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Different Noetherian Rings in Some Axiomatic Relations

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§ 1. Introduction. Statement of the axiomatic relations.

The purpose of the present paper is to consider some ideal-theoretic relations between different Noetherian rings which are, as shown below, in some axiomatic relations.

(I). Let A and B be Noetherian rings such that A is a subring of B and that they have a common identity. We assume that the pair (A, B) satisfies the following three conditions:

(P_1) $\mathfrak{a}B \cap A = \mathfrak{a}$,

(P_2) $(\mathfrak{a}:\mathfrak{b})B = \mathfrak{a}B : \mathfrak{b}B$,

(P_3) $(\mathfrak{a} \cap \mathfrak{b})B = \mathfrak{a}B \cap \mathfrak{b}B$,

where $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of A; namely, the mapping $\mathfrak{a} \to \mathfrak{a}B$ is an isomorphism with respect to all the ideal-operations $(+, \cdot, :, \cap)$. It should be noted that in this case, for any ideal $\mathfrak{c}$ of A, the pair $(A/\mathfrak{c}, B/\mathfrak{c}B)$ also satisfies the three conditions, and further, the pair (A_M, B_M) satisfies them too, where M is a multicatively closed, non-empty subset of A which does not contain zero.

(II). Let A, A', and B be Noetherian rings such that A and A' are subrings of B and that they have a common identity. We assume that $(A, A'; B)$ satisfies the next conditions: If $\mathfrak{a}$ and $\mathfrak{a}'$ are ideals of A and A', respectively, then

(P_1^*) $(\mathfrak{a}B + \mathfrak{a}'B) \cap A = \mathfrak{a}$, $(\mathfrak{a}B + \mathfrak{a}'B) \cap A' = \mathfrak{a}'$,

(P_2^*) each pair of $(A/\mathfrak{a}, B/(\mathfrak{a}B + \mathfrak{a}'B))$ and $(A'/\mathfrak{a}', B/(\mathfrak{a}B + \mathfrak{a}'B))$ satisfies (P_1), (P_2), (P_3).

The above statements suggest a category of pairs of Noetherian rings, and that of classes of three Noetherian rings, which is closely related to the former category. In the following will be given the examples of Noetherian rings which belong to these categories and are also important in the theory of algebraic geometry.

Examples: (1) A Noetherian ring R and the polynomial ring $R[x_1, \cdots, x_n]$ in letters $x_1, \cdots, x_n$.

(2) A Noetherian ring R and the formal power series ring $R\{x_1, \cdots, x_n\}$ in letters $x_1, \cdots, x_n$.

(3) A Zariski ring and its completion.

Let R and R' be two Noetherian rings with a common subfield k, then

(4) R and the tensor product $R\otimes R'$ over k[1] (assumed to be Noetherian);

(4′) R, R', and $R\otimes R'$.

Let S be a semi-local ring and S' be a Zariski ring, assumed to have a common subfield k, then

(5) S and the complete tensor product $S\hat{\otimes}S'$ over k (assumed to be Noetherian);

(5′) S, S', and $S\hat{\otimes}S'$, where both S and S' are semi-local.

Now, in the previous paper [9], we considered some relations between the ideals of an $\mathfrak{m}$-adic Zariski ring S and those of its completion $\hat{S}$. In that paper, the relations (except Proposition 2) were derived, with the aid of the properties of general Noetherian rings, from the very facts that for any ideal $\mathfrak{c}$ of S, (i) $\hat{S}/\mathfrak{c}\hat{S}$ is the completion of $(\mathfrak{m}+\mathfrak{c})/\mathfrak{c}$-adic Zariski ring $S/\mathfrak{c}$, and hence, (ii) the pair $(S/\mathfrak{c}, \hat{S}/\mathfrak{c}\hat{S})$ satisfies the three conditions (P_1), (P_2), (P_3). Consequently, it will be seen that the theorems of [9] (except Proposition 2) are valid for (A, B) before mentioned, and in turn, these theorems are also valid for each case given as the examples, without any individual considerations. In §2, for the sake of completeness, we shall give those theorems, with brief sketch of proofs, for (A, B).

In §3, theorems similar to that of §2 will be stated for $(A, A'; B)$, and in §4, it will be shown that the examples (5) and (5′) really belong to the categories before mentioned.

As an application of the results obtained in §3, §5 will be devoted to the calculations of the multiplicities of primary ideals[2] of the rings shown in the example (4′) and of open primary ideals of those shown in the example (5′). In the latter example we shall give a more precise relation than that shown in [8], Chap. VI, 1. Our relation in the case where S and S' are local rings is as follows:

Let S and S' be local rings with a common subfield k; $\mathfrak{m}$ and $\mathfrak{m}'$ be maximal ideals of S and S', respectively; $S/\mathfrak{m}$ be canonically isomorphic with an algebraic extension over k. Suppose that the complete tensor product $T=S\hat{\otimes}S'$ over k is Noetherian, then it becomes a complete $(\mathfrak{m}T+\mathfrak{m}'T)$-adic semi-local ring. Further, let c_i be the length of the $\mathfrak{M}_i$-primary component of $\mathfrak{m}T+\mathfrak{m}'T (i=1,\cdots,n)$, where $\mathfrak{M}_1,\cdots,\mathfrak{M}_n$ are the maximal ideals of T. Then $c_1=\cdots=c_n(=c)$ and $\operatorname{rank}\mathfrak{M}_i=\dim S+\dim S'$ $(i=1,\cdots,n)$. In these situations, if $\mathfrak{d}$ and $\mathfrak{d}'$ are $\mathfrak{m}$- and $\mathfrak{m}'$-primary ideals of S and S', respectively, and if $\mathrm{e}(\mathfrak{d})$, $\mathrm{e}(\mathfrak{d}')$, and $\mathrm{e}(\mathfrak{d}T+\mathfrak{d}'T)$ denote the multiplicities of $\mathfrak{d}$, $\mathfrak{d}'$, and $\mathfrak{d}T+\mathfrak{d}'T$, respectively, then

$$\mathrm{e}(\mathfrak{d}T+\mathfrak{d}'T)=nc\,\mathrm{e}(\mathfrak{d})\,\mathrm{e}(\mathfrak{d}').$$

Finally in §6 a remark on *minimal bases* will be given.

1) We understand that R and R' have been canonically embedded in $R\otimes R'$. The same for (5).

2) We mean the multiplicity introduced by P. Samuel.

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§2. (A, B).

Let A and B be two Noetherian rings such that A is a subring of B, and let A and B have a common identity. Assume that the pair (A, B) satisfies the three conditions (P_1), (P_2), (P_3) given in the introduction. In this section, unless otherwise explicitly stated, A and B will denote these rings; $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{p}$, $\mathfrak{q}$ will denote ideals of A and $\mathfrak{P}$, $\mathfrak{Q}$ ideals of B.

Theorem 1. (Prime divisors). *Let $\mathfrak{p}$ be any prime divisor of $\mathfrak{a}$. Then every prime divisor of $\mathfrak{p}B$ is also a prime divisor of $\mathfrak{a}B$, and conversely, any prime divisor of $\mathfrak{a}B$ is a prime divisor of $\mathfrak{p}B$ for some prime divisor $\mathfrak{p}$ of $\mathfrak{a}$. Moreover, if $\mathfrak{p}$ is isolated, then every isolated prime divisor of $\mathfrak{p}B$ is also an isolated prime divisor of $\mathfrak{a}B$, and conversely, any isolated prime divisor of $\mathfrak{a}B$ is an isolated prime divisor of $\mathfrak{p}B$ for some isolated prime divisor $\mathfrak{p}$ of $\mathfrak{a}$.*

The second part of the theorem will be proved by the fact that if $\mathfrak{P}$ is a prime divisor of $\mathfrak{p}B$, where $\mathfrak{p}$ is any prime ideal, then $\mathfrak{P}\cap A=\mathfrak{p}$, which is an immediate consequence of (P_2). To see the first part, let $\mathfrak{p}$ be any prime divisor of $\mathfrak{a}$, then it will follow that every prime divisor of $\mathfrak{p}B$ is also a prime divisor of $\mathfrak{a}B$, from (P_2) and the following fact: In a Noetherian ring R, a prime ideal $\mathfrak{p}$ of R is a prime divisor of an ideal $\mathfrak{a}$ of R if and only if $\mathfrak{p}=\mathfrak{a}:pR$ for some element $p\notin\mathfrak{a}$. To see the converse it will be enough, by (P_3), to prove the following: Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal, then $\mathfrak{q}B$ and $\mathfrak{p}B$ have the same prime divisors. This statement will be proved by induction on the length of $\mathfrak{q}$.

Remark 1. Let $\mathfrak{q}$ be a primary ideal and let $\mathfrak{q}B=\mathfrak{Q}_1\cap\cdots\cap\mathfrak{Q}_n$ be a normal decomposition of $\mathfrak{q}B$ into primary components. Then we have $\mathfrak{Q}_i\cap A=\mathfrak{q}$ $(i=1,\cdots,n)$ ([9], p. 95, Lemma 4).

Proposition 1. *Let $\mathfrak{p}$ be a prime ideal, then every isolated prime divisor of $\mathfrak{p}B$ has the same rank as $\mathfrak{p}$.*

Let $\mathfrak{p}_0\subset\mathfrak{p}_1\subset\cdots\subset\mathfrak{p}_r=\mathfrak{p}$ (rank $\mathfrak{p}=r$) be a chain of prime ideals, and let $\mathfrak{P}$ be any isolated prime divisor of $\mathfrak{p}B$. Then there exists a chain of prime ideals $\mathfrak{P}_0\subset\mathfrak{P}_1\subset\cdots\subset\mathfrak{P}_r=\mathfrak{P}$ such that $\mathfrak{P}_i$ is an isolated prime divisor of $\mathfrak{p}_iB$ $(i=0,1,\cdots,r)$. From this and Krull's Primidealkettensatz, Proposition 1 will follow by virtue of Theorem 1.

Corollary 1. rank $\mathfrak{a}=$rank $\mathfrak{a}B$.

Corollary 2. *If the unmixedness theorem holds in B, it also holds in A*[3].

3) For the unmixedness theorem, see, p. 211, [7].

Theorem 2. **(Maximal chains of prime ideals).** *Let* $\mathfrak{p}_0\subset\mathfrak{p}_1\subset\cdots\subset\mathfrak{p}_r$ *be a chain of prime ideals such that* rank $\mathfrak{p}_i/\mathfrak{p}_{i-1}=1$ $(i=1,\cdots,r)$. *Then there exists a chain of prime ideals* $\mathfrak{P}_0\subset\mathfrak{P}_1\subset\cdots\subset\mathfrak{P}_r$ *such that* $\mathfrak{P}_i\cap A=\mathfrak{p}_i$, $\mathfrak{P}_i$ *is an isolated prime divisor of* $\mathfrak{p}_iB$ $(i=0,1,\cdots,r)$, *and* rank $\mathfrak{P}_j/\mathfrak{P}_{j-1}=1$ $(j=1,\cdots,r)$.

Denote by $\mathrm{L}(\mathfrak{q})$ and $\mathrm{e}(\mathfrak{q})$ the length and the multiplicity of an arbitrary primary ideal $\mathfrak{q}$, respectively, then

Theorem 3. **(Transition theorem).** *Let* $\mathfrak{q}$ *be a* $\mathfrak{p}$*-primary ideal and let* $\mathfrak{P}$ *be an isolated prime divisor of* $\mathfrak{p}B$. *Then* $\mathrm{L}(\mathfrak{q})\mathrm{L}(\mathfrak{p}B_{\mathfrak{P}})=\mathrm{L}(\mathfrak{q}B_{\mathfrak{P}})$ *and* $\mathrm{e}(\mathfrak{q})\mathrm{L}(\mathfrak{p}B_{\mathfrak{P}})=\mathrm{e}(\mathfrak{q}B_{\mathfrak{P}})$.

Let $\mathfrak{q}_1$ and $\mathfrak{q}_2(\mathfrak{q}_1\supset\mathfrak{q}_2)$ be $\mathfrak{p}$-primary such that no further $\mathfrak{p}$-primary ideal can be inserted between $\mathfrak{q}_1$ and $\mathfrak{q}_2$, and let $\mathfrak{Q}_0$, $\mathfrak{Q}_1$, and $\mathfrak{Q}_2$ be the isolated $\mathfrak{P}$-primary components of $\mathfrak{p}B$, $\mathfrak{q}_1B$, and $\mathfrak{q}_2B$, respectively. Then for a suitable element $p\in\mathfrak{q}_1$, $\notin\mathfrak{q}_2$, $\mathfrak{Q}_0=\mathfrak{Q}_2:pB$, and $\mathfrak{Q}_2+pB$ has $\mathfrak{Q}_1$ as the isolated $\mathfrak{P}$-primary component. Passing to $(B/\mathfrak{Q}_2)_{\mathfrak{P}/\mathfrak{Q}_2}$, we have $\mathrm{L}(\mathfrak{Q}_2)-\mathrm{L}(\mathfrak{Q}_1)=\mathrm{L}(\mathfrak{Q}_0)$, and from this follows the first relation of the theorem. The second relation concerning multiplicities will follow from the relation $\mathrm{L}(\mathfrak{q}^{(n)})\mathrm{L}(\mathfrak{p}B_{\mathfrak{P}})=\mathrm{L}(\mathfrak{Q}^{(n)})$, where $\mathfrak{q}^{(n)}$ is the n-th symbolic power of $\mathfrak{q}$ and $\mathfrak{Q}^{(n)}$ is the one of the isolated $\mathfrak{P}$-primary component $\mathfrak{Q}$ of $\mathfrak{q}B$.

§3. (A, A′; B).

Let A, A', and B be Noetherian rings such that A and A' are subrings of B and that they have a common identity. We assume that (P_1^*), (P_2^*) (briefly (P^*)) given in the introduction are satisfied by $(A, A'; B)$. In this section, unless otherwise explicitly stated, A, A', and B will denote these rings; $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{p}$, $\mathfrak{q}$ will denote ideals of A, $\mathfrak{a}'$, $\mathfrak{b}'$, $\mathfrak{p}'$, $\mathfrak{q}'$ ideals of A', and $\mathfrak{P}$, $\mathfrak{Q}$ ideals of B.

Theorem 1*. *Let* $\mathfrak{p}$ *and* $\mathfrak{p}'$ *be any prime divisors of* $\mathfrak{a}$ *and* $\mathfrak{a}'$, *respectively. Then every prime divisor of* $\mathfrak{p}B+\mathfrak{p}'B$ *is also a prime divisor of* $\mathfrak{a}B+\mathfrak{a}'B$, *and conversely, any prime divisor of* $\mathfrak{a}B+\mathfrak{a}'B$ *is a prime divisor of* $\mathfrak{p}B+\mathfrak{p}'B$ *for some prime divisors* $\mathfrak{p}$ *of* $\mathfrak{a}$ *and* $\mathfrak{p}'$ *of* $\mathfrak{a}'$. *Moreover, if* $\mathfrak{p}$ *and* $\mathfrak{p}'$ *are isolated, then every isolated prime divisor of* $\mathfrak{p}B+\mathfrak{p}'B$ *is also an isolated prime divisor of* $\mathfrak{a}B+\mathfrak{a}'B$, *and conversely, any isolated prime divisor of* $\mathfrak{a}B+\mathfrak{a}'B$ *is an isolated prime divisor of* $\mathfrak{p}B+\mathfrak{p}'B$ *for some isolated prime divisors* $\mathfrak{p}$ *of* $\mathfrak{a}$ *and* $\mathfrak{p}'$ *of* $\mathfrak{a}'$.

Proof. Let $\mathfrak{p}$ and $\mathfrak{p}'$ be any prime ideals. Since, by (P^*), the pair $(A/\mathfrak{p}, B/(\mathfrak{p}B+\mathfrak{p}'B))$ satisfies (P_2), each non-zero element in $A/\mathfrak{p}$ is not a zero-divisor in $B/(\mathfrak{p}B+\mathfrak{p}'B)$; hence, if $\mathfrak{P}$ is a prime divisor of $\mathfrak{p}B+\mathfrak{p}'B$, then $\mathfrak{P}\cap A=\mathfrak{p}$; similarly, $\mathfrak{P}\cap A'=\mathfrak{p}'$. From these results we may easily verify the second part of the theorem. To see the first part of the theorem, let $\mathfrak{p}$ and $\mathfrak{p}'$ be any prime divisors of $\mathfrak{a}$ and $\mathfrak{a}'$, respectively. Consider the pair $(A, B/\mathfrak{p}'B)$. Then it will be seen by Theorem 1 that any prime divisor of $(\mathfrak{p}B+\mathfrak{p}'B)/\mathfrak{p}'B$ is also a prime divisor of $(\mathfrak{a}B+\mathfrak{p}'B)/\mathfrak{p}'B$; this shows that any prime divisor of $\mathfrak{p}B+\mathfrak{p}'B$ is also a prime divisor of $\mathfrak{a}B+\mathfrak{p}'B$. Next consider the pair $(A', B/\mathfrak{a}B)$, then by a similar reasoning, we see that any prime divisor of $\mathfrak{a}B+\mathfrak{p}'B$ is also a prime divisor of $\mathfrak{a}B+\mathfrak{a}'B$; therefore every prime divisor of $\mathfrak{p}B+\mathfrak{p}'B$ is also a prime divisor of $\mathfrak{a}B+\mathfrak{a}'B$. The converse of this result will be seen by the following two lemmas.

Lemma 1. *Let $\mathfrak{q}$ and $\mathfrak{q}'$ be $\mathfrak{p}$- and $\mathfrak{p}'$-primary ideals, respectively. Then $\mathfrak{q}B+\mathfrak{q}'B$ and $\mathfrak{p}B+\mathfrak{p}'B$ have the same prime divisors.*

Proof. With the same reasoning as above, we can see this assertion by making use of Lemma 3, p. 94, [9].

Lemma 2. $(\mathfrak{a}\cap\mathfrak{b})B+\mathfrak{a}'B=(\mathfrak{a}B+\mathfrak{a}'B)\cap(\mathfrak{b}B+\mathfrak{a}'B)$, $\mathfrak{a}B+(\mathfrak{a}'\cap\mathfrak{b}')B=(\mathfrak{a}B+\mathfrak{a}'B)\cap(\mathfrak{a}B+\mathfrak{b}'B)$.

Proof. These will be seen by (P^*) and (P_3).

Remark 1*. In Lemma 1, let $\mathfrak{q}B+\mathfrak{q}'B=\mathfrak{Q}_1\cap\cdots\cap\mathfrak{Q}_n$ be a normal decomposition of $\mathfrak{q}B+\mathfrak{q}'B$ into primary components. Then $\mathfrak{Q}_i\cap A=\mathfrak{q}$, $\mathfrak{Q}_i\cap A'=\mathfrak{q}'$ $(i=1,\cdots,n)$.

Proposition 1*. *Let $\mathfrak{p}$ and $\mathfrak{p}'$ be prime ideals, then the rank of any isolated prime divisor of $\mathfrak{p}B+\mathfrak{p}'B$ is equal to* $\operatorname{rank}\mathfrak{p}+\operatorname{rank}\mathfrak{p}'$.

Proof. Let $\mathfrak{p}_0\subset\mathfrak{p}_1\subset\cdots\subset\mathfrak{p}_r=\mathfrak{p}$ $(\operatorname{rank}\mathfrak{p}=r)$ and $\mathfrak{p}'_0\subset\mathfrak{p}'_1\subset\cdots\subset\mathfrak{p}'_s=\mathfrak{p}'$ $(\operatorname{rank}\mathfrak{p}'=s)$ be chains of prime ideals in A and A', respectively. If we consider the pair $(A', B/\mathfrak{p}_rB)$, then from Theorem 2 it follows that there exists a chain of prime ideals $\mathfrak{P}_r\subset\cdots\subset\mathfrak{P}_{r+s}$ such that $\mathfrak{P}_{r+j}\cap A'=\mathfrak{p}'_j$ and $\mathfrak{P}_{r+j}$ is an isolated prime divisor of $\mathfrak{p}_rB+\mathfrak{p}'_jB$ $(j=0,1,\cdots,s)$. Next consider the pair $(A, B/\mathfrak{p}'_0B)$. Then, by the fact that $\mathfrak{P}_r\cap A=\mathfrak{p}_r$ and $\mathfrak{P}_r$ is an isolated prime divisor of $\mathfrak{p}_rB+\mathfrak{p}'_0B$, we can obtain, again by Theorem 2, a chain of prime ideals $\mathfrak{P}_0\subset\cdots\subset\mathfrak{P}_r$ such that $\mathfrak{P}_i\cap A=\mathfrak{p}_i$ $(i=0,1,\cdots,r)$. From these results and Krull's Primidealkettensatz, Proposition 1* will follow by virtue of Theorem 1*.

Corollary. $\operatorname{rank}(\mathfrak{a}B+\mathfrak{a}'B)=\operatorname{rank}\mathfrak{a}+\operatorname{rank}\mathfrak{a}'$.

From the proof of Proposition 1*, we see

Theorem 2*. *Let $\mathfrak{p}_0\subset\mathfrak{p}_1\subset\cdots\subset\mathfrak{p}_r$ and $\mathfrak{p}'_0\subset\mathfrak{p}'_1\subset\cdots\subset\mathfrak{p}'_s$ be chains of prime ideals in A and A', respectively, such that* $\operatorname{rank}\mathfrak{p}_i/\mathfrak{p}_{i-1}=1$ $(i=1,\cdots,r)$ *and* $\operatorname{rank}\mathfrak{p}'_j/\mathfrak{p}'_{j-1}=1$ $(j=1,\cdots,s)$. *Then there exists a chain of prime ideals*

$\mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_{r+s}$ *in* B *such that* $\mathfrak{P}_i \cap A = \mathfrak{p}_i$, $\mathfrak{P}_{r+j} \cap A' = \mathfrak{p}'_j$ $(i=0,1,\cdots,r;$ $j=0,1,\cdots,s)$, *and* rank $\mathfrak{P}_k/\mathfrak{P}_{k-1}=1$ $(k=1,\cdots,r+s)$.

Now let $\mathfrak{q}$ and $\mathfrak{q}'$ be $\mathfrak{p}$- and $\mathfrak{p}'$-primary, respectively, and let $\mathfrak{P}$ be an isolated prime divisor of $\mathfrak{p}B+\mathfrak{p}'B$. Let further $\mathfrak{Q}_0$, $\mathfrak{Q}_1$, $\mathfrak{Q}_2$ be the isolated $\mathfrak{P}$-primary components of $\mathfrak{p}B+\mathfrak{p}'B$, $\mathfrak{p}B+\mathfrak{q}'B$, $\mathfrak{q}B+\mathfrak{q}'B$, respectively. Applying Theorem 3 to the pairs $(A', B/\mathfrak{p}B)$ and $(A, B/\mathfrak{q}'B)$ respectively, we get $\mathrm{L}(\mathfrak{q}')\mathrm{L}(\mathfrak{Q}_0)=\mathrm{L}(\mathfrak{Q}_1)$ and $\mathrm{L}(\mathfrak{q})\mathrm{L}(\mathfrak{Q}_1)=\mathrm{L}(\mathfrak{Q}_2)$; consequently $\mathrm{L}(\mathfrak{Q}_2)=\mathrm{L}(\mathfrak{q})\mathrm{L}(\mathfrak{q}')\mathrm{L}(\mathfrak{Q}_0)$. This is the transition theorem on lengths of primary ideals, which we shall state in the following form:

Theorem 3*. *Let* $\mathfrak{q}$ *and* $\mathfrak{q}'$ *be* $\mathfrak{p}$*- and* $\mathfrak{p}'$*-primary ideals, respectively. If* $\mathfrak{P}$ *is an isolated prime divisor of* $\mathfrak{p}B+\mathfrak{p}'B$, *then* $\mathrm{L}(\mathfrak{q})\mathrm{L}(\mathfrak{q}')\mathrm{L}((\mathfrak{p}B+\mathfrak{p}'B)B_{\mathfrak{P}})=\mathrm{L}((\mathfrak{q}B+\mathfrak{q}'B)B_{\mathfrak{P}})$.

For future use we shall add two lemmas.

Lemma 3. *Notations being the same as in Theorem 3**, $\mathrm{L}(\mathfrak{q}^{(i)})\mathrm{L}(\mathfrak{q}'^{(j)})\mathrm{L}((\mathfrak{p}B+\mathfrak{p}'B)B_{\mathfrak{P}})=\mathrm{L}((\mathfrak{q}^iB+\mathfrak{q}'^jB)B_{\mathfrak{P}})$, *where* $\mathfrak{q}^{(i)}$ *and* $\mathfrak{q}'^{(j)}$ *are the i-th and j-th symbolic powers of* $\mathfrak{q}$ *and* $\mathfrak{q}'$, *respectively.*

Proof. Since $\mathfrak{q}^{(i)}$ and $\mathfrak{q}'^{(j)}$ are the isolated $\mathfrak{p}$- and $\mathfrak{p}'$-primary components of $\mathfrak{q}^i$ and $\mathfrak{q}'^j$, respectively, it follows from Theorem 1* that $\mathfrak{q}^iB+\mathfrak{q}'^jB$ and $\mathfrak{q}^{(i)}B+\mathfrak{q}'^{(j)}B$ have the same isolated $\mathfrak{P}$-primary component. Hence by Theorem 3*, $\mathrm{L}(\mathfrak{q}^{(i)})\mathrm{L}(\mathfrak{q}'^{(j)})\mathrm{L}((\mathfrak{p}B+\mathfrak{p}'B)B_{\mathfrak{P}})=\mathrm{L}((\mathfrak{q}^iB+\mathfrak{q}'^jB)B_{\mathfrak{P}})$.

Lemma 4. $(\mathfrak{a}:\mathfrak{b})B+\mathfrak{a}'B=(\mathfrak{a}B+\mathfrak{a}'B):\mathfrak{b}B$, $\mathfrak{a}B+(\mathfrak{a}':\mathfrak{b}')B=(\mathfrak{a}B+\mathfrak{a}'B):\mathfrak{b}'B$.

Proof. Consider the pair $(A, B/\mathfrak{a}'B)$, then the first relation will be proved by (P_2); next consider the pair $(A', B/\mathfrak{a}B)$, then the second one will be proved again by (P_2).

Corollary. $(\mathfrak{a}B+\mathfrak{a}'B):(\mathfrak{b}B+\mathfrak{b}'B)=((\mathfrak{a}:\mathfrak{b})B+\mathfrak{a}'B)\cap(\mathfrak{a}B+(\mathfrak{a}':\mathfrak{b}')B)$.

Remark 2. Let R and R' be two commutative rings with a common subfield k, and let $R\otimes R'$[4] be the tensor product of R and R' over k. Since $R\otimes R'/(\mathfrak{a}\otimes R'+R\otimes\mathfrak{a}')$ is isomorphic with $(R/\mathfrak{a})\otimes(R'/\mathfrak{a}')$ where $\mathfrak{a}$ and $\mathfrak{a}'$ are ideals of R and R' respectively, we can prove, by making use of suitable k-basis, that $(R, R';\, R\otimes R')$ satisfies (P*), namely, the assumption that R, R', and $R\otimes R'$ are Noetherian is unnecessary for the validity of (P*) for $(R, R';\, R\otimes R')$. Hence, Lemma 4 is also valid for this class $(R, R';\, R\otimes R')$. This remark will be necessary later on.

4) When we construct $\otimes$ and $\hat{\otimes}$ over k, we shall omit k.

§4. $S\hat{\otimes}S'$.

In this section we shall verify that the pair of rings given as example (5) in the introduction really satisfies the conditions (P_1), (P_2), (P_3).

Let S and S' be, respectively, $\mathfrak{m}$- and $\mathfrak{m}'$-adic Zariski rings with a common subfield k. Set $T_1=S\otimes S'$. A direct proof of the well-known fact that T_1 is a general $(\mathfrak{m}T_1+\mathfrak{m}'T_1)$-adic ring[5], is as follows: Let $\xi\in\bigcap\mathfrak{m}^nT_1$ and let $\xi=r_1r_1'+\cdots+r_tr_t'$, where $r_i\in S$, $r_i'\in S'$ $(i=1,\cdots,t)$, and r_i' are k-linearly independent. Since $\xi\in\mathfrak{m}^nT_1$ and $\mathfrak{m}^nT_1$ is isomorphic with $\mathfrak{m}^n\otimes S'$, $r_i\in\mathfrak{m}^n$ $(n=1,2,\cdots)$; hence $r_i=0$. This shows that $\bigcap\mathfrak{m}^nT_1=(0)$. From this reasoning it follows that $\bigcap_n(\mathfrak{m}^nT_1+\mathfrak{m}'^iT_1)=\mathfrak{m}'^iT_1$, because $T_1/\mathfrak{m}'^iT_1$ is isomorphic with $S\otimes(S'/\mathfrak{m}'^i)$. Thus we see : $\bigcap(\mathfrak{m}T_1+\mathfrak{m}'T_1)^n=\bigcap(\mathfrak{m}^nT_1+\mathfrak{m}'^nT_1)\subseteq\bigcap_i\{\bigcap_n(\mathfrak{m}^nT_1+\mathfrak{m}'^iT_1)\}=\bigcap_i\mathfrak{m}'^iT_1=(0)$. Now the completion of T_1 is the complete tensor product $S\hat{\otimes}S'$, which we denote by T.

Let T be Noetherian[6], then it becomes a $(\mathfrak{m}T+\mathfrak{m}'T)$-adic Zariski ring. It seems to be desirable to state here Proposition 2, p. 65, [8], as a lemma, since our discussions depend much on it.

Lemma 5. *Notations being the same as above, if* $\mathfrak{a}$ *and* $\mathfrak{a}'$ *are ideals of* S *and* S', *respectively, then* $(\mathfrak{a}T+\mathfrak{a}'T)\cap S=\mathfrak{a}$, $(\mathfrak{a}T+\mathfrak{a}'T)\cap S'=\mathfrak{a}'$, *and* $T/(\mathfrak{a}T+\mathfrak{a}'T)$ *is isomorphic with* $(S/\mathfrak{a})\hat{\otimes}(S'/\mathfrak{a}')$.

This lemma shows that the pair (S, T) satisfies (P_1). We now assume that S is $\mathfrak{m}$-adic semi-local. There remains to verify that the pair (S, T) satisfies (P_2), (P_3), and for this purpose it will be enough to show that $(\mathfrak{a}:aS)T=\mathfrak{a}T:aT$, where a is an element in S, because S is Noetherian and hence, we can apply the same method as was shown on p. 54, [4], to the present case by virtue of Lemma 5.

Obviously $(\mathfrak{a}:aS)T\subseteq\mathfrak{a}T:aT$. To prove the converse inequality, let $\xi\in\mathfrak{a}T:aT$, and put $\xi=b_n+\overline{m}_n$, where $b_n\in T_1$, $\overline{m}_n\in\mathfrak{m}^nT+\mathfrak{m}'^nT$. Then $ab_n\in(\mathfrak{a}T+a(\mathfrak{m}^nT+\mathfrak{m}'^nT))\cap T_1$. Since $(\mathfrak{a}T+a(\mathfrak{m}^nT+\mathfrak{m}'^nT))\cap T_1$ is the closure of $\mathfrak{a}T_1+a(\mathfrak{m}^nT_1+\mathfrak{m}'^nT_1)$ in T_1, $ab_n\in\mathfrak{a}T_1+a(\mathfrak{m}^nT_1+\mathfrak{m}'^nT_1)+\mathfrak{m}^NT_1+\mathfrak{m}'^NT_1$ for any positive integer N. Hence $a(b_n-m_n)\in\mathfrak{a}T_1+\mathfrak{m}^NT_1+\mathfrak{m}'^NT_1$ for some $m_n\in\mathfrak{m}^nT_1+\mathfrak{m}'^nT_1$; consequently $b_n-m_n\in((\mathfrak{a}+\mathfrak{m}^N)T_1+\mathfrak{m}'^NT_1):aT_1$. By Remark 2, $((\mathfrak{a}+\mathfrak{m}^N)T_1+\mathfrak{m}'^NT_1):aT_1=((\mathfrak{a}+\mathfrak{m}^N):aS)\ T_1+m'^NT_1$; since S is semi-local, $((\mathfrak{a}+\mathfrak{m}^N):aS)T_1+\mathfrak{m}'^NT_1\subset(\mathfrak{a}:aS)T_1+\mathfrak{m}^{\sigma(N)}T_1+\mathfrak{m}'^NT_1$, where $\sigma(N)\to\infty$ as $N\to\infty$; hence $b_n-m_n\in(\mathfrak{a}:aS)T_1+\mathfrak{m}^NT_1+\mathfrak{m}'^NT_1$ for any positive integer N. This shows that $b_n-m_n\in(\mathfrak{a}:aS)T_1$, because $(\mathfrak{a}:aS)T_1$ is closed (see the proof of

5) Given a commutative ring R with an identity and an ideal $\mathfrak{m}$ of R such that $\bigcap\mathfrak{m}^n=(0)$, we may topologize R by adopting $\{\mathfrak{m}^n:n=1,2\ldots\}$ as a fundamental system of neighbourhoods of zero. This topologized ring is referred to as a general $\mathfrak{m}$-adic ring provided that $\mathfrak{m}$ has a finite ideal basis.

6) A necessary and sufficient condition for T to be Noetherian is that $(S/\mathfrak{m})\otimes(S'/\mathfrak{m}')$ be Noetherian.

Lemma 5). Therefore $\xi \in (\mathfrak{a}:aS)T$, because ξ is the limit of the sequence $\{b_n - m_n\}$; this is what we had to prove.

So far we have seen that if S is semi-local, then the pair $(S, S\hat{\otimes}S')$ satisfies (P_1), (P_2), (P_3). From this it will be seen, by Lemma 5, that if both S and S' are semi-local, then $(S, S'; S\hat{\otimes}S')$ satisfies (P^*).

Further results: (1). For future use, we shall prove here the following: If, moreover, S and S' are local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$, respectively, then the prime divisors of $\mathfrak{m}T + \mathfrak{m}'T$ are all of them isolated and the lengths of the (isolated) primary components of that ideal are equal to each other. Our assertion, however, will be reduced, by Lemma 5, to the following: Let K and K' be fields with a common subfield k, and suppose that $H = K\otimes K'$ is Noetherian, then every prime divisor of zero ideal of H is isolated and the lengths of the (isolated) primary components of that ideal are equal to each other. To see this assertion let $(x_i) = (x_1, x_2, \cdots)$ and $(x'_j) = (x'_1, x'_2, \cdots)$ be subsets of K and K', respectively, which are algebraically independent over k. Suppose that both (x_i) and (x'_j) are infinite in number, then there exists, in $k((x_i))\otimes k((x'_j)) = H_0$, a strictly ascending sequence of prime ideals: $(x_1 - x'_1)H_0 \subset (x_1 - x'_1)H_0 + (x_2 - x'_2)H_0 \subset \cdots$. On the other hand, since H is Noetherian and since $H = K\otimes_{k((x_i))}(k((x_i))\otimes K')$ (canonically identified), by Remark 2, $k((x_i))\otimes K'$ must be Noetherian ((P_1)); consequently, we see, in a similar manner, that H_0 is Noetherian, and this is a contradiction. Henceforth we assume $\dim_k K < \infty$ and denote by $(x_i) = (x_1, \cdots, x_m)$ a transcendental basis of K over k. Then H is integral over $k((x_i))\otimes K'$ (canonically embedded), which is a Noetherian integral domain, and from this we may conclude that any prime divisor of zero ideal of H is isolated (by (P_2)). Let now $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be the normal decomposition of zero ideal of H into primary components and let $\mathfrak{p}_i$ be the prime divisor of $\mathfrak{q}_i$ $(i = 1, \cdots, n)$. Our next object is to show that $L(\mathfrak{q}_1) = \cdots = L(\mathfrak{q}_n)$. Put $L(\mathfrak{q}_1) = l_1$ and let $\mathfrak{q}_1 = \mathfrak{q}_{11} \subset \mathfrak{q}_{12} \subset \cdots \subset \mathfrak{q}_{1l_1} = \mathfrak{p}_1$ be a chain of $\mathfrak{p}_1$-primary ideals. Then there exist elements $\xi_{1i} \in \mathfrak{q}_{1i}$ such that $\xi_{1i} \notin \mathfrak{p}_2, \cdots, \mathfrak{p}_n, \mathfrak{q}_{1\,i-1}$ $(i = 1, \cdots, l_1)$. In fact, since $\mathfrak{q}_{1\,i-1} \not\subseteq \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_n$, there exists an element $\xi \in \mathfrak{q}_{1\,i-1}$ such that no $\mathfrak{p}_j$ contains ξ $(j = 2, \cdots, n)$; let $\zeta \in \mathfrak{q}_{1i}, \notin \mathfrak{q}_{1i-1}$, and let $\eta \in \mathfrak{p}_2, \cdots, \mathfrak{p}_n, \notin \mathfrak{p}_1$, then $\eta\zeta \in \mathfrak{q}_{1i}, \mathfrak{p}_2, \cdots, \mathfrak{p}_n, \notin \mathfrak{q}_{1i-1}$; put $\xi_{1i} = \xi + \eta\zeta$, then this is such an element as we required. For each pair $(\mathfrak{q}_r, \mathfrak{p}_r)$ $(r = 2, \cdots, n)$ select elements $\{\xi_{rj}\}$ which have the same properties as $\{\xi_{1i}\}$ and take a suitable finite algebraic extension $K_1 (\subseteq K)$ over $k((x_i))$ such that all the ξ_{ij} are contained in $H_1 = K_1 \otimes K'$. Since $H = K\otimes_{K_1} H_1$ (canonically identified), by Theorem 1 and 3, we get the normal decomposition of zero ideal of H_1 into primary components: $(0) = \bar{\mathfrak{q}}_1 \cap \cdots \cap \bar{\mathfrak{q}}_n$, where $\bar{\mathfrak{q}}_i = \mathfrak{q}_i \cap H_1$ and $L(\bar{\mathfrak{q}}_i) = L(\mathfrak{q}_i)$ $(i = 1, \cdots, n)$. By construction, K_1 is the so-called algebraic function field over k, and hence it is known that $L(\bar{\mathfrak{q}}_1) = L(\bar{\mathfrak{q}}_2) = \cdots = L(\bar{\mathfrak{q}}_n)$ ([3], p. 44, Theorem 2.10, which is stated as: Let K be an algebraic function field over a field k and let K' be an arbitrary overfield of k. Then the total quotient ring of $K\otimes K'$ is a direct sum of primary rings and the lengths of the components of the direct sum are equal to each other). The common length is referred to as the order of inseparability of K_1 over k with respect to K'. Thus our proof is complete.

We shall state the above observations in the following form, which is a modification of the quoted theorem of [3].

Theorem 4. *Let R and R' be Noetherian rings with a common subfield k and assume that the tensor product $U=R\otimes R'$ over k is Noetherian. Let $\mathfrak{p}$ and $\mathfrak{p}'$ be prime ideals of R and R', respectively. Then the rank of any prime divisor of $\mathfrak{p}U+\mathfrak{p}'U$ is equal to* rank $\mathfrak{p}$+rank $\mathfrak{p}'$ *and the lengths of the (isolated) primary components of $\mathfrak{p}U+\mathfrak{p}'U$ are equal to each other.*

(2). Next we shall state propositions which correspond to Proposition 9a, p. 78, [1], which says: Let Z be any overfield of a field K and let $K\{x_1,\cdots,x_n\}$ be a formal power series ring in letters $x_1,\cdots,x_n$. Then, if $\mathfrak{p}$ is a prime ideal of dimension r of $K\{x_1,\cdots,x_n\}$, every prime divisor of the ideal generated by $\mathfrak{p}$ in $Z\{x_1,\cdots,x_n\}$ is of dimension r.

Our propositions are as follows.

Proposition 2. *Let S and S' be complete local rings with the same basic field k and let $T=S\hat{\otimes}S'$ be the complete tensor product of S and S' over k. Then, if $\mathfrak{p}$ and $\mathfrak{p}'$ are prime ideals of S and S', respectively, for any prime divisor $\mathfrak{P}$ of $\mathfrak{p}T+\mathfrak{p}'T$,* rank $\mathfrak{P}$=rank $\mathfrak{p}$+rank $\mathfrak{p}'$.

Proof. It will be enough to prove that $\mathfrak{P}$ is isolated. Now by Lemma 5 we may reduce the assertion to the case where S, S' are complete local domains and $\mathfrak{p}$, $\mathfrak{p}'$ are zero ideals of S, S', respectively. These having been done, let $x_1,\cdots,x_n$ be a system of parameters in S, then x_i are analytically independent over k, so that $R=k\{x_1,\cdots,x_n\}$ is a formal power series ring; put $(x_1,\cdots,x_n)R=\mathfrak{n}$. As is well known, R is a subspace of S and S is a finite R-module, say $S=R+Ry_1+\cdots+Ry_m$; we renumber y_i so that $\{1, y_1,\cdots,y_t\}$ is a maximal subset of R-linearly independent elements of the set $\{1, y_1,\cdots,y_m\}$, then there exists a non-zero element $c\in R$ such that $cS\subseteq R+Ry_1+\cdots+Ry_t$. For S', the corresponding ones to the above will be denoted by prime notation.

Suppose that $U_1=R\otimes R'$ has been canonically embedded in $T_1=S\otimes S'$. Then, as is easily seen, (i) T_1 is a finite U_1-module: $T_1=\sum_{i=0}^{m}\sum_{j=0}^{m'}U_1y_iy'_j$ ($y_0=y'_0=1$), $cc'T_1\subseteq\sum_{i=0}^{t}\sum_{j=0}^{t'}U_1y_iy'_j$, where $y_iy'_j$ ($i=0,\cdots,t$; $j=0,\cdots,t'$) are linearly independent over U_1; (ii) the general $(\mathfrak{n}U_1+\mathfrak{n}'U_1)$-adic ring U_1 is a subspace of the general $(\mathfrak{m}T_1+\mathfrak{m}'T_1)$-adic ring T_1, which is also general $(\mathfrak{n}T_1+\mathfrak{n}'T_1)$-adic, and hence in T may be canonically embedded the completion $U=R\hat{\otimes}R'$ of the general $(\mathfrak{n}U_1+\mathfrak{n}'U_1)$-adic ring U_1, which is isomorphic with $k\{x_1,\cdots,x_n,$

$x'_1, \cdots, x'_{n'}\}$. It should be noted, on the other hand, that cc' is not a zero-divisor in T. In fact, $(0):cT=((0):cS)T=(0)$ (S is an integral domain); similarly $(0):c'T=(0)$; hence cc' is not a zero-divisor in T.

We want to prove that every non-zero element in U is not a zero-divisor in T. Fortunately, we are in a situation which enables us to apply the same method as was shown in h, p. 17, [8]. By the method, we see: first, $T=\sum_{i=0}^{m}\sum_{j=0}^{m'} Uy_iy'_j$ and $cc'T\subseteq\sum_{i=0}^{t}\sum_{j=0}^{t'} Uy_iy'_j$; secondarily, $y_iy'_j$ $(i=0,\cdots,t;$ $j=0,\cdots,t')$ are U-linearly independent (since U is an integral domain). Finally, by making use of the above expression of T, we shall see our assertion, because U is an integral domain and cc' is not a zero-divisor in T. (From this result we can see that T_1 and U are linearly disjoint over U_1.)

Since T is a finite U-module and hence, integral over U, Proposition 2 will follow from the above result.

In Proposition 2, let S' be an arbitrary overfield of k. In this case, if we consider $k\{x_1,\cdots,x_n\}\otimes S'$ instead of U_1, we see, in a similar manner,

Proposition 2′. *Let S be a complete local ring with a basic field k, let K be any overfield of k, and put $T=S\hat{\otimes}K$. If $\mathfrak{p}$ is a prime ideal of rank r of S, then for any prime divisor $\mathfrak{P}$ of $\mathfrak{p}T$,* rank $\mathfrak{P}=$ rank $\mathfrak{p}$.

Remark 3. Notations being the same as in the proof of Proposition 2, let $\mathfrak{m}$ and $\mathfrak{m}'$ be the maximal ideals of S and S', respectively. Since $\mathfrak{m}^nT_1$ is closed with respect to $(\mathfrak{m}T_1+\mathfrak{m}'T_1)$-adic topology on T_1, the statement given in b, p. 8, [8], will be applied to T_1, and we see that the completion $\hat{T}_1$ of general $\mathfrak{m}T_1$-adic ring T_1 may be canonically embedded in T. On the other hand, by Lemma 1, p. 272, [5], $\hat{T}_1$ coincides with T as a ring. Thus we see that T may also be regarded as the completion of general $\mathfrak{m}T_1$-adic ring T_1.

§ 5. Multiplicities.

As applications of our results obtained in §2 and 3, we shall give some relations concerning the multiplicities of primary ideals, introduced by P. Samuel, in the cases shown as examples (4′) and (5′) in the introduction. These cases will be treated separately.

Case I. Let R and R' be Noetherian rings with a common subfield k and assume that $U=R\otimes R'$ is Noetherian. In this Case I, unless otherwise explicitly stated, $\mathfrak{a}$, $\mathfrak{p}$, $\mathfrak{q}$ will denote ideals of R, $\mathfrak{a}'$, $\mathfrak{p}'$, $\mathfrak{q}'$ ideals of R', and $\mathfrak{P}$, $\mathfrak{Q}$ ideals of U.

We shall begin with lemmas. It should be noted that for the following lemmas the assumption that R, R', and U are Noetherian is superfluous (this note will be needed in Case II).

Lemma 6. $\mathfrak{a}\mathfrak{a}'U=\mathfrak{a}U\cap\mathfrak{a}'U$.

Proof. Let ξ be any element in $\mathfrak{a}U\cap\mathfrak{a}'U$, then it can be expressed in the following form: $\xi=a_1r_1'+\cdots+a_sr_s'=a_1'r_1+\cdots+a_t'r_t$, where $a_i\in\mathfrak{a}$, $r_j\in R$, $a_j'\in\mathfrak{a}'$, $r_i'\in R'$ $(i=1,\cdots,s;\ j=1,\cdots,t)$ and a_i are k-linearly independent. Renumbering them if necessary, it may be assumed that $\{a_1,\cdots,a_s,r_1,\cdots,r_u\}$ is a maximal subset of k-linearly independent elements of the set $\{a_1,\cdots a_s, r_1,\cdots,r_t\}$. Then ξ can also be expressed in the following form: $\xi=a_1a_1''+\cdots+a_sa_s''+r_1a_{s+1}''+\cdots+r_ua_{s+u}''$, where $a_i''\in\mathfrak{a}'$ $(i=1,\cdots,s+u)$. From $a_1r_1'+\cdots+a_sr_s'=a_1a_1''+\cdots+r_ua_{s+u}''$, we get $r_i'\in\mathfrak{a}'$ $(i=1,\cdots,s)$, and hence $\xi\in\mathfrak{a}\mathfrak{a}'U$.

Lemma 7. $(\mathfrak{a}^{n-1}\mathfrak{a}'U+\mathfrak{a}^{n-2}\mathfrak{a}'^2U+\cdots+\mathfrak{a}^{n-i+1}\mathfrak{a}'^{i-1}U)\cap(\mathfrak{a}^{n-2}\mathfrak{a}'^2U+\cdots+\mathfrak{a}^{n-i+1}\mathfrak{a}'^{i-1}U+\mathfrak{a}^{n-i}\mathfrak{a}'^iU)=\mathfrak{a}^{n-2}\mathfrak{a}'^2U+\cdots+\mathfrak{a}^{n-i+1}\mathfrak{a}'^{i-1}U$ $(3\leqslant i<n)$.

Proof. First take a k-basis of $\mathfrak{a}'^i$. Extending this k-basis, next take a k-basis of $\mathfrak{a}'^{i-1}$, and so on. Then we get a k-basis of R', and by making use of this k-basis, the lemma will be proved easily.

Now, let $\mathfrak{q}$ and $\mathfrak{q}'$ be $\mathfrak{p}$- and $\mathfrak{p}'$-primary ideals and let $\mathfrak{P}$ be a prime divisor of $\mathfrak{p}U+\mathfrak{p}'U$. Further let $\mathfrak{Q}_{(i,j)}$ and $\mathfrak{Q}_{i\cdot j}$ be the isolated $\mathfrak{P}$-primary components of $\mathfrak{q}^iU+\mathfrak{q}'^jU$ and $(\mathfrak{q}^iU+\mathfrak{q}'^nU)\cap(\mathfrak{q}^nU+\mathfrak{q}'^jU)$, respectively $(1\leqslant i,j\leqslant n)$. Passing to $U_{\mathfrak{P}}$ and applying the second isomorphism theorem, we have $\mathrm{L}(\mathfrak{Q}_{i\cdot j})=\mathrm{L}(\mathfrak{Q}_{(i,n)})+\mathrm{L}(\mathfrak{Q}_{(n,j)})-\mathrm{L}(\mathfrak{Q}_{(i,j)})$, because $\mathfrak{q}^iU+\mathfrak{q}'^jU$ is the ideal sum of $\mathfrak{q}^iU+\mathfrak{q}'^nU$ and $\mathfrak{q}^nU+\mathfrak{q}'^jU$. Put $\mathrm{L}_1(i)=\mathrm{L}(\mathfrak{q}^{(i)})$ and $\mathrm{L}_2(j)=\mathrm{L}(\mathfrak{q}'^{(j)})$, respectively, then by Lemma 3, $\mathrm{L}(\mathfrak{Q}_{i\cdot j})=c(\mathrm{L}_1(i)\mathrm{L}_2(n)+\mathrm{L}_1(n)\mathrm{L}_2(j)-\mathrm{L}_1(i)\mathrm{L}_2(j))$, where $c=\mathrm{L}((\mathfrak{p}U+\mathfrak{p}'U)U_{\mathfrak{P}})$.

Let us calculate the length of the isolated $\mathfrak{P}$-primary component of $(\mathfrak{q}U+\mathfrak{q}'U)^n=\mathfrak{q}^nU+\mathfrak{q}^{n-1}\mathfrak{q}'U+\cdots+\mathfrak{q}'^nU$. It may be assumed that $\mathfrak{q}^n=(0)$ and $\mathfrak{q}'^n=(0)$. Under this assumption $(\mathfrak{q}U+\mathfrak{q}'U)^n=\mathfrak{q}^{n-1}\mathfrak{q}'U+\cdots+\mathfrak{q}\mathfrak{q}'^{n-1}U$ and $\mathfrak{Q}_{i\cdot j}$ is the isolated $\mathfrak{P}$-primary component of $\mathfrak{q}^iU\cap\mathfrak{q}'^jU=\mathfrak{q}^i\mathfrak{q}'^jU$. Since by Lemma 6, $\mathfrak{q}^{n-1}\mathfrak{q}'U\cap\mathfrak{q}^{n-2}\mathfrak{q}'^2U=\mathfrak{q}^{n-1}\mathfrak{q}'^2U$, it will be seen by the same reasoning as above that the length of the isolated $\mathfrak{P}$-primary component of $\mathfrak{q}^{n-1}\mathfrak{q}'U+\mathfrak{q}^{n-2}\mathfrak{q}'^2U$ is equal to $\mathrm{L}(\mathfrak{Q}_{n-1\cdot 1})+\mathrm{L}(\mathfrak{Q}_{n-2\cdot 2})-\mathrm{L}(\mathfrak{Q}_{n-1\cdot 2})=c\{\mathrm{L}_1(n)\mathrm{L}_2(1)+\mathrm{L}_1(n-1)(\mathrm{L}_2(2)-\mathrm{L}_2(1))+\mathrm{L}_1(n-2)(\mathrm{L}_2(n)-\mathrm{L}_2(2))\}$. Next we calculate the length of the isolated $\mathfrak{P}$-primary component of $\mathfrak{q}^{n-1}\mathfrak{q}'U+\mathfrak{q}^{n-2}\mathfrak{q}'^2U+\mathfrak{q}^{n-3}\mathfrak{q}'^3U$. By Lemma 7, $(\mathfrak{q}^{n-1}\mathfrak{q}'U+\mathfrak{q}^{n-2}\mathfrak{q}'^2U)\cap(\mathfrak{q}^{n-2}\mathfrak{q}'^2U+\mathfrak{q}^{n-3}\mathfrak{q}'^3U)=\mathfrak{q}^{n-2}\mathfrak{q}'^2U$, and obviously, the ideal sum of $\mathfrak{q}^{n-1}\mathfrak{q}'U+\mathfrak{q}^{n-2}\mathfrak{q}'^2U$ and $\mathfrak{q}^{n-2}\mathfrak{q}'^2U+\mathfrak{q}^{n-3}\mathfrak{q}'^3U$ is $\mathfrak{q}^{n-1}\mathfrak{q}'U+\mathfrak{q}^{n-2}\mathfrak{q}'^2U+\mathfrak{q}^{n-3}\mathfrak{q}'^3U$. Hence again with the same reasoning as above, the following relation will be obtained: the required length $=c\{\mathrm{L}_1(n)\mathrm{L}_2(1)+\mathrm{L}_1(n-1)(\mathrm{L}_2(2)-\mathrm{L}_2(1))+\mathrm{L}_1(n-2)(\mathrm{L}_2(3)-\mathrm{L}_2(2))+\mathrm{L}_1(n-3)(\mathrm{L}_2(n)-\mathrm{L}_2(3))\}$. If we proceed in this manner, we shall last of all obtain the length l of the isolated $\mathfrak{P}$-primary component of $(\mathfrak{q}U+\mathfrak{q}'U)^n$: $l=c\{\mathrm{L}_1(n)\mathrm{L}_2(1)+\mathrm{L}_1(n-1)(\mathrm{L}_2(2)-\mathrm{L}_2(1))+\cdots+\mathrm{L}_1(2)(\mathrm{L}_2(n-1)-\mathrm{L}_2(n-2))+\mathrm{L}_1(1)(\mathrm{L}_2(n)-\mathrm{L}_2(n-1))\}$. Put $f(i)=\mathrm{L}_1(i+1)-\mathrm{L}_1(i)$ and $g(j)=\mathrm{L}_2(j+1)-\mathrm{L}_2(j)$. Then, since $\mathrm{L}_1(n-i)=f(n-i-1)+\cdots+f(0)$, $l=c(\sum_{i+j<n} f(i)g(j))$.

This formula enables us to calculate the multiplicity of the isolated $\mathfrak{P}$-

primary component of $(\mathfrak{q}U+\mathfrak{q}'U)^n$ by the same methods as were shown on p. 65, [8] or on p. 223, [7]. Thus we get

Theorem 5. *Let R and R' be two Noetherian rings with a common subfield k and let $U=R\otimes R'$ be the tensor product of R and R' over k, which is assumed to be Noetherian. Let further $\mathfrak{q}$ and $\mathfrak{q}'$ be $\mathfrak{p}$- and $\mathfrak{p}'$-primary ideals and $\mathfrak{P}$ be a prime divisor of $\mathfrak{p}U+\mathfrak{p}'U$. Then,* rank $\mathfrak{P}=$ rank $\mathfrak{p}+$rank $\mathfrak{p}'$ *and*

$$\mathrm{e}((\mathfrak{q}U+\mathfrak{q}'U)U_{\mathfrak{P}})=\mathrm{e}(\mathfrak{q})\mathrm{e}(\mathfrak{q}')\mathrm{L}((\mathfrak{p}U+\mathfrak{p}'U)U_{\mathfrak{P}}),$$

where $\mathrm{L}((\mathfrak{p}U+\mathfrak{p}'U)U_{\mathfrak{P}})$ *is constant for any* $\mathfrak{P}$.

Case II. In the notations given at the beginning of §4, let S and S be local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$, respectively. Then there exists, between open ideals of T_1 and those of T, a one-to-one correspondence with respect to the operations: extension and contraction. In addition, residue rings of T_1 and T relative to corresponding open ideals are isomorphic with each other. Let us now consider open primary ideals (i.e. $\mathfrak{m}$- and $\mathfrak{m}'$-primary) $\mathfrak{v}$ and $\mathfrak{v}'$ of S and S', respectively. Then it will be seen from the following three facts that Theorem 5 is also valid for those ideals $\mathfrak{v}$ and $\mathfrak{v}'$: (α) By the above remark $T_1/(\mathfrak{v}T_1+\mathfrak{v}'T_1)^n$ is isomorphic with $T/(\mathfrak{v}T+\mathfrak{v}'T)^n$ for any integer n; (β) Lemma 6 and 7 hold for $(S, S'; T_1)$; (γ) Lemma 3 is valid for $(S, S'; T)$ (T is assumed to be Noetherian). Thus we get

Theorem 6. *Let S and S' be local rings with a common subfield k and let $\mathfrak{m}$ and $\mathfrak{m}'$ be maximal ideals of S and S', respectively. Let further $T=S\hat{\otimes}S'$ be the complete tensor product of S and S' over k, which is assumed to be Noetherian. Then, if $\mathfrak{P}$ is a prime divisor of $\mathfrak{m}T+\mathfrak{m}'T$,* rank $\mathfrak{P}=$rank $\mathfrak{m}+$rank $\mathfrak{m}'$ *and* $\mathrm{L}((\mathfrak{m}T+\mathfrak{m}'T)T_{\mathfrak{P}})$ *is constant for $\mathfrak{P}$. In these situations, let $\mathfrak{v}$ and $\mathfrak{v}'$ be $\mathfrak{m}$- and $\mathfrak{m}'$-primary ideals of S and S', respectively. Then $\mathfrak{m}T+\mathfrak{m}'T$ and $\mathfrak{v}T+\mathfrak{v}'T$ have the same prime divisors, and*

$$\mathrm{e}((\mathfrak{v}T+\mathfrak{v}'T)T_{\mathfrak{P}})=\mathrm{e}(\mathfrak{v})\mathrm{e}(\mathfrak{v}')\mathrm{L}((\mathfrak{m}T+\mathfrak{m}'T)T_{\mathfrak{P}}).$$

If furthermore $S/\mathfrak{m}$ is algebraic over k (canonically embedded), then every prime ideal of $(S/\mathfrak{m})\otimes(S'/\mathfrak{m}')$ is, as easily seen, maximal, and hence, in this case, T is a $(\mathfrak{m}T+\mathfrak{m}'T)$-adic complete semi-local ring, because T is a $(\mathfrak{m}T+\mathfrak{m}'T)$-adic Zariski ring. In these circumstances, we have

$$\mathrm{e}(\mathfrak{v}T+\mathfrak{v}'T)=nc\,\mathrm{e}(\mathfrak{v})\mathrm{e}(\mathfrak{v}'),$$

where n is the number of the maximal prime ideals of T and $c=\mathrm{L}((\mathfrak{m}T+\mathfrak{m}'T)T_{\mathfrak{P}})$.

So far we have observed the case where S and S' are local rings, but in the case where S and S' are semi-local, the corresponding relations will easily be derived from the former, and further, relations concerning the so-called relative multiplicities with respect to the basic fields may also be calculated by our theorem.

§6. Minimal bases.

Let S be a Noetherian ring with an identity and let T be an overring of S. Suppose that $T=St_1+\cdots+St_n$ and $t_1,\cdots,t_n$ are S-linearly independent (minimal basis). Then (S,T) satisfies (P_1), (P_2), (P_3). In fact, let $1=s_1t_1+\cdots+s_nt_n$. Then, if $\mathfrak{s}$ is the ideal which is generated by $s_1,\cdots,s_n$ in S, $\mathfrak{s}$ is unit ideal. For, as is easily seen, $t_i=s_1t_it_1+\cdots+s_nt_it_n=s_{i1}t_1+\cdots+s_{in}t_n$, where $s_{ij}\in\mathfrak{s}$; hence

$$0=\begin{vmatrix} 1-s_{11} & -s_{12}\cdots & -s_{1n} \\ -s_{21} & 1-s_{22}\cdots & -s_{2n} \\ & & \\ -s_{n1} & -s_{n2}\cdots & 1-s_{nn} \end{vmatrix} t_1=(1-s)t_1 \quad (s\in\mathfrak{s}),$$

and $1\in\mathfrak{s}$. Now let $\mathfrak{a}$ be an ideal of S and let $a\in\mathfrak{a}T\cap S$. Then $a=a_1t_1+\cdots+a_nt_n$, where $a_i\in\mathfrak{a}$. On the other hand, $a=as_1t_1+\cdots+as_nt_n$; hence $as_i=a_i$ $(i=1,\cdots,n)$, and $a\in\mathfrak{a}$. This shows that $\mathfrak{a}=\mathfrak{a}T\cap S$. The rest will be easily verified. Since now (S,T) satisfies (P_1), (P_2), (P_3), we see that if $\mathfrak{p}$ is a prime ideal of rank r of S, then any prime divisor of $\mathfrak{p}T$ is of rank r ((P_2), integral dependence, and Proposition 1), and further that the unmixedness theorem holds in T if and only if it holds in S (see §7, [7], and also confer Theorem 23, [2]).

In the case where S is a local ring, we can prove the converse of the above statement, namely; If T is a finite S-module and if (S,T) satisfies (P_1), (P_2), (P_3), then T has a minimal basis over S. In fact, if we denote by $\mathfrak{m}$ the maximal ideal of S, then, by (P_1), S is a subspace of $\mathfrak{m}T$-adic semi-local ring T. Therefore, the completion $\hat{S}$ of S can be canonically embedded in the completion $\hat{T}$ of T, and $\hat{T}$ is a finite $\hat{S}$-module. Take elements $t_1,\cdots,t_n$ in T such that they give rise modulo $\mathfrak{m}T$ to elements which form a $(\hat{S}/\mathfrak{m}\hat{S})$-basis for $\hat{T}/\mathfrak{m}\hat{T}$. Then, as is well known, $\hat{T}=\hat{S}t_1+\cdots+\hat{S}t_n$. Let t be an element in T and put $t=\hat{s}_1t_1+\cdots+\hat{s}_nt_n$. If we take elements s_i in S such that $s_i\in\hat{s}_i+\mathfrak{m}\hat{S}$, then $t-(s_1t_1+\cdots+s_nt_n)\in\mathfrak{m}\hat{T}\cap T=\mathfrak{m}T$. This shows that $T=St_1+\cdots+St_n+\mathfrak{m}T$. Hence, by Corollary 2, p. 69, [6], we have: $T=St_1+\cdots+St_n$.

Next we shall show that $t_1,\cdots,t_n$ are S-linearly independent. Assume the contrary. Then there exists, by a suitable reordering, a relation such as follows: $s_1t_1+\cdots+s_mt_m=0$ $(m\leqslant n)$, where s_i are some non zero elements in S. Since the residues of $t_1,\cdots,t_n$ modulo $\mathfrak{m}T$ form a $(S/\mathfrak{m}S)$-basis for $T/\mathfrak{m}T$, no s_i can be unit in S.

Suppose that $s_j\in s_1S$ $(j=2,\cdots,\mathfrak{m})$. Then $s_1(t_1+s_2't_2+\cdots)=0\,(s_2'\in S)$; this shows that $(t_1+s_2't_2+\cdots)\in(0):s_1T=((0):s_1S)T\subseteq\mathfrak{m}T$; consequently, if we pass to the residue ring $T/\mathfrak{m}T$, a relation contrary to the selection of t_i will be obtained. Next, therefore, suppose that—renumbering them if necessary—$s_i\in s_1S$ $(i=1,\cdots,m_1)$ and $s_j\notin s_1S$ $(j=m_1+1,\cdots,m)$. If we pass from (S,T)

to $(S/s_1S,\ T/s_1T)$, then $\bar{s}_{m_1+1}\bar{t}_{m_1+1}+\cdots+\bar{s}_m\bar{t}_m=0$, where $\bar{s}_j$ are non zero elements in S/s_1S (bars indicate residue classes mod s_1T). For this relation we may repeat the same considerations as above, and if we proceed in this manner, then, last of all, we shall reach the following: For an element in s_i, say s_m, it holds that $s_mt_m \in \mathfrak{a}T$, $s_m \notin \mathfrak{a}$, where $\mathfrak{a}$ is the ideal which is generated by $s_1,\cdots,s_{m-1}$ in S. But this is a contradiction because it means that $t_m \in \mathfrak{a}T : s_mT=(\mathfrak{a}:s_mS)T\subseteq \mathfrak{m}T$. Thus we have in all cases contradictions, and this completes the proof.

Proposition3. *Let S be a local ring and T be an overring of S. Assume that T is a finite S-module. Then (S, T) satisfies* (P_1), (P_2), (P_3) *if and only if T has a minimal basis over S.*

Any two minimal bases have the same number of elements which is equal to the dimension of $T/\mathfrak{m}T$ over $S/\mathfrak{m}$.

References

[1] Chevalley, C., *Some properties of ideals in rings of power series*, Trans. Amer. Math. Soc., **55** (1944), 68–84.

[2] Cohen, I. S., *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc., **59** (1946), 54–106.

[3] Nakai, Y. and Nagata, M., *Algebraic geometry* (in Japanese), Series of Modern Math., **4-C**, Kyoritsu-Schuppan Co., Tokyo, 1957.

[4] Nagata, M., *Some remarks on local rings*, Nagoya Math. J., **6** (1953), 53–58.

[5] ———, *Note on complete local integrity domains*, Memoirs Kyoto Univ. (A), **28** (1954), 271–278.

[6] ———, *Basic theorems on general commutative rings*, ibid. **29** (1955), 59–77.

[7] ———, *The theory of multiplicity in general local rings*, Proc. Inter. Symp. Alg. Number Theory, Tokyo-Nikko 1955 (1956), 191–226.

[8] Samuel, P., *Algèbre locale*, Mémor. Sci. Math., No. **123** (1953).

[9] Satô, H., *Some remarks on Zariski rings*, J. Sci. Hiroshima Univ., (A), **20** (1957), 93–99.

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