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On a Theorem in an (LF) Space

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A locally convex space E is distinguished ([1] p. 78) if every $\sigma(E'', E')$-bounded subset of the bidual E'' is contained in the $\sigma(E'', E')$-closure of a bounded subset of E, or if the strong dual E' is barrelled (French; espace tonnelé). J. Dieudonné and L. Schwartz raised a question ([1] p. 98): is an (F) or (LF) space always distinguished? This is negatively answered by A. Grothendieck ([3] pp. 88–89). He has shown there by an example the existence of a non distinguished closed subspace of a certain distinguished (F) space. We know that the strong dual of a distinguished (F) or reflexive (LF) space is bornological ([2] p. 342). In this paper we shall generalize this in the following way: if E is a strict inductive limit of an increasing sequence of metrisable locally convex spaces E_n and is distinguished, then the strong dual E' is bornological. In this statement if each E_n is distinguished, then E is distinguished ([3] p. 85), therefore our statement contains as a special case the theorem of A. Grothendieck ([3] p. 85) to the effect that the strong dual of a strict inductive limit of distinguished metrisable spaces is bornological.

1. In the sequel we mean by a locally convex space a topological linear locally convex Hausdorff space. *In a locally convex space E the following statements for an absolutely convex subset U are equivalent:*

(1) *U absorbs every bounded subset of E;*
(2) *U absorbs every bounded sequence of E;*
(3) *U absorbs every sequence converging to 0 of E.*

If follows therefore that in a bornological locally convex space any absolutely convex subset absorbing every sequence converging to 0 is a neighborhood of 0. This remark will be used for the proof of the following Lemma. Let E be the projective limit of the sequence of locally convex spaces E_n with respect to the mappings ϕ_n, where each ϕ_n is a continuous linear mapping of E_{n+1} into E_n. Let u_n be the projection of E into E_n. We assume that u_n is onto, and therefore so also for ϕ_n. Then we have

LEMMA. *If every sequence converging to 0 in E_n (every bounded subset of E_n) is an image of a bounded sequence (bounded subset) of E, then E is bornological.*

Proof. Let U be any absolutely convex subset of E absorbing every bounded subset of E. Let N_n denote the kernel of u_n. We show that U contains N_n for sufficiently large n. Suppose the contrary. Then, since $N_n \not\subset nU$ for $n=1,2,\cdots$, there exists a sequence $\{x_n\}$ of E such that $x_n \in N_n \cap C(nU)$. If follows from $u_1x_n=u_2x_n=\cdots=u_nx_n=0$ that $\{x_n\}$ is bounded, and therefore there exists a positive integer m such that $x_n \in mU$, $n=1,2,\cdots$. Hence we have $x_m \in mU$. This is a contradiction.

Let $N_n \subset U$, and we set $V=\{x \mid x+N_n \subset U\}$. Then V is an absolutely convex subset of E and we have $U \supset V \supset \frac{1}{2}U$. If we put $V_n=u_n(V)$, then V_n is absolutely convex and absorbs every sequence of E_n converging to 0. Indeed, let $\{x_p^{(n)}\}$ be such a sequence. Then from our hypothesis, there exists a bounded sequence $\{x_p\}$ with $u_n(x_p)=x_p^{(n)}$. $\{x_p\}$ is absorbed by U, and so by V, and therefore $\{x_p^{(n)}\}$ by V_n. Since E_n is bornological, it follows that V_n is a neighborhood of 0 in E_n. It follows from $U \supset u_n^{-1}(V_n)=V$ that U is a neighborhood of 0, completing the proof.

This Lemma is an analogue to the results established by A. Grothendieck ([3] pp. 85–86) and J. S. e. Silva ([4] p. 408). As a simple application of this Lemma, we indicate the following well-known theorem: *If E is the direct product of a countable number of bornological spaces E_n, then E is bornological.* In fact E is the projective limit of $\{E_1 \times \cdots \times E_n\}$, and the assumption of Lemma is easily verified.

2. In this section E denotes the strict inductive limit of a sequence of metrisable spaces E_n, that is, (1) E is the union of E_n; (2) each E_n is a closed subspace of E_{n+1}; (3) the locally convex topology of E is the inductive limit of those of E_n. Then E is bornological and each E_n is a closed subspace of E and any bounded subset of E is contained in an E_n ([3] pp. 83–84). Let u_n and ϕ_n be the identical mappings of E_n into E and E_{n+1} respectively. Since the strong topology of E' is the topology of uniform convergence on the bounded subsets of E, the strong dual E' is the projective limit of $\{E'_n\}$ with respect to the mapping $\{\phi'_n\}$ and u'_n becomes the projection of E' onto E'_n. u''_n and ϕ''_n are the isomorphic mappings of E''_n into E'' and E''_{n+1} respectively ([3] p. 84), and hold the relations $u''_{n+1} \circ \phi''_n = u''_n$ Hence we can identify E''_n with $u''_n(E''_n)$. We ignore that E'' is bornological except for special cases. The topology of E'' is weaker than the inductive limit of those of E''_n.

THEOREM. *If E, the strict inductive limit of a sequence of metrisable spaces E_n, is distinguished, then the strong dual E' is bornological.*

Proof. Let us denote by H_n the space E'_n with the topology of the uniform convergence on the bounded subsets of E''_n. Then E'_n and H_n have the same bounded subset and H_n is bornological ([3] p. 73). First we show that E' is the projective limit of H_n with respect to the mappings ϕ'_n. ϕ'_n maps any bounded subset of E'_{n+1} into a bounded subset of E'_n. It follows from the remark just given that ϕ'_n is a continuous linear mapping of H_{n+1} onto H_n. By our hypothesis E is distinguished, namely E' is barrelled, so that a fundamental system $\mathfrak{U}$ of neighborhoods of 0 in E' is given by the polars of the bounded subsets of E''. Since any bounded subset of E'' is contained in the weak closure of a bounded subset of E and therefore of a bounded subset of E_n, we may assume that $\mathfrak{U}$ consists of the polars of the bounded subsets of E''_n, $n=1,2,\cdots$. This shows that E' is the projective limit of $\{H_n\}$ with respect to $\{\phi'_n\}$. Next we shall show by making use of Lemma that E' is bornological. Let $\{x'^{(n)}_p\}$ be any sequence converging to 0 in H_n. Since the topology of H_n is stronger than that of E'_n, $\{x'^{(n)}_p\}$ also converges to 0 in E'_n. Since E_n is metrisable and u_n is an isomorphic mapping, it follows that $\{x'^{(n)}_p\}$ is equi-continuous and an image of an equi-continuous sequence $\{x'_p\}$ of E'. Hence by Lemma E' is bornological. The proof is completed.

References

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