}, \tag{37}$$

follows that $\beta<l_p$, and, from (36) follows

$$D_\mu(z) = c_0 z^\sigma + o(z^\sigma),$$

where $\sigma = l_p - \beta > 0$. On the other hand, due to (37), from (26), follows

$$C_\mu(z) = a_{i_p j_p} + O(z).$$

Thus the equation (F_μ) becomes that of the form as follows:

$$\frac{dz}{dy_\mu} = \frac{-z\{c_0 z^\sigma + o(z^\sigma) + d(y_\mu, z)\}}{y_\mu\{a_{i_p j_p} + O(z) + c(y_\mu, z)\}}.$$

Now, since μ is an integer, $c(y_\mu, z)$ and $d(y_\mu, z)$ are analytic in y_μ and z at the origin. Hence, by applying Keil's theorem to the above equation, we have the assertion as follows:

the integral curve of (F_μ) tending to the origin in the direction of the y_μ-axis is only $z=0$;

when σ is even, if $c_0 a_{i_p j_p} < 0$, all the integral curves of (F_μ) except for $z=0$ tend to the origin in the direction of the z-axis and, if $c_0 a_{i_p j_p} > 0$, the integral curves of (F_μ) tending to the origin are only $z=0$ and $y_\mu = 0$;

when σ is odd, if $c_0 a_{i_p j_p} < 0$, all the integral curves lying above the y_μ-axis tend to the origin in the direction of the z-axis and only the integral curve $y_\mu = 0$ can tend to the origin from below the z-axis; when σ is odd, if $c_0 a_{i_p j_p} > 0$, the integral curves behave in the same manner but the sides of the z-axis are interchanged.

If we return to the equation (E_μ), from the above assertion, there is obtained

Theorem 4. *The integral curve of order μ_p in x and of the magnitude infinity can exist only when*

$$\mu_p = b_{k_p l_p} / a_{i_p j_p}$$

and $B_{\mu_p}(y_{\mu_p})$ is of the form

$$B_{\mu_p}(y_{\mu_p}) = c_0 y_{\mu_p}^{\beta} + c_1 y_{\mu_p}^{\beta-1} + \cdots + c_\beta \qquad (c_0 \neq 0,\ \beta < l_p).$$

When $\sigma = l_p - \beta$ is even, if $c_0 a_{i_p j_p} < 0$, such integral curves exist infinitely many on both sides of the x-axis, but, if $c_0 a_{i_p j_p} > 0$, such an integral curve does not exist.

When $\sigma = l_p - \beta$ is odd, such integral curves exist infinitely many only on the upper or lower side of the x-axis according as $c_0 a_{i_p j_p} < 0$ or > 0 and there exists none on the opposite side.

2°. The integral curve of order μ_p in x and of the magnitude zero.

By the result of § **3**, the integral curve of (5) of order μ_p in x and of the magnitude zero can appear only as the one yielded by an integral curve of (E_μ) tending to the origin in the direction of the y_{μ_p}-aixs for the vertex V_{p+1} $(1 \leqq p \leqq s)$ of **Case III, 4°** such that $B_{\mu_p}(y_{\mu_p}) \not\equiv 0$. But, by the reasonings of § **2**, such an integral curve of (E_{μ_p}) is yielded by the one of (F_{μ_p}) tending to the origin. So, in this section, let us seek for such an integral curve of (F_{μ_p}).

For brevity, put $\mu=\mu_p$. Then, from

$$b_{k_{p+1}l_{p+1}}/a_{i_{p+1}j_{p+1}}=\mu,$$

$A_\mu(y_\mu)$ and $B_\mu(y_\mu)$ become the polynomials of the forms as follows:

$$A_\mu(y_\mu)=a_{i_{p+1}j_{p+1}}y_\mu^{j_{p+1}}+o(y_\mu^{j_{p+1}}),$$
$$B_\mu(y_\mu)=o(y_\mu^{l_{p+1}}).$$

Now, on account of $B_\mu(y_\mu)\neq 0$, we can put

$$B_\mu(y_\mu)=d_0y_\mu^\gamma+o(y_\mu^\gamma) \qquad (d_0\neq 0,\ \gamma>l_{p+1}).$$

Then, since

$$\{y_\mu A_\mu(y_\mu)-B_\mu(y_\mu)\}y_\mu^{-l_{p+1}}=a_{i_{p+1}j_{p+1}}+O(y_\mu),$$

and

$$B_\mu(y_\mu)y_\mu^{-l_{p+1}}=d_0y_\mu^\delta+o(y_\mu^\delta) \qquad (\delta=\gamma-l_{p+1}),$$

by (24), the equation (F_μ) becomes that of the form as follows:

$$\frac{dz}{dy_\mu}=\frac{z\{a_{i_{p+1}j_{p+1}}+O(y_\mu)+a(y_\mu, z)\}}{y_\mu\{d_0y_\mu^\delta+o(y_\mu^\delta)+b(y_\mu, z)\}}.$$

But, since μ is an integer, $a(y_\mu, z)$ and $b(y_\mu, z)$ are analytic in y_μ and z at the origin. Hence, if we apply Keil's thorem to the above equation, there is obtained the assertion as follows:

the integral curve of (F_μ) tending to the origin in the direction of the z-axis is only $y_\mu=0$;

when δ is even, if $d_0a_{i_{p+1}j_{p+1}}>0$, all the integral curves of (F_μ) except for $y_\mu=0$ tend to the origin in the direction of the y_μ-axis and, if $d_0a_{i_{p+1}j_{p+1}}<0$, the integral curves of (F_μ) tending to the origin are only $z=0$ and $y_\mu=0$;

when δ is odd, if $d_0a_{i_{p+1}j_{p+1}}>0$, all the integral curves lying on the right side of the z-axis tend to the origin in the direction of the y_μ-axis and only the integral curve $z=0$ can tend to the origin on the left side of the z-axis; when δ is odd, if $d_0a_{i_{p+1}j_{p+1}}<0$, the integral curves behave in the same manner but the sides of the z-axis are interchanged.

From this assertion, returning to the equation (E_μ), we obtain

Theorem 5. *The integral curve of (5) of order μ_p in x and of the magnitude zero can exist only when*

$$\mu_p=b_{k_{p+1}l_{p+1}}/a_{i_{p+1}j_{p+1}}$$

and $B_{\mu_p}(y_{\mu_p})$ is of the form

$$B_{\mu_p}^\gamma(y_{\mu_p})=d_0y_{\mu_p}^\gamma+o(y_{\mu_p}^\gamma) \qquad (d_0\neq 0,\ \gamma>l_{p+1}).$$

When $\delta=\gamma-l_{p+1}$ is even, if $d_0a_{i_{p+1}j_{p+1}}>0$, such integral curves exist infinitely many on both sides of the x-axis, but, if $d_0a_{i_{p+1}j_{p+1}}<0$, such an integral aurve does not exist.

When $\delta=\gamma-l_{p+1}$ is odd, such integral curves exist infinitely many only on the upper or lower side of the x-axis according as $d_0a_{i_{p+1}j_{p+1}}>0$ or <0 and there exists none on the opposite side.

3°. The integral curve of order μ_p in x and of a finite non-zero magnitude.

The integral curve $y=y(x)$ of order μ_p in x and of a finite non-zero magnitude is also evidently that of the same type even when the transformation $w=x^{1/\lambda_0}$ used in the proof of lemma 1 is not acted. So, in this section, we suppose that the equation (5) is the initial equation itself, namely the equation for which the transformation $w=x^{1/\lambda_0}$ is not yet acted. Consequently, of course, in this section, the values of μ_p's are not necessarily integers.

Since the value of μ_p is a rational number, μ_p can be exprssed as q_1/p_1 with positive integers p_1 and q_1. Then the substitution of $x=t_1^{p_1}$ and $y=u\,t_1^{q_1}$ into (5) entails

$$\frac{du}{dt_1}=\frac{p_1B_{\mu_p}(u)+t_1g(t_1, u)}{t_1\{A_{\mu_p}(u)+t_1f(t_1, u),\}}\,. \tag{38}$$

It is evident that the integral curve of (5) of order μ_p in x and of a finite non-zero magnitude ρ is yielded by the integral curve of (38) tending to the point $(0, \rho)$ as $t_1\to+0$.

When $B_{\mu_p}(u)\not\equiv 0$, let the non-zero real roots of $B_{\mu_p}(u)=0$, if any, be ρ_j's $(j=1, 2, \cdots, f)$, and let the multiplicty of ρ_j be κ_j. Then, as is seen from the form of (38), the integral curve tending to any point $(0, \rho)$ such that $\rho\neq 0$, $\neq\rho_j$'s is only $t_1=0$, in other words, there exists no integral curve of (5) of order μ_p in x and of magnitude ρ such that $\rho\neq 0$, $\neq\rho_j$'s.

But, for $\rho=\rho_j$, when $A_{\mu_p}(\rho_j)\neq 0$, the equation (38) becomes

$$\frac{dv}{dt_1}=\frac{\dfrac{1}{\kappa_j!}\,p_1B_{\mu_p}^{(\kappa_j)}(\rho_j)v^{\kappa_j}+o(v^{\kappa_j})+t_1g(t_1, \rho_j+v)}{A_{\mu_p}(\rho_j)t_1+t_1\cdot O(v)+t_1^2f(t_1, \rho_j+v)}\,,$$

where $u=\rho_j+v$. Then, when $\kappa_j=1$, as the origin is a simple critical point, about the integral curves of this equation, we have the assertion as follows:

if $A_{\mu_p}(\rho_j)B'_{\mu_p}(\rho_j)>0$, all the integral curves lying on the right side of the v-axis tend to the origin;

if $A_{\mu_p}(\rho_j)B'_{\mu_p}(\rho_j)<0$, one and only one integral curve tends to the origin in the direction $\{p_1B'_{\mu_p}(\rho_j)-A_{\mu_p}(\rho_j)\}v-g(0,\rho_j)t_1=0$ and any other integral curve lying on the right side of the v-axis does not tend to the origin.

When $\kappa_j\geqq 2$, the behavior of integral curves is known by Keil's theorem as follows:

only one integral curve tends to the origin in the direction $A_{\mu_p}(\rho_j)v-g(0, \rho_j)t_1=0$ (we shall denote this integral curve by $v=v_0(t_1)$);

when κ_j is odd, if $A_{\mu_p}(\rho_j)B_{\mu_p}^{(\kappa)}(\rho_j)<0$, only the integral curve $t_1=0$ tends to the origin on each side of the integral curve $v=v_0(t_1)$ and, if $A_{\mu_p}(\rho_j)B_{\mu_p}^{(\kappa)}(\rho_j)>0$, all the integral curves tend to the origin on both sides of the integral curve $v=v_0(t_1)$;

when κ_j is even, if $A_{\mu_p}(\rho_j)B_{\mu_p}^{(\kappa)}(\rho_j)<0$, only the integral curve $t_1=0$ tends to the origin on the upper side of the integral curve $v=v_0(t_1)$ and all the integral curves tend to the origin on the lower side of the integral curve $v=v_0(t_1)$; if $A_{\mu_p}(\rho_j)B_{\mu_p}^{(\kappa)}(\rho_j)>0$, the behaviors of the integral curves on each side of the integral curve $v=v_0(t_1)$ are interchanged.

From this result is known the behavior of the integral curves of (5) of order μ_p in x and of magnitude ρ_j such that $A_{\mu_p}(\rho_j) \neq 0$.

When $B_{\mu_p}(u) \equiv 0$, the equation (38) becomes

$$\frac{du}{dt_1} = \frac{g(t_1, u)}{A_{\mu_p}(u) + t_1 f(t_1, u)}. \tag{39}$$

But, as is readily seen from the definition, $A_{\mu_p}(u)$ and $B_{\mu_p}(u)$ cannot vanish identically at the same time, so, at present, $A_{\mu_p}(u) \not\equiv 0$. Hence let the non-zero real roots of $A_{\mu_p}(u)=0$, if any, be σ_i's $(i=1, 2, \cdots, g)$. Then, since the point $(0, \rho)$ such that $\rho \neq 0, \neq \sigma_i$'s is an ordinary point of (39), there exists one and only one integral curve of (39) passing through such a point. Evidently such an integral curve is distinct from $t_1=0$. This says that, for any ρ such that $\rho \neq 0, \neq \sigma_i$'s, there exists one and only one integral curve of (5) of order μ_p in x and of magnitude ρ.

Thus, summarizing the above results, we obtain

Theorem 6. *The non-zero value ρ such that $B_{\mu_p}(\rho) \neq 0$ cannot be a magnitude of the integral curve of (5) of order μ_p in x.*

For the non-zero value ρ such that $B_{\mu_p}(\rho)=0$ but $A_{\mu_p}(\rho) \neq 0$, when $B_{\mu_p}(u) \equiv 0$, there exists one and only one integral curve of (5) of order μ_p in x and of magnitude ρ, and when $B_{\mu_p}(u) \not\equiv 0$, if we denote the multiplicity of the root ρ by κ, the portrait of the integral curves of (5) of order μ_p in x and of magnitude ρ is expressed as follows:

when κ is odd and $A_{\mu_p}(\rho)B^{(\kappa)}_{\mu_p}(\rho)<0$, there exists one and only one integral curve of order μ_p in x and of magnitude ρ;

when κ is odd and $A_{\mu_p}(\rho)B^{(\kappa)}_{\mu_p}(\rho)>0$, there exist infinitely many such integral curves;

when κ is even, there exists one and only one integral curve—say Γ—of the form

$$y=y(x)=(\rho+v_1(x))x^{\mu_p}, \tag{40}$$

where $\quad v_1(x)=O(x^{1/p_1}) \quad$ *as* $x \to +0 \quad (\mu_p=q_1/p_1)$;

if $A_{\mu_p}(\rho)B^{(\kappa)}_{\mu_p}(\rho)<0$, on the upper side of Γ, there does not exist any integral curve of order μ_p in x and of maguitude ρ, but, on the lower side of Γ, there exist infinitely many such integral curves, all of which are of the form

$$y=y(x)=(\rho+v_2(x))x^{\mu_p}, \tag{41}$$

where $\quad v_2(x) \to 0 \quad$ *but* $\quad |v_2(x)|/x^{1/p_1} \to \infty \quad$ *as* $x \to +0$;

if $A_{\mu_p}(\rho)B^{(\kappa)}_{\mu_p}(\rho)>0$, the portraits of the integral curves on one side of Γ are mutually interchanged with those on the other side.

For the non-zero value ρ such that $B_{\mu_p}(\rho)=0$ and $A_{\mu_p}(\rho)=0$, if we put $u=\rho+v$, then (38) or (39) becomes the equation of the same form as (5), but, in this case, it may happen that the origin is not a critical point.

When the origin is not a critical point of the equation with respect to v, the behavior of the integral curves of that equation, consequently of the equation (5) is known readily.

When the origin is a critical point of the equation with respect to v, we can construct again the Newton polygon of that equation. Then, by theorems 1–6, we can know the behavior of integral curves of that equation tending to the origin in the fixed directions as $t_1 \to +0$, except for those for which $B_{\mu_p}(u) = 0$ and $A_{\mu_p}(u)=0$ have real non-zero common roots. For such integral curves, we again repeat the above process. Then, as is shown by M. Urabe [6], in a finite number of steps, the initial equation becomes that of either of the forms:

$$\frac{dv}{dt} = \frac{av+pt+\cdots}{t^{\kappa}}\,; \qquad \kappa \geqq 2,\ a \neq 0, \tag{42}$$

$$\frac{dv}{dt} = \frac{v^{\lambda}(av^n+pt^m+\cdots)}{t}\,; \qquad a,\ p \neq 0,\ n,\ \lambda \geqq 0,\ m>0, \text{ and } \lambda \geqq 1 \text{ when } n=0. \tag{43}$$

The behaviors of the integral curves of (42) is readily known by Keil's theorem as follows:

the integral curves tending to the origin in the direction of the v-axis is only $t=0$; when $a<0$, one and only one integral curve tends to the origin in the direction $av+pt=0$ and any other integral curve lying on the right side of the v-axis does not tend to the origin; when $a>0$, all the integral curves lying on the right side of the v-axis tend to the origin in the direction $av+pt=0$.

By the same reasonings as in the proof of theorem 6, it is seen that the behavior of integral curves of (43) is as follows:

when $n+\lambda$ is odd, if $a<0$, one and only one integral curve tends to the origin on the right side of the v-axis and, if $a>0$, all the integral curves lying on the right side of the v-axis tend to the origin;

when $n+\lambda$ is even, there exists one and only one integral curve of the form $v=v(t)=O(t)$ and, if $a<0$, only the integral curve $t=0$ tends to the origin on the upper side of the integral curve $v=v(t)$ and all the integral curves lying on the lower side of the integral curve $v=v(t)$ tend to the origin; if $a>0$, the behaviors of the integral curves on each side of the integral curve $v=v(t)$ are mutually interchanged.

Thus, combined with theorems 1–6, we see that the behavior of all the integral curves of (5) tending to the origin in the fixed directions is completely known by the process consisting of a finite number of steps.

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