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On the Exchange Formula for Distributions

By

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Let R^n denote the n-dimensional Eulcidean space with points $x=(x_1, x_2, \cdots, x_n)$. The theory of Fourier transform of temperate distributions (elements of the space $(\mathcal{S}')$) on R^n has been developed by L. Schwartz [3]. The Fourier transform of any temperate distribution is defined as another temperate distribution. For any distribution $S\in(\mathcal{S}')$ the Fourier and inverse Fourier transforms of S are denoted by $\mathcal{F}(S)$ and $\overline{\mathcal{F}}(S)$ respectively. Let $\{\rho_k\}$ and $\{\rho'_k\}$ be any sequences of regularizations and let $r_k=\overline{\mathcal{F}}(\rho_k)$. Now let S and T be any two distributions of $(\mathcal{S}')$ such that $S*T$ is defined and belongs to $(\mathcal{S}')$. r_kT is an element of $(\mathcal{O}'_c)$ since r_k belongs to $(\mathcal{S})$. Owing to the basic exchange formula due to L. Schwartz [3], we have $\mathcal{F}(S*(r_kT))=\mathcal{F}(S)\cdot(\mathcal{F}(T)*\rho_k)$. It follows then as noted by R. E. Edwards [2] that the exchange formula

$$\mathcal{F}(S*T)=\mathcal{F}(S)\cdot\mathcal{F}(T) \tag{1}$$

holds under the following conditions:

(α) $S*(r_kT)$ converges to $S*T$ in $(\mathcal{S}')$ as $k\to\infty$

(β) $\mathcal{F}(S)\cdot\mathcal{F}(T)$ is defined and $\mathcal{F}(S)\cdot(\mathcal{F}(T)*\rho_k)$ converges to $\mathcal{F}(S)\cdot\mathcal{F}(T)$ in $(\mathcal{D}')$ as $k\to\infty$.

Our present purpose of this paper is to eliminate, in a certain sense, the two conditions (α) and (β) in the above statement. To this end we first introduce the concept of $(\mathcal{S}')$-convolution $S*T$ (§1). We show that if $S*T$ is defined, then it belongs to $(\mathcal{S}')$ and coincides with the ordinary convolution in the sense of C. Chevalley [1]. Secondly we propose to define the muliplicative product $A\cdot B$ of two distributions A, B as the common limit of sequence $(A*\rho_k)B$ and $A(B*\rho'_k)$ in $(\mathcal{D}')$ as $k\to\infty$ provided these limits exist and coincide. In §2 we show that if, for any two distributions $S, T\in(\mathcal{S}')$, the $(\mathcal{S}')$-convolution $S \circledast T$ is defined, then the exchange formula (1) holds.

Concerning distributions, we adopt the notations of L. Shwartz [3] unless otherwise specifically mentioned.

1. Let S and T be any two distributions. Following C. Chevalley [1] we say that the convolution $S*T$ is defined if the following condition is satisfied:

$(*)$ $(S*\varphi)\cdot(\check{T}*\psi)$ belongs to L for any $\varphi, \psi\in(\mathcal{D})$,

where L is the space of summable functions defined on R^n.
Then there exists a unique distribution denoted by $S*T$ such that

$$\langle (S*T)*\varphi, \psi\rangle = \int (S*\varphi)(x)\cdot(\check{T}*\psi)(x)dx \text{ for any } \varphi, \psi \in (\mathscr{D}).$$

Now assume that S and T are temperate. We shall define the $(\mathscr{S}')$-convolution of S and T by replacing $(\mathscr{D})$ by $(\mathscr{S})$ in the above condition $(*)$. We say that the $(\mathscr{S}')$-convolution $S \circledast T$ is defined if the following condition is satisfied:

$(\circledast)$ $\quad S*\varphi\cdot\check{T}*\psi$ belongs to L for any $\varphi, \psi \in (\mathscr{S})$.

Then the bilinear form $B(\varphi, \psi) = \int (S*\varphi)(x)\cdot(\check{T}*\psi)(x)dx$ defined on $(\mathscr{S})\times(\mathscr{S})$ is separately continuous, and therefore continuous on $(\mathscr{S})\times(\mathscr{S})$ since $(\mathscr{S})$ is a space of *type* (F). Let $\mathscr{L}$ be the continuous linear application of $(\mathscr{S})$ into $(\mathscr{S}')$ determined by $B(\varphi, \psi) = \langle \mathscr{L}(\varphi), \psi\rangle$. We show that $\mathscr{L}$ is a composition operator [4], that is, $\tau_h\mathscr{L}(\varphi) = \mathscr{L}(\tau_h\varphi)$ for any translation τ_h and $\varphi \in (\mathscr{D})$. We have

$$\begin{aligned}\langle \tau_h\mathscr{L}(\varphi), \psi\rangle &= B(\varphi, \tau_{-h}\psi)\\ &= \int (S*\varphi)(x)\cdot(\check{T}*\tau_{-h}\psi)(x)dx\\ &= \int (S*(\tau_h\varphi))(x)\cdot(\check{T}*\psi)(x)dx\\ &= B(\tau_h\varphi, \psi)\\ &= \langle \mathscr{L}(\tau_h\varphi), \psi\rangle, \text{ for any } \varphi, \psi \in (\mathscr{D}).\end{aligned}$$

Accordingly, $\tau_h\mathscr{L} = \mathscr{L}\tau_h$, as desired. It follows then by Theorem 4 [4] that there exists a unique distribution U such that $\mathscr{L}(\varphi) = U*\varphi$, and U must belongs to $(\mathscr{S}')$. We say that this distribution U is $(\mathscr{S}')$-*convolution* of S and T, and denote by $S\circledast T$. From this definition it is clear that $S*T$ is also defined and coincide with $S*T$.

Examples. (a) Let $S \in (\mathscr{O}'_c)$ and $T \in (\mathscr{S}')$. Then $S*\varphi \in (\mathscr{S})$, $T*\psi \in (\mathscr{O}_M)$ for any $\varphi, \psi \in (\mathscr{S})$, therefore the condition $(\circledast)$ is satisfied, so that $S\circledast T$ is defined.

(b) Let $S \in (\mathscr{D}'_{L^p})$ and $T \in (\mathscr{D}'_{L^q})$ with $\frac{1}{p}+\frac{1}{q} \geqq 1$. Then $S*\varphi \in L^p$ and $T*\varphi \in L^q$ for any $\varphi, \psi \in (\mathscr{S})$. It follows then that $(S*\varphi)(\check{T}*\psi) \in L$, and therefore $S\circledast T$ is defined.

(c) Let $S = \mathrm{Pf.}\, r^p$ and $T = \mathrm{Pf.}\, r^q$ with $\mathscr{R}(p+q) < -n$. We show that $S\circledast T$ is defined. To this end it suffices to show that, for any $\varphi \in (\mathscr{S})$, $S*\varphi = 0(|x|^{\mathscr{R}(p)})$ for large $|x|$. The proof is divided into the two casses: (i) $\mathscr{R}(p) \geqq 0$ and (ii) $\mathscr{R}(p) < 0$. We use M, with or without subscripts, to denote constants depending on φ.

ad (i): For any $\varphi \in (\mathscr{S})$,

$$\begin{aligned}|S*\varphi(x)| &\leqq \int |x-y|^{\mathscr{R}(p)}|\varphi(y)|dy\\ &\leqq M\int (|x|+|y|)^{\mathscr{R}(p)}\frac{1}{1+|y|^{\mathscr{R}(p)+n+1}}dy\end{aligned}$$

$$\leqq 2^{\mathcal{R}(p)} M \int \frac{|x|^{\mathcal{R}(p)}+|y|^{\mathcal{R}(p)}}{1+|y|^{\mathcal{R}(p)+n+1}}\, dy = O(|x|^{\mathcal{R}(p)}).$$

ad (ii): $S_i(i=1,2)$ be defined by $S_1=r^p$ for $r\geqq 1$ and $=0$ for $r<1$, and $S_2=\mathbf{Pf.}\, r^p - S_1$. As the support of S_2 is compact, so $S_2 * \varphi\in(\mathcal{S})$ for any $\varphi\in(\mathcal{S})$. This implies that $S_2*\varphi=O(r^{\mathcal{R}(p)})$. As to S_1, we have

$$|S_1*\varphi(x)|\leqq \int_{|x-y|\geqq 1} |x-y|^{\mathcal{R}(p)}|\varphi(y)|\,dy$$
$$=\int_{|y|\geqq 1} |y|^{\mathcal{R}(p)}|\varphi(x-y)|\,dy$$
$$=\int_{1\leqq|y|\leqq\frac{x}{2}} |y|^{\mathcal{R}(p)}|\varphi(x-y)|\,dy+\int_{|y|\geqq\frac{|x|}{2}} |y|^{\mathcal{R}(p)}|\varphi(x-y)|\,dy.$$

If $1\leqq|y|\leqq\frac{|x|}{2}$, then $|x-y|\geqq|x|-|y|\geqq 2|y|-|y|=|y|\geqq 1$ and $|x-y|\geqq|x|-\frac{|x|}{2}=\frac{|x|}{2}$. Then we have

$$|y|^{\mathcal{R}(p)}|\varphi(x-y)|\leqq\frac{M_1}{|y|^{\sigma}|x-y|^{\sigma+n+1}}$$
$$=\frac{M_1}{|x|^{\sigma}|x-y|^{n+1}|y|^{\sigma}}$$
$$\leqq\frac{M_1}{\left(\frac{|x|}{2}\right)^{\sigma}|y|^{\sigma+n+1}},$$

where $\sigma=-\mathcal{R}(p)$.

If $|y|\geqq\frac{|x|}{2}$, then

$$|y|^{\mathcal{R}(p)}|\varphi(x-y)|\leqq\frac{M_2}{|y|^{\mathcal{R}(p)}(1+|x-y|^{\sigma+n+1})}$$
$$\leqq\frac{M_1}{\left(\frac{|x|}{2}\right)^{\sigma}(1+|x-y|^{\sigma+n+1})}.$$

It results from these inequalities that

$$|S_1*\varphi(x)|\leqq\left(\frac{|x|}{2}\right)^{\mathcal{R}(p)}\left\{M_1\int\frac{1}{|y|^{\sigma+n+1}}dy+M_2\int\frac{1}{1+|x-y|^{\sigma+n+1}}dy\right\}$$
$$=O(|x|^{\mathcal{R}(p)}).$$

It is not difficult to see that if $S*T$ is defined, then for any distribution $W\in(\mathcal{E}')$, the associative law for convolutions is valid, that is, $W*(S*T)=(W*S)*T$. Similarly we have

Proposition 1. *If $S\circledast T$ is defined and $W\in(\mathcal{O}_c')$, then*
$$W*(S*T)=(W*S)*T,$$

PROOF. $W\in(\mathcal{O}'_c)$ implies that $W*\varphi\in(\mathcal{S})$ for any $\varphi\in(\mathcal{S})$, and therefore $(W*S)*\varphi=S*(W*\varphi)\in S*(\mathcal{S})$. Now, by assumption, $S \circledast T$ is defined, so that

$$((W*S)*\varphi)\cdot(\check{T}*\psi)\in L \quad \text{for any} \quad \varphi, \psi\in(\mathcal{S})$$

that is, $(W*S)\circledast T$ is defined. Consider the linear form on $(\mathcal{O}'_c)$ defined by

$$(1) \qquad \chi(W)=\langle (W*S)*T*\varphi, \psi\rangle = \int ((W*S)*\varphi)(x)\cdot(\check{T}*\psi)(x)dx.$$

It follows by Theorem 2 [4] that χ is continuous on $(\mathcal{O}'_c)$. We may regard χ as an element of $(\mathcal{O}_c)$, the dual of $(\mathcal{O}'_c)$ and we can write $\chi(W)=\langle W, \chi\rangle$. For any $\varphi_1\in(\mathcal{D})$, we get

$$\langle \varphi_1, \chi\rangle = \int ((S*\varphi_1)*\varphi)(x)\cdot(\check{T}*\psi)(x)dx = \langle \varphi_1, (S*T)^{\vee}*\check{\varphi}*\psi\rangle.$$

Therefore $\chi=(S*T)^{\vee}*\check{\varphi}*\psi$. From this

$$(2) \qquad \langle W, \chi\rangle = \langle W, (S*T)^{\vee}*\check{\varphi}*\psi\rangle = \langle W*(S*T)*\varphi, \psi\rangle \text{ for any } \varphi, \psi\in(\mathcal{S}).$$

From (1) and (2) it results that $W*(S*T)=(W*S)*T$, as desired.

Most of the results contained in C. Chevally [1] (P. 117–123) concerning the convolutions are also valid with obvious modifications for our $(\mathcal{S}')$-convolutions. For example, if $S\circledast(T*\xi)$ is defined for any $\xi\in(\mathcal{D})$, then $S\circledast T$ is defined. To see this, it is enough to prove that $S*\varphi\cdot\check{T}*\psi$ belongs to L for any $\varphi, \psi\in(\mathcal{S})$. Consider the bilinear form

$$\lambda(\psi, \xi)=\int (S*\varphi)(x)\cdot(\check{T}*\psi*\xi)(x)dx.$$

For any relatively compact open subset Ω in R^n, $\lambda(\psi, \xi)$ is continuous on $(\mathcal{S})\times(\mathcal{D}_{\bar{\Omega}})$. Therefore there exists an integer $m>0$ such that

$$\int (S*\varphi)(x)\cdot(\check{T}*\psi*u)(x)dx<+\infty$$

for any $\psi\in(\mathcal{S})$ and $u\in(\mathcal{D}^m_{\bar{\Omega}})$.

By a suitable choice of a parametrix $u\in(\mathcal{D}^m_{\bar{\Omega}})$ such that $\delta=\Delta^h u+\rho$ where $\rho\in(\mathcal{D}_{\bar{\Omega}})$, we have

$$\psi=\Delta^h\psi*u+\psi*\rho.$$

Hence it follows immediately that

$$S*\varphi\cdot\check{T}*\psi \qquad \text{belongs to } L,$$

as desired.

2. According to L. Schwartz [3], the multiplicative product αA of distributions $\alpha\in(\mathcal{E})$ and $A\in(\mathcal{D}')$ is the distribution determined by $\langle \alpha A, \varphi\rangle = \langle A, \alpha\varphi\rangle$ for any $\varphi\in(\mathcal{D})$. We shall extend this definition for a larger class of distributions as follows: Let $\{\rho_k\}$ and $\{\rho'_k\}$ be any sequences of regularizations. If, for any distributions A and B, the sequences of the multiplica-

tive products $(A*\rho_k)\cdot B$ and $A(B*\rho'_k)$ converge to the same distribution in the sense of $(\mathcal{D}')$ as $k\to\infty$, then this common limit distribution is termed to be the *multiplicative* product of A and B, and denoted by $A\cdot B$.

THEOREM. *If* $S\circledast T$ *is defined, then*

$$\mathcal{F}(S*T)=\mathcal{F}(S)\cdot\mathcal{F}(T).$$

PROOF. It is sufficient to show that $\mathcal{F}(S)(\mathcal{F}(T)*\rho_k)\to\mathcal{F}(S*T)$ in $(\mathcal{D}')$ as $k\to\infty$, for any sequence of regularizations $\{\rho_k\}$. This is equivalent to show that

$$(1)\quad \lim_{k\to\infty}\langle\mathcal{F}(S)\cdot(\mathcal{F}(T)*\rho_k),\varphi\rangle=\langle\mathcal{F}(S*T),\varphi\rangle \text{ for any } \varphi\in(\mathcal{D}).$$

Let Ω be a relatively compact open subset of R^n containing the support of φ. We choose a function $\psi\in(\mathcal{D})$ such that $\check{\psi}(x)\equiv1$ on Ω. Let $\Phi=\overline{\mathcal{F}}(\varphi)$, $\Psi=\overline{\mathcal{F}}(\psi)$, and $r_k=\overline{\mathcal{F}}(\rho_k)$. Then Φ and Ψ belong to $(\mathcal{S})$, $r_k\to1$ in $(\mathcal{E})$ and $|r_k|\le1$. As $S\circledast T$ is defined, so $S*\Phi\cdot T*\Psi\in L$. Then

$$(2)\qquad \int(S*\Phi)(x)\cdot(\check{T}*\Psi)(x)dx=\lim_{k\to\infty}\int(S*\Phi)(x)\cdot(\check{T}*\Psi)(x)r_k(x)dx.$$

Let $\{\alpha_k\}$ be any sequence of multiplicators, that is, $\alpha_k\in(\mathcal{D})$ and $\{\alpha_k\}$ is bounded in $(\mathcal{B})$ and converges in $(\mathcal{E})$ to 1 as $k\to\infty$. Now by the aid of Parseval's formula, we have

$$(3)\qquad \begin{aligned}&\langle\alpha_m\cdot(S*\Phi),\ (T*\Psi)\cdot r_k\rangle\\ &=\langle\mathcal{F}(\alpha_m\cdot(S*\Phi)),\ \overline{\mathcal{F}}((T*\Psi)\cdot r_k)\\ &=\langle\mathcal{F}(\alpha_m)*(\mathcal{F}(S)\cdot\varphi),\ (\mathcal{F}(\check{T})\cdot\psi)^{\vee}*\rho_k\rangle.\end{aligned}$$

If follows since $(\mathcal{F}(\check{T})\cdot\psi)^{\vee}*\rho_k\in(\mathcal{S})$ and $\mathcal{F}(\alpha_m)\cdot(\mathcal{F}(S)\cdot\varphi)$ converges to $\mathcal{F}(S)\cdot\varphi$ in $(\mathcal{S}')$ as $k\to\infty$, that (3) yields

$$(4)\qquad \langle S*\Phi,(T*\Psi)\cdot r_k\rangle=\langle\mathcal{F}(S)\cdot\varphi,(\mathcal{F}(T)\cdot\check{\psi})*\rho_k\rangle.$$

As $\check{\psi}$ is taken to be 1 on a neighbourhood of the support of φ, so the second member of the equation (4) becomes $\langle\mathcal{F}(S)\cdot\varphi,\mathcal{F}(T)*\rho_k\rangle$ for all sufficiently large k. Then

$$\begin{aligned}\langle\mathcal{F}(S*T),\varphi\rangle&=\langle S*T,\mathcal{F}(\varphi)\rangle\\ &=\langle S*T,\mathcal{F}(\varphi\cdot\check{\psi})\rangle\\ &=\langle S*T,\mathcal{F}(\varphi)*\mathcal{F}(\psi)^{\vee}\rangle\\ &=\langle S*T*\mathcal{F}(\varphi)^{\vee},\mathcal{F}(\psi)^{\vee}\rangle\\ &=\langle S*T*\Phi,\Psi\rangle.\end{aligned}$$

Now using (2) and (4), we have

$$\begin{aligned}&\langle S*T*\Phi,\Psi\rangle\\ &=\lim_{k\to\infty}\langle S*\Phi,(\check{T}*\Psi)\cdot r_k\rangle\\ &=\lim_{k\to\infty}\langle\mathcal{F}(S)(\mathcal{F}(T)*\rho_k),\varphi\rangle.\end{aligned}$$

From this we have (1), as desired.

References

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