JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 22, NO. 3, DECEMBER 1958

A Note on Homeomorphisms and Fundamental Groups

By

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(Received September, 20, 1958)

§ 1. Introduction

Let Γ be a properly discontinuous group generated by the following mappings S_i, which are defined in the $(n+1)$-dimensional Euclidean space R^{n+1}:

$$(1.1)\quad \begin{cases} S_i: & (x_1, \cdots x_n, x_{n+1}) \to (x_1, \cdots, x_i+1, \cdots, x_n, x_{n+1}) \quad (i=1, 2, \cdots, n) \\ S_{n+1}: & (x_1, \cdots, x_n, x_{n+1}) \to (x_1', \cdots, x_n', x_{n+1}+1) \end{cases}$$

where $x_i'=\sum_{j=1}^{n} a_{ji}x_i$, $A=(a_{ij})$ is an integral matrix of order n and det. $A=1$. As is well known, we may regard Γ as a group generated by the following matrices of order $n+2$[1]:

$$(1.2)\quad S_i=\begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ \vdots & & \ddots & 0 & \\ 1 & & & \ddots & \\ \vdots & 0 & & & \ddots \\ 0 & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow (i+1)\text{-th} \\ \\ \\ \end{matrix} (i=1, 2, \cdots, n) \text{ and } S_{n+1}=\begin{pmatrix} 1 & 0\cdots\cdots0 & 0 \\ 0 & & 0 \\ \vdots & A^t & \vdots \\ 0 & & 0 \\ 1 & 0\cdots\cdots0 & 1 \end{pmatrix}^{2)}.$$

Let M_Γ be a manifold which is generated by the classes of those points in R^{n+1} that are equivalent under the group Γ. Then, the fundamental group G of M_Γ is isomorphic to Γ and a generator C_i of G corresponds to a mapping S_i of Γ. Since there exist the following relations:

$$\begin{cases} \text{(i)} & S_j\cdot S_i=S_i\cdot S_j \qquad (1\leq i, j\leq n) \\ \text{(ii)} & S_{n+1}\cdot S_i=S_1^{a_{i1}}\cdots S_n^{a_{in}}\cdot S_{n+1} \qquad (i=1, 2, \cdots, n), \end{cases}$$

we have

$$(1.3)\quad \begin{cases} \text{(i)} & c_i+c_j=c_j+c_i \qquad (1\leq i, j\leq n) \\ \text{(ii)} & c_i+c_{n+1}=c_{n+1}+\sum_{j=1}^{n} a_{ij}c_j \qquad (i=1, 2, \cdots, n). \end{cases}$$

When $n=2$, H. Poincaré has proved for two groups G, G' defined by (1.3) that the n.a.s.c. in order to be $G\cong G'$ is that there exists an integral unimodular matrix B such that $BA^tB^{-1}=A'$. And according to this theorem

1) In this place, any point of R^{n+1} is expressed by $\begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_{n+1} \end{pmatrix}$.

2) We shall denote by A^t the transpose of A.

3) H. Poincaré: *Analysis situs*, J. d. l'Ecole Polyte. **14** (1895).

he has proved that it is impossible to topologically characterize the manifold M_r simply by using only Betti-number, because M_r has only three kinds of Betti-numbers.

The purpose of this paper is to investigate the n.a.s.c. to be $G\cong G'$ for $n\geq 2$, and to consider the geometrical characterization of M_r whose fundamental group is defined by the similar relations to (1.3).

§ 2. Isomorphisms of fundamental groups

We assume that the group G considered in this paragraph has the finitely generated presentation $G(C; A)$, where $C=\{c_1, \cdots, c_{n+1}\}$ is a system of generators of G and $A=(a_{ij})$ is an integral matrix of the coefficients in the fundamental relations (1.3) of G. Assuming that another group G' has such a presentation $G'(C'; A')$ as (1.3), we seek for the n.a.s.c. in order to be $G'\cong G$. Now, let f be a given isomorphism from G' onto G, then it is usually written[4] by the following form:

$$(2.1)\qquad f(c_i')=b_i c_{n+1}+\sum_{j=1}^{n} b_{ij} c_j \qquad (i=1, 2, \cdots, n+1),$$

where b_i and b_{ij} are integers. In order to be $G'\cong G$, it is necessary that the fundamental relations (1.3) of G must be derived from those of G', that is,

$$(2.2)\qquad \begin{cases} f(c_i')+f(c_j')=f(c_j')+f(c_i') & (1\leq i, j\leq n) \\ f(c_i')+f(c_{n+1}')=f(c_{n+1}')+\sum_{j=1}^{n} a_{ij}' f(c_j') & (i=1, 2, \cdots, n). \end{cases}$$

For the study of these conditions, let us prove the following two lemmas about the calculation in G.

Lemma 1. $$\sum_{i=1}^{n} h_i c_i + h c_{n+1} = h c_{n+1} + \sum_{i,j=1}^{n} h_i \overset{(h)}{a}_{ij} c_j,$$

where h_i $(i=1, 2, \cdots, n)$ *and* h *are arbitrary integers and* $A^h=(\overset{(h)}{a}_{ij})$.

Proof. We prove by induction.

(i) $h\geqq 0$: it is trivial for $h=1$. Next, assume that the above relation holds for $h-1$, then we have

$$\sum_{i=1}^{n} h_i c_i + h c_{n+1} = (h-1)c_{n+1} + \sum_{i,j=1}^{n} h_i \overset{(h-1)}{a}_{ij} c_j + c_{n+1} = h c_{n+1} + \sum_{i,j,k=1}^{n} h_i \overset{(h-1)}{a}_{ij} a_{jk} c_k$$
$$= h c_{n+1} + \sum_{i,j=1}^{n} h_i \overset{(h)}{a}_{ij} c_j,$$

(ii) $h\leq 0$: if we put $h'=-h$ and use $c_i - c_{n+1} = -c_{n+1} + \sum_{j=1}^{n} \overset{(-1)}{a}_{ij} c_j$,

then

$$\sum_{i=1}^{n} h_i c_i + h c_{n+1} = \sum_{i=1}^{n} h_i c_i + h'(-c_{n+1}) = h'(-c_{n+1}) + \sum_{i,j=1}^{n} h_i \overset{(-h')}{a}_{ij} c_j = h c_{n+1} + \sum_{i,j=1}^{n} h_i \overset{(h)}{a}_{ij} c_j.$$

Lemma 2. $$k(h c_{n+1} + \sum_{i=1}^{n} h_i c_i) = k h c_{n+1} + L(c_1, \cdots, c_n)$$

4) The elements of G are written in the form $ac_{n+1}+bc_1+\cdots+dc_n$.

where h_i $(i=1, 2, \cdots, n)$, h *and* k *are arbitrary integers and* $L(c_1, \cdots, c_n)$ *is a linear combination of* $c_1, \cdots, c_n$.

Proof. (i) $k \geqq 0$:

$$k(hc_{n+1}+\sum^n h_i c_i)=\overbrace{(hc_{n+1}+\sum_{i=1}^n h_i c_i)+\cdots+(hc_{n+1}+\sum_{i=1}^n h_i c_i)}^{k}$$

$$=\overbrace{(hc_{n+1}+\sum_{i=1}^n h_i c_i)+\cdots+(hc_{n+1}+\sum_{i=1}^n h_i c_i)}^{k-2}+2hc_{n+1}+\sum_{i,j=1}^n h_i \overset{(h)}{a}_{ij} c_j +\sum h_i c_i$$

$$=\cdots\cdots$$

$$=khc_{n+1}+\sum_{i=1}^n h_i(\overset{(kh-h)}{a_{ij}}+\cdots+\overset{(h)}{a}_{ij}+\delta_{ij})c_j=khc_{n+1}+L(c_1, \cdots, c_n)$$

(ii) $k \leq 0$. If we set $k'=-k$,

$$k(hc_{n+1}+\sum_{i=1}^n h_i c_i)=k'(-\sum_{i=1}^n h_i c_i-hc_{n+1})=-k'hc_{n+1}-\sum_{i,j=1}^n h_i(\overset{-h'h)}{a_{ij}}+\overset{(-k'h-h)}{a_{ij}}+\cdots+\overset{(-h)}{a}_{ij})c_j$$
$$=khc_{n+1}+L(c_1, \cdots, c_n).$$

In the first place, let us treat the special case where the isomorphism f is written as follows:

$$(2.2)' \quad \begin{cases} f(c_i')=\sum_{j=1}^n b_{ij} c_j & (i=1, 2, \cdots, n) \\ f(c_{n+1}')=\varepsilon c_{n+1}+\sum_{j=1}^n b_{n+1,j} c_j \end{cases}$$

where $\varepsilon=+1$ or -1. Since $f(c_i')$ $(i=1, 2, \cdots, n+1)$ must be the generators of G, $B=(b_{ij})$ $(i, j=1, 2, \cdots, n)$ is the integral unimodular matrix. If we observe that

$$\begin{cases} f(c_i')+f(c_{n+1}')=\varepsilon c_{n+1}+\sum_{j,k=1}^n b_{ij}\overset{(\varepsilon)}{a}_{jk} c_k+\sum_{i=1}^n b_{n+1,j} c_j \\ f(c_{n+1}')+\sum_{j=1}^n a_{ij}' f(c_j')=\varepsilon c_{n+1}+\sum_{j,k=1}^n a_{ij}' b_{jk} c_k+\sum_{j=1}^n b_{n+1,j} c_j, \end{cases}$$

from the second equations of (2.2) we have

$$(2.3) \qquad BA^{\varepsilon}B^{-1}=A'.$$

Conversely, if there exist an integral unimodular matrix B and an integer ε satisfying (2.3), then the linear mapping f: $G' \to G$ defined by (2.1), where we may take $b_{n+1,j}$ $(j=1, 2, \cdots, n)$ arbitrarily, is homomorphic on account of (2.3), and is univalent and onto because B is unimodular, so f defines an isomorphism from G' onto G.

On the other hand, if $|A'-E| \neq 0$, b_i $(i=1, 2, \cdots, n)$ in (2.1) are all zero and hence $b_{n+1}=\varepsilon$ for an arbitrary isomorphism f, because we have, from (2.1),

$$\begin{cases} f(c_i')+f(c_{n+1}')=(b_i+b_{n+1})c_{n+1}+L_i(c_1, \cdots, c_n) \\ f(c_{n+1}')+\sum_{j=1}^n a_{ij}' f(c_j')=(b_{n+1}+\sum_{j=1}^n a_{ij}' b_j)c_{n+1}+L_{n+1}(c_1, \cdots, c_n), \end{cases}$$

so comparing the coefficients of c_{n+1} in both equations, we find that

$$\sum_{j=1}^{n}(a'_{ij}-\delta_{ij})b_j=0 \qquad (i=1, 2, \cdots, n),$$

hence $b_i=0$ $(i=1, 2, \cdots, n)$ by our assumption $|A'-E|\neq 0$.
Also even if $|A-E|\neq 0$, we are able to reach the same results as (2.3) by considering f^{-1} in place of f.

Thus summarizing the results, we have

Theorem 1. *Assume that* $|A-E|\neq 0$ *or* $|A'-E|\neq 0$, *then the n.a.s.c. in order to be* $G'\cong G$ *is that there exists an integral unimodular matrix* B *such that* $BA^{\varepsilon}B^{-1}=A'$ *where* $\varepsilon=+1$ *or* -1.

In the next place, let us consider the problem of reduction of the representation (2.1) of a given isomorphism f, in which $(b_1, b_2, \cdots, b_n)\neq(0, 0, \cdots, 0)$. There exists an integral unimodular matrix Ω' such that

$$\Omega'\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}=\begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where b is G.C.M. of $b_1, b_2, \cdots, b_n$. Define that

$$\begin{pmatrix} \tilde{c}'_1 \\ \tilde{c}'_2 \\ \vdots \\ \tilde{c}'_n \end{pmatrix}=\Omega'\begin{pmatrix} c'_1 \\ c'_2 \\ \vdots \\ c'_n \end{pmatrix} \quad \text{and} \quad \tilde{c}'_{n+1}=c'_{n+1},$$

then $\tilde{C}'=\{\tilde{c}'_1, \tilde{c}'_2, \cdots, \tilde{c}'_{n+1}\}$ is a system of the generators of G' because of det. $\Omega'=\pm 1$, and its fundamental relations turn into

(i) $\tilde{c}'_i+\tilde{c}'_j=\tilde{c}'_j+\tilde{c}'_i \qquad (1\leq i, j\leq n)$

(ii) $\tilde{c}'_i+\tilde{c}'_{n+1}=\tilde{c}'_{n+1}+\sum_{j=1}^{n}\tilde{a}'_{ij}\tilde{c}'_j \qquad (i=1, 2, \cdots, n)$

where $\tilde{A}'=(\tilde{a}'_{ij})=\Omega'A'\Omega'^{-1}$, and it is easily proved that

$$\begin{cases} f(\tilde{c}'_1)=bc_{n+1}+\sum_{j=1}^{n}d_{ij}c_j \\ f(\tilde{c}'_i)=\qquad\quad \sum_{j=1}^{n}d_{ij}c_j \qquad (i=2, 3, \cdots, n) \\ f(\tilde{c}'_{n+1})=b_{n+1}c_{n+1}+\sum_{j=1}^{n}b_{n+1,j}c_j \end{cases}$$

where $D=(d_{ij})$ is an integral matrix. Since the above transformation of the generators C' keeps the generators C of G fixed, we have the similar results with regard to

$$f^{-1}(c_i)=b'_i\tilde{c}'_{n+1}+\sum_{j=1}^{n}b'_{ij}\tilde{c}'_j \qquad (i=1, 2, \cdots, n+1).$$

We find, therefore,

Lemma 3. *As for* $G'(C'; A')$, $G(C; A)$ *and the mapping* f *defined by* (2.1), *there exist the generators* $\tilde{C}'$, $\tilde{C}$ *of* G', G *respectively, such that*

$$(2.4)\qquad \begin{cases} f(\tilde{c}'_1)=bc_{n+1}+\sum_{j=1}^{n}\tilde{b}_{ij}\tilde{c}_j \\ f(\tilde{c}'_i)= \sum_{j=1}^{n}\tilde{b}_{ij}\tilde{c}_j \qquad (i=2,3,\cdots,n) \\ f(\tilde{c}'_{n+1})=b_{n+1}c_{n+1}+\sum_{j=1}^{n}\tilde{b}_{n+1,j}\tilde{c}_j \end{cases}$$

$$(2.5)\qquad \begin{cases} f^{-1}(\tilde{c}_1)=b'c'_{n+1}+\sum_{j=1}^{n}\tilde{b}'_j\tilde{c}'_j \\ f^{-1}(\tilde{c}_i)= \sum_{j=1}^{n}\tilde{b}'_{ij}\tilde{c}'_j \qquad (i=2,3,\cdots,n) \\ f^{-1}(\tilde{c}_{n+1})=b'_{n+1}\tilde{c}'_{n+1}+\sum_{j=1}^{n}\tilde{b}'_{n+1,j}\tilde{c}'_j \end{cases}$$

and that $G'(\tilde{C}';\tilde{A}')$ *and* $G(\tilde{C};\tilde{A})$, *where* $\tilde{A}'=\Omega'A'\Omega'^{-1}$, $\tilde{A}=\Omega A\Omega^{-1}$ *and* Ω, Ω' *are the integral unimodular matrices.*

From (2.4) follows

$$\tilde{c}'_i=\sum_{j=1}^{n}\tilde{b}_{ij}(\delta_{j1}b\tilde{c}'_{n+1}+\sum_{k=1}^{n}\tilde{b}'_{jk}\tilde{c}'_k) \qquad (i=2,3,\cdots,n),$$

consequetly, we obtain

$$(2.6)\qquad \tilde{b}_{i1}=0 \ (i=2,3,\cdots,n) \text{ and } \sum_{j=1}^{n}\tilde{b}_{ij}\tilde{b}'_{jk} \ (i=2,3,\cdots,n).$$

Similarly, from (2.5) we have

$$(2.7)\qquad \tilde{b}'_{i1}=0 \ (i=2,3,\cdots,n) \text{ and } \sum_{j=1}^{n}\tilde{b}'_{ij}\tilde{b}_{ik}=\delta_{ik} \ (i=2,3,\cdots,n).$$

Furthermore, if we use the following generators $\hat{C}'=\{\hat{c}'_1,\hat{c}'_2,\cdots,\hat{c}'_{n+1}\}$ of G';

$$\begin{pmatrix} \hat{c}'_1 \\ \hat{c}'_2 \\ \vdots \\ \hat{c}'_n \end{pmatrix}=\tilde{\Omega}'\begin{pmatrix} \tilde{c}'_1 \\ \tilde{c}'_2 \\ \vdots \\ \tilde{c}'_n \end{pmatrix} \quad \text{and} \quad \hat{c}'_{n+1}=\tilde{c}'_{n+1},$$

where

$$\tilde{\Omega}'=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \tilde{b}'_{22} & \cdots & \tilde{b}'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & \tilde{b}'_{n1} & \cdots & \tilde{b}'_{nn} \end{pmatrix}$$

is the integral unimodular matrix from (2.7), (2.4) reduces to

$$(2.4)'\qquad \begin{cases} f(\hat{c}'_1)=b\tilde{c}_{n+1}+\sum_{j=1}^{n}b_{1j}\tilde{c}_j \\ f(\hat{c}'_i)=\tilde{c}_i \qquad (i=2,3,\cdots,n) \\ f(\hat{c}'_{n+1})=b_{n+1}\tilde{c}_{n+1}+\sum_{j=1}^{n}\tilde{b}_{n+1j}\tilde{c}_j. \end{cases}$$

Finally, define the generators $\bar{C}'=\{\bar{c}'_1,\bar{c}'_2,\cdots,\bar{c}'_{n+1}\}$ of G' by

$$\begin{pmatrix} \bar{c}'_1 \\ \bar{c}'_2 \\ \vdots \\ \bar{c}'_n \end{pmatrix}=\hat{\Omega}'\begin{pmatrix} \hat{c}'_1 \\ \hat{c}'_2 \\ \vdots \\ \hat{c}'_n \end{pmatrix} \quad \text{and} \quad \bar{c}'_{n+1}=\hat{c}'_{n+1}-\sum_{j=2}^{n}\tilde{b}_{n+1j}\hat{c}'_j,$$

where

$$\hat{\Omega}'=\begin{pmatrix} 1 & -\tilde{b}_{12}\cdots & -\tilde{b}_{1n} \\ 0 & 1 & \\ \vdots & \ddots & 0 \\ \vdots & 0 & \ddots \\ 0 & & 1 \end{pmatrix},$$

then the presentation $G'(\hat{C}';\hat{A}')$ turns into $G'(\bar{C}';\hat{\Omega}'\hat{A}'\hat{\Omega}'^{-1})$ and (2.4)$'$ is reduced to

$$(2.4)'' \qquad \begin{cases} f(\bar{c}_1')=b\tilde{c}_{n+1}+\tilde{b}_{11}\tilde{c}_1 \\ f(\bar{c}_i')=\tilde{c}_i \qquad (i=2,\cdots,n) \\ f(\bar{c}_{n+1}')=b_{n+1}\tilde{c}_{n+1}+\tilde{b}_{n+1,1}\tilde{c}_1. \end{cases}$$

Thus, summarizing the above considerations, we find

Lemma 4. *As for $G'(C';A')$, $G(C;A)$ and the mapping f defined by* (2.1), *there exist the generators $\bar{C}'$, $\tilde{C}$ of G', G respectively, such that $G'(\bar{C}';\bar{\Omega}'A'\bar{\Omega}'^{-1})$, $G(\tilde{C};\Omega A\Omega^{-1})$ where $\bar{\Omega}'$, and Ω are the integral unimodular matrices, and* (2.1) *is reduced to*

$$(2.8) \qquad \begin{cases} f(\bar{c}_1')=\alpha\tilde{c}_{n+1}+\beta\tilde{c}_1 \\ f(\bar{c}_i')=\tilde{c}_i \qquad (i=2,3,\cdots,n) \\ f(\bar{c}_{n+1}')=\gamma\tilde{c}_{n+1}+\delta\tilde{c}_1 \end{cases}$$

where $\alpha,\beta,\gamma,\delta$ are integers.

Let us now consider the case of $(b_1,b_2,\cdots,b_n)\neq(0,0,\cdots,0)$ making use of the reduced form (2.8). We set $\tilde{A}=(\tilde{a}_{ij})=\Omega A\Omega^{-1}$ and $\bar{A}'=(\bar{a}_{ij}')=\bar{\Omega}'A'\bar{\Omega}'^{-1}$. We have the following relations;

$$(2.9) \quad \begin{cases} f(\bar{c}_1')+f(\bar{c}_i')=\alpha\tilde{c}_{n+1}+\beta\tilde{c}_1+\tilde{c}_i \\ f(\bar{c}_i')+f(\bar{c}_1')=\alpha\tilde{c}_{n+1}+\sum_{j=1}^{n}\overset{(\alpha)}{\tilde{a}}_{ij}\tilde{c}_j+\beta\tilde{c}_1 \qquad (i=2,3,\cdots,n) \end{cases}$$

$$(2.10) \quad \begin{cases} f(\bar{c}_i')+f(\bar{c}_{n+1}')=\gamma\tilde{c}_{n+1}+\sum_{j=1}^{n}\overset{(\gamma)}{\tilde{a}}_{ij}\tilde{c}_j+\delta\tilde{c}_1 \\ f(\bar{c}_{n+1}')+\sum_{j=1}^{n}\bar{a}_{ij}'f(\bar{c}_j')=\gamma\tilde{c}_{n+1}+\delta\tilde{c}_1+\bar{a}_{i1}'(\alpha\tilde{c}_{n+1}+\beta\tilde{c}_1)+\sum_{j=2}^{n}\bar{a}_{ij}'\tilde{c}_j \end{cases}$$
$$(i=2,3,\cdots,n),$$

and

$$(2.11) \quad \begin{cases} f(\bar{c}_i')+f(\bar{c}_{n+1}')=(\alpha+\gamma)\tilde{c}_{n+1}+\sum_{j=1}^{n}\beta\overset{(\gamma)}{\tilde{a}}_{1j}\tilde{c}_j+\delta\tilde{c}_1 \\ f(\bar{c}_{n+1}')+\sum_{j=1}^{n}\bar{a}_{1j}'f(\bar{c}_j')=\gamma\tilde{c}_{n+1}+\delta\tilde{c}_1+\bar{a}_{11}'(\alpha\tilde{c}_{n+1}+\beta\tilde{c}_1)+\sum_{j=2}^{n}\bar{a}_{1j}'\tilde{c}_j. \end{cases}$$

Then, the relations (2.9) yield

$$(2.12) \qquad \overset{(\alpha)}{\tilde{a}}_{ij}=\delta_{ij} \ (i=2,3,\cdots,n,\ j=1,2,\cdots,n) \text{ and } \overset{(\alpha)}{\tilde{a}}_{11}=1,$$

the relations (2.10) imply

$$(2.13) \qquad \bar{a}_{i1}'=\tilde{a}_{i1}=\delta_{i1} \ (i=2,3,\cdots,n) \text{ and } \bar{a}_{ij}'=\overset{(\gamma)}{\tilde{a}}_{ij} \ (i,j=2,3,\cdots,n),$$

and from (2.11) we have

$$(2.14) \qquad \overset{(\gamma)}{\tilde{a}}_{11}=\bar{a}_{11}'=\tilde{a}_{11}=1 \text{ and } \bar{a}_{1j}'=\beta\overset{(\gamma)}{\tilde{a}}_{1j}-\delta\overset{(\alpha)}{\tilde{a}}_{1j} \qquad (j=2,3,\cdots,n).$$

Also we can prove from $\overset{(\alpha)}{\tilde{a}}_{11}=\overset{(\gamma)}{\tilde{a}}_{11}=1$ that the n.a.s.c. in order that f is onto and univalent is $\alpha\delta-\beta\gamma=\pm 1$.

Thus, summarizing (2.12), (2.13) and (2.14) and taking the case of $(b_1, b_2, \cdots, b_n)=(0, 0, \cdots, 0)$ into consideration, we have

Theorem 2. *The n.a.s.c. in order to be* $G'(C'; A')\cong G(C; A)$ *where* $|A'-E|=0$ *and* $|A-E|=0$, *is that*

(i) *there exists an integral unimodular matrix B such that*

$$BA^{t}B^{-1}=A'$$

or

(ii) *there exist two integral unimodular matrices* $\bar{\Omega}'$ *and* Ω *such that*

$$\tilde{A}=\Omega A\Omega^{-1}=\begin{pmatrix} 1 & \tilde{a}_{12}\cdots\tilde{a}_{1n} \\ 0 & \tilde{a}_{22}\cdots\tilde{a}_{2n} \\ \vdots & \vdots \qquad \vdots \\ 0 & \tilde{a}_{n2}\cdots\tilde{a}_{nn} \end{pmatrix},\quad \tilde{A}^{\alpha}=\begin{pmatrix} 1 & \overset{(\alpha)}{\tilde{a}}_{12}\cdots\overset{(\alpha)}{\tilde{a}}_{1n} \\ 0 & 1 \quad\quad 0 \\ \vdots & \quad\ddots \\ 0 & 0 \quad\quad 1 \end{pmatrix} \text{ and}$$

$$\bar{A}'=\bar{\Omega}'A'\bar{\Omega}'^{-1}=\begin{pmatrix} 1 & \bar{a}'_{12}\cdots\bar{a}'_{1n} \\ 0 & \overset{(\gamma)}{\tilde{a}}_{22}\cdots\overset{(\gamma)}{\tilde{a}}_{2n} \\ \vdots & \vdots \qquad \vdots \\ 0 & \overset{(\gamma)}{\tilde{a}}_{n2}\cdots\overset{(\gamma)}{\tilde{a}}_{nn} \end{pmatrix}$$

where $\bar{a}'_{1j}=\beta\tilde{a}_{1j}-\delta\tilde{a}_{1j}$ $(j=2\cdots n)$, $\alpha\neq 0$ *and* $\alpha\delta-\beta\gamma=\pm 1$.

Remark 1. If the characteristic roots of A are all simple, we can prove from (2.9) that A^{α} is the unit matrix. Hence, in the case where the characteristic roots of A are all simple in Theorem 2, the n.a.s.c. in order to be $G'\cong G$ is (i) in Theorem 2 or (ii)′; the condition (ii)′ is obtained from (ii) by replacing $\tilde{A}^{\alpha}$ with the unit matrix.

§ 3. Geometrical meanings of isomorphisms

Let $\Gamma=\{S_i\}$ and $\Gamma'=\{S_i'\}$ be the groups corresponding to A and A' by (1.2) respectively, then we shall consider the two manifolds M_{Γ} and $M_{\Gamma'}$ corresponding to Γ and Γ' respectively, when Γ and Γ' are isomorphic. We investigate into such a matrix $U=(u_{ij})$ corresponding to an isomorphism $f\colon \Gamma'\to\Gamma$ that $US_i'U^{-1}=f(S_i')$ $(i=1, 2, \cdots, n+1)$ and its first row is $(1, 0, \cdots, 0)$. If such a matrix U exists, M_{Γ} and $M_{\Gamma'}$ are mutually transformable by a coordinate transformation, so that all the manifolds whose fundamental groups are isomorphic to G and the same type with G, are geometrically characterized by M_{Γ}.

In the first place, assume that the conditions of Theorem 1 are satisfied, then there exists an isomorphism $f:\Gamma'\to\Gamma$ defined by

$$\text{(3.1)}\qquad \begin{cases} f(S_i')=S_1^{b_{i1}}S_2^{b_{i2}}\cdots S_n^{b_{in}} & (i=1, 2, \cdots, n) \\ f(S_{n+1}')=S_1^{b_{n+1,1}}S_2^{b_{n+1,2}}\cdots S_n^{b_{n+1,n}}S_{n+1}^{\varepsilon} \end{cases}$$

where $BA^{\varepsilon}B^{-1}=A'$. From $US'_i U^{-1}=f(S'_i)$ $(i=1, 2, \cdots, n)$ i.e.

$$(u_{ij})\begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & & 0 & \\ \vdots & & \ddots & & \\ 1 & & & \ddots & \\ \vdots & 0 & & & \\ 0 & & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & & & \\ b_{i1} & 1 & & 0 & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ b_{in} & & 0 & & \\ 0 & & & & 1 \end{pmatrix}(u_{ij}),$$

it follows that

$$U=\begin{pmatrix} 1 & 0\cdots\cdots 0 & 0 \\ u_{21} & & u_{2,n+2} \\ \vdots & B^t & \vdots \\ u_{n+1,1} & & u_{n+1,n+2} \\ u_{n+2,1} & 0\cdots\cdots 0 & u_{n+1,n+2} \end{pmatrix}. \tag{3.2}$$

Also from $US'_{n+1}U^{-1}=f(S'_{n+1})$ i.e.

$$(u_{ij})\begin{pmatrix} 1 & 0\cdots\cdots 0 & 0 \\ 0 & & 0 \\ \vdots & A'^t & \vdots \\ 0 & & 0 \\ 0 & 0\cdots\cdots 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0\cdots\cdots 0 & 0 \\ b_{n+1,1} & & 0 \\ \vdots & (A^{\varepsilon})^t & \vdots \\ b_{n+1,n} & & 0 \\ \varepsilon & 0\cdots\cdots 0 & 1 \end{pmatrix}(u_{ij})$$

where (u_{ij}) is the matrix (3.2), we have $u_{n+2,n+2}=1$ and

$$u_{k1}+u_{k,n+2}=b_{n+1,k-1}+\sum_{j=1}^{n}\overset{(\varepsilon)}{a}_{j,k-1}u_{j+1,1} \qquad (k=2, 3, \cdots, n+1) \tag{3.3}$$

$$u_{k,n+1}=\sum_{j=1}^{n}\overset{(\varepsilon)}{a}_{j,k-1}u_{j+1,n+2} \qquad (k=2, 3, \cdots, n+1). \tag{3.4}$$

From (3.4) and $|A^{\varepsilon}-E|\neq 0$, we have $u_{j+1,n+2}=0$ $(j=1, 2, \cdots, n)$, and (3.3) is, therefore, written in the form

$$b_{n+1,k-1}=\sum_{j=1}^{n}(\delta_{j,k-1}-\overset{(\varepsilon)}{a}_{j,k-1})u_{j+1,1}, \tag{3.5}$$

consequently $u_{j+1,1}$ $(j=1, 2, \cdots, n)$ are uniquely determined as the solutions of the equations (3.5). Therefore, U is of the form

$$U=\begin{pmatrix} 1 & 0\cdots\cdots 0 & 0 \\ u_{21} & & 0 \\ \vdots & B^t & \vdots \\ u_{n+1,1} & & 0 \\ u_{n+2,1} & 0\cdots\cdots 0 & 1 \end{pmatrix} \tag{3.6}$$

where $u_{n+2,1}$ is an arbitrary number and $u_{21}, \cdots, u_{n+1,1}$ are the solutions of (3.5). Since

$$T=\begin{pmatrix} 1 & 0 & & 0 \\ u_{21} & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & \ddots \\ u_{n+2,1} & & 0 & & 1 \end{pmatrix}$$

is a translation in R^{n+1} and it holds that

$$U=T\begin{pmatrix} 1 & 0\cdots 0 & 0 \\ 0 & & 0 \\ \vdots & B^t & \vdots \\ 0 & 0\cdots 0 & 1 \end{pmatrix},$$

we have

Theorem 3. *If* $|A-E|\neq 0$ *or* $|A'-E|\neq 0$, *there exists such a matrix* U *corresponding to a given isomorphism* $f:\Gamma'\to\Gamma$ *that* $US_i'U^{-1}=f(S_i')$ $(i=1, 2, \cdots, n+1)$ *and its first row is* $(1, 0, \cdots, 0)$, *and this matrix* U *is unimodular* and is *uniquely determined except for a translation.*

Now, let us consider the case in which $|A-E|=|A'-E|=0$ and the characteristic roots of A are all simple. Then it suffices for our purpose to investigate the case of $(b_1, b_2, \cdots, b_n)\neq(0, \cdots, 0)$, because the case of $(b_1, b_2, \cdots, b_n)=(0, \cdots, 0)$ has been investigated in Theorem 3. It is easily shown in the similar way to Theorem 2 that

$$\tilde{a}'_{i1}=\tilde{a}_{i1}=\delta_{i1} \qquad (i=1, 2, \cdots, n). \tag{3.7}$$

Then we have

$$\begin{cases} f(\tilde{c}'_i)+f(\tilde{c}'_{n+1})=(\delta_{i1}b+b_{n+1})\tilde{c}_{n+1}+\sum_{j,k=1}^{n}\tilde{b}_{ij}\overset{(b_{n+1})}{\tilde{a}_{jk}}\tilde{c}_k+\sum_{j=1}^{n}\tilde{b}_{n+1,j}\tilde{c}_j \\ f(\tilde{c}'_{n+1})+\sum_{j=1}^{n}\tilde{a}'_{ij}f(\tilde{c}'_j)=(\delta_{i1}b+b_{n+1})\tilde{c}_{n+1}+\sum_{j,k=1}^{n}\tilde{a}'_{ij}\tilde{b}_{jk}\tilde{c}_k+\sum\tilde{b}_{n+1,j}\tilde{c}_j. \end{cases}$$

Hence

$$\tilde{B}\tilde{A}^{b_{n+1}}\tilde{B}^{-1}=\tilde{A}', \tag{3.8}$$

where $\tilde{B}=(\tilde{b}_{ij})$ and det. $\tilde{B}\neq 0$. Hence, there exists an isomorphism $f:\Gamma'\to\Gamma$ defined by

$$\begin{cases} f(\tilde{S}'_1)=\tilde{S}_1^{\tilde{b}_{11}}\tilde{S}_2^{\tilde{b}_{12}}\cdots\tilde{S}_n^{\tilde{b}_{1n}}\tilde{S}_{n\div 1}^{b} \\ f(\tilde{S}'_1)= \qquad \tilde{S}_2^{\tilde{b}_{i2}}\cdots\tilde{S}_n^{\tilde{b}_{in}} \qquad (i=2, 3, \cdots, n) \\ f(\tilde{S}^1_{n+1})=\tilde{S}_1^{\tilde{b}_{n+1,1}}\tilde{S}_2^{\tilde{b}_{n+1,2}}\cdots\tilde{S}_n^{\tilde{b}_{n+1,n}}\tilde{S}_{n+1}^{b_{n+1}} \end{cases}$$

where $\tilde{A}^b=E$ and $\tilde{B}\tilde{A}^{b_{n+1}}B^{-1}=\tilde{A}'$.

Theorem 2 means that a suitable coordinate transformation corresponds to that of commutative generators of G, and consequently we may assume that the mappings S_i corresponding to the generators $\tilde{c}_i$ of G are of the following forms:

$$\tilde{S}_i=\begin{pmatrix} 1 & 0 & & & 0 \\ 0 & 1 & & & \\ \vdots & & \ddots & & \\ 1 & & & \ddots & \\ \vdots & 0 & & & \ddots \\ 0 & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow (i+1)\text{-th} \\ \\ \\ \end{matrix} \quad (i=1, 2, \cdots, n) \text{ and } \tilde{S}_{n+1}=\begin{pmatrix} 1 & 0\cdots\cdots 0 & 0 \\ 0 & & 0 \\ \vdots & \tilde{A}^t & \vdots \\ 0 & & 0 \\ 1 & 0\cdots\cdots 0 & 1 \end{pmatrix}$$

and that the mappings S_i' $(i=1, 2, \cdots, n+1)$ are of the similar forms to the above.

From $U\tilde{S}'_i U^{-1}=f(\tilde{S}'_i)$ $(i=1, 2, \cdots, n)$, we have

$$(3.9)\qquad U=\begin{pmatrix} 1 & 0\cdots\cdots 0 & 0 \\ u_{21} & & u_{2,\,n+2} \\ \vdots & \tilde{B}^t & \vdots \\ u_{n+1,\,1} & & u_{n+1,\,n+2} \\ u_{n+2,\,1} & b_1\ 0\cdots\cdots 0 & u_{n+2,\,n+2} \end{pmatrix}.$$

Moreover, comparing the corresponding elements of both sides of $U\tilde{S}'_{n+1}=f(\tilde{S}'_{n+1})U$, i. e.

$$(u_{ij})\begin{pmatrix} 1 & 0\cdots\cdots 0 & 0 \\ 0 & & 0 \\ \vdots & (\bar{A}')^t & \vdots \\ 0 & & 0 \\ 1 & 0\cdots\cdots 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0\cdots\cdots 0 & 0 \\ b_{n+1,\,1} & & 0 \\ \vdots & (\tilde{A}^{b_{n+1}})^t & \vdots \\ b_{n+1,\,n} & & 0 \\ b_{n+1} & 0\cdots\cdots 0 & 1 \end{pmatrix}(u_{ij}),$$

we have $u_{2,\,n+2}=b_{n+1,\,1}$, $u_{n+2,\,n+2}=b_{n+1}$ and

$$(3.10)\qquad u_{k1}+u_{k,\,n+2}=b_{n+1,\,k-1}+\sum_{i=1}^{n}\overset{(b_{n+1})}{\tilde{a}_{i,\,k-1}}u_{i+1,\,1}\quad (k=2, 3, \cdots, n+1)$$

$$(3.11)\qquad u_{k,\,n+2}=\sum_{i=1}^{n}\overset{(b_{n+1})}{\tilde{a}_{i,\,k-1}}u_{i+1,\,n+2}\quad (k=2, 3, \cdots, n+1).$$

From (3.11)

$$(3.12)\qquad \sum_{i=1}^{n}(\overset{(b_{n+1})}{\tilde{a}_{ik}}-\delta_{ik})u_{i+1,\,n+2}=-\overset{(b_{n+1})}{\tilde{a}_{1k}}b_{n+1,\,1}\quad (k=2, 3, \cdots, n).$$

Since $\det.(\overset{(b_{n+1})}{\tilde{a}_{ik}}-\delta_{ik})_{i,\,k=2,\cdots,n}\neq 0$ because of $(b, b_{n+1})=1$, $\tilde{A}^b=E$ and $a_{i1}=\delta_{i1}$ $(i=1, 2, \cdots, n)$, $u_{i,\,n+2}$ $(i=3, \cdots, n+2)$ are uniquely determined as the solutions of (3.12).

And also from (3.10), we have

$$u_{k+1,\,n+2}-b_{n+1,\,k}-\overset{(b_{n+1})}{\tilde{a}_{1k}}u_{21}=\sum_{i=2}^{n}(\overset{(b_{n+1})}{\tilde{a}_{ik}}-\delta_{ik})u_{i+1,\,1}\quad (k=2, 3, \cdots, n),$$

hence, $u_{i+1,\,1}$ $(i=2, \cdots, n)$ are uniquely determined involving u_{21} as a parameter. So U is of the form

$$U=\begin{pmatrix} 1 & 0\cdots\cdots 0 & 0 \\ u_{21} & & \tilde{b}_{n+1,\,1} \\ \vdots & \tilde{B}^t & u_{3,\,n+2} \\ \vdots & & \vdots \\ u_{n+1,\,1} & & u_{n+1,\,n+2} \\ u_{n+2,\,1} & b_1\ 0\cdots\cdots 0 & b_{n+1} \end{pmatrix}.$$

Thus we conclude

Theorem 4. *If $|A-E|=|A'-E|=0$ and the characteristic roots of A are all simple, there exists such a matrix U corresponding to a given isomorphism $f:\Gamma'\to\Gamma$ that $US'_iU^{-1}=f(S'_i)$ $(i=1, 2, \cdots, n+1)$ and its first row is $(1, 0, \cdots, 0)$, and this matrix U is unimodular and is uniquely* *determined, except for a translation.*

Finally, let us consider the case in which $|A-E|=|A'-E|=0$ and the characteristic equation of A has not necessarily simple roots. In this case, the matrix U corresponding to the given f does not exist. For example, if we put

$$A=\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix} \quad \text{and} \quad A'=\begin{pmatrix}1 & -h\\ 0 & 1\end{pmatrix} \quad (h\neq 0)$$

and take a mapping $f:\Gamma'\rightarrow\Gamma$ defined by

$$\begin{cases} f(S_1')=S_1^{\beta}S_3^{\alpha}\\ f(S_2')=S_2\\ f(S_3')=S_1^{\delta}S_3^{\gamma}\end{cases}$$

where $\alpha\cdot\beta\neq 0$ and $\alpha\delta-\beta\gamma=1$, then, f is an isomorphism from Γ' onto Γ. We cannot, however, construct such a matrin U as $US_i'U^{-1}=f(S_i')$ $(i=1, 2, 3)$. As a matter of fact, from $US_2'U^{-1}=f(S_2')$ we have

$$U=\begin{pmatrix}1 & 0 & 0 & 0\\ u_{21} & u_{22} & 0 & u_{24}\\ u_{31} & u_{32} & 1 & u_{34}\\ u_{41} & u_{42} & 0 & u_{44}\end{pmatrix}$$

and since $US_1'U^{-1}=f(S_1')$ i.e. $U(S_1'-E)U^{-1}=f(S_1')-E$, comparing (2.1)-and (3.2)-elements of both sides of

$$(u_{ij})\begin{pmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & 0\\ \beta & 0 & 0 & 0\\ 0 & \alpha h & 0 & 0\\ \alpha & 0 & 0 & 0\end{pmatrix}(u_{ij}),$$

we get $u_{22}=\beta$ and $\alpha h u_{22}=0$, and consequently, we obtain $\alpha\cdot\beta=0$. This contradicts our assumption. Hence it is not always possible to construct a matrix U corresponding to the given isomorphism f.

From this example, we have

Theorem 5. *If $G'\cong G$, $|A-E|=|A'-E|=0$ and the characteristic roots of A are not all simple, we cannot always construct such a matrix U corresponding to the given isomorphism f as $US_i'U^{-1}=f(S_i')$ $(i=1, 2, n+1)$.*

Remark 2. Especially if $n=2$, by Theorem 2 there exists an integral unimodular matrix $B=(b_{ij})$ satisfying

$$BA^{\varepsilon}B^{-1}=A'.$$

In our example, $\varepsilon=1$ and B is of the form

$$B=\begin{pmatrix}\varepsilon' & \tau\\ 0 & -\varepsilon'\end{pmatrix}$$

where $\varepsilon'=1$ or -1 and τ is an arbitrary integer, so that a mapping $g:\Gamma'\rightarrow\Gamma$ defined by

$$\begin{cases} g(S_1')=S_1^{\varepsilon'}\cdot S_2^{\tau}\\ g(S_2')=S_2^{-\varepsilon'}\\ g(S_3')=S_3^{\varepsilon}S_1^{x}S_2^{y}\end{cases}$$

$x, y,$ being arbitrary integers, is an isomorphism from Γ' onto Γ by Theorem 2. Then we can construct U corresponding to g[5].

From the above disccusion, the manifolds of which the fundamental groups are defined by (1.3) are geometrically characterized by the manifold M_r in the following three cases; (i) $n=2$, and as for $n \geqq 3$, (ii) $|A-E| \neq 0$ or $|A'-E| \neq 0$, or (iii) $|A-E|=|A'-E|=0$ and the characteristic roots of A or A' are all simple.

In conclusion, the author wishes to express his sincere thanks to Prof. K. Morinaga for his kind guidance.

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5) As for $n \geqq 3$, the author does not know whether such a favourable isomorphism exists or not.