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On a Ring whose Principal Right Ideals Generated by Idempotents Form a Lattice

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Introduction

In his algebraic study of operator algebras [4], Kaplansky considered a ring (resp. *-ring) with unity in which the right annihilator of any *subset* is the principal right ideal generated by an idempotent (resp. a projection = self-adjoint idempotent). Then the left annihilator of any *subset* has the similar property. He called such a ring (resp. *-ring) a Baer ring (resp. Baer *-ring).

In a ring $\mathfrak{A}$, we denote by $\mathcal{R}_I(\mathfrak{A})$ the set of all principal right ideals generated by idempotents, which is partially ordered by set-inclusion. We know that if $\mathfrak{A}$ is a Baer ring then $\mathcal{R}_I(\mathfrak{A})$ (equal to the set of all right annihilators) forms a complete lattice. On the other hand, if $\mathfrak{A}$ is a regular ring of von Neumann then $\mathcal{R}_I(\mathfrak{A})$ (equal to the set of all principal right ideals) forms a complemented modular lattice. These two rings both satisfy the following conditions:

(R_r) The right annihilator of any *element* is the principal right ideal generated by an idempotent,

(R_l) The left annihilator of any *element* is the principal left ideal generated by an idempotent.

As will be stated below, to imply that $\mathcal{R}_I(\mathfrak{A})$ forms a lattice it is sufficient that $\mathfrak{A}$ satisfies these two conditions.

When $\mathfrak{A}$ is a *-ring, we consider the following condition:

(R_r^*) The right annihilator of any *element* is the principal right ideal generated by a projection.

It is obvious that the similar condition (R_l^*) for the left annihilators is equivalent to (R_r^*). These conditions (R_r^*) and (R_l^*) were treated first by Rickart [8] in the case of B^*-algebras: a B_p^*-algebra of Rickart is a B^*-algebra satisfying (R_r^*). In this paper, a *-ring is called a Rickart *-ring if it satisfies (R_r^*), and a ring is called a Rickart ring if it satisfies (R_r) and (R_l). These rings have many examples in the literatures on operator algebras and continuous geometries, i. e., Baer *-rings, *-regular rings, B_p^*-algebras and the *-rings treated by Berberian [1, § 3] are Rickart *-rings; Baer rings and regular rings are Rickart rings. This paper is devoted to the study of Rickart rings and Rickart *-rings.

In § 1, we shall prove that if $\mathfrak{A}$ is a Rickart ring then $\mathcal{R}_I(\mathfrak{A})$ forms a lattice, and then it follows from the result of my previous paper [7, § 3] that $\mathcal{R}_I(\mathfrak{A})$ is a relatively semi-orthocomplemented lattice. We shall examine the relation between the structures of a Rickart ring $\mathfrak{A}$ and the lattice $\mathcal{R}_I(\mathfrak{A})$. In § 2, we shall show that there is a natural correspondence between the set of central idempotents of $\mathfrak{A}$ and the center of $\mathcal{R}_I(\mathfrak{A})$. In § 3, we shall prove that when a lattice is relatively semi-orthocomplemented, the quotient lattice can be defined by a neutral ideal and it is also relatively semi-orthocomplemented. In § 4, we shall examine the perspectivity in $\mathcal{R}_I(\mathfrak{A})$ as a preparation for § 5, where we shall examine the correspondence between the quotient rings of $\mathfrak{A}$ and the quotient lattices of $\mathcal{R}_I(\mathfrak{A})$. An ideal $\mathfrak{J}$ of a ring will be called an *AP*-ideal if it satisfies the following condition: If two elements x, y have the same right or left annihilator then $x \in \mathfrak{J}$ implies $y \in \mathfrak{J}$. We shall show that if $\mathfrak{J}$ is an *AP*-ideal of a Rickart ring $\mathfrak{A}$ then $\mathfrak{A}/\mathfrak{J}$ is also a Rickart ring and $\mathcal{R}_I(\mathfrak{A}/\mathfrak{J})$ is lattice-isomorphic to a quotient lattice of $\mathcal{R}_I(\mathfrak{A})$. Especially, when $\mathfrak{A}$ is a regular ring, any ideal is an *AP*-ideal and there is a one-to-one correspondence between the set of all ideals $\mathfrak{J}$ of $\mathfrak{A}$ and the set of all neutral ideals J of $\mathcal{R}_I(\mathfrak{A})$ such that $\mathcal{R}_I(\mathfrak{A}/\mathfrak{J})$ is isomorphic to $\mathcal{R}_I(\mathfrak{A})/J$.

In § 6, we shall show that when $\mathfrak{A}$ is a Rickart *-ring, the set $P(\mathfrak{A})$ of all projections forms a lattice isomorphic to $\mathcal{R}_I(\mathfrak{A})$; and by making use of the results in the preceding sections we shall have several theorems concerning relations between the structures of $\mathfrak{A}$ and $P(\mathfrak{A})$. It will be shown that any *AP*-ideal $\mathfrak{J}$ of a Rickart *-ring $\mathfrak{A}$ is self-adjoint and then $\mathfrak{A}/\mathfrak{J}$ is also a Rickart *-ring where $P(\mathfrak{A}/\mathfrak{J})$ is isomorphic to a quotient lattice of $P(\mathfrak{A})$. If, moreover, $\mathfrak{A}$ satisfies the condition ($\bar{a}$) given in my previous paper [6] (especially, $\mathfrak{A}$ is a Baer *-ring satisfying the *EP*-and *SR*-axioms of Kaplansky [4] or $\mathfrak{A}$ is a Berberian's *-ring) then any ideal generated by projections is an *AP*-ideal.

§ 1. Rickart rings

Let $\mathfrak{A}$ be a ring. The principal right (resp. left) ideal generated by $x \in \mathfrak{A}$ is denoted by $(x)_r$ (resp. $(x)_l$) and the right (resp. left) annihilator of x is denoted by $(x)^r$ (resp. $(x)^l$).

DEFINITION 1.1. An idempotent e of a ring $\mathfrak{A}$ is called a *right* (resp. *left*) *idempotent* of an element $x \in \mathfrak{A}$ if e and x have the same right (resp. left) annihilator, i. e. $(x)^r=(e)^r$ (resp. $(x)^l=(e)^l$). If $\mathfrak{A}$ has the unity element, $(x)^r=(e)^r$ is equivalent to $(x)^r=(1-e)_r$. A ring with unity is called a *Rickart ring* if every element has its right idempotent (not necessarily unique) and its left idempotent. It is obvious that a ring is a Rickart ring if and only if it satisfies the conditions (R_r) and (R_l) in the introduction.

An element x of a ring $\mathfrak{A}$ is called to be relatively regular if there exists $y \in \mathfrak{A}$ such that $x=xyx$ (Rickart [8, § 3]). Then, it is obvious that yx and xy are respectively a right idempotent and a left idempotent of x. If $\mathfrak{A}$ is a regular ring of von Neumann, then since every $x \in \mathfrak{A}$ is relatively regular, $\mathfrak{A}$ is a

Rickart ring. It is obvious that a Baer ring of Kaplansky [4] is also a Rickart ring.

Hereafter we use the following notations: $I(\mathfrak{A})$ is the set of all idempotents of a ring $\mathfrak{A}$, $\mathcal{R}_I(\mathfrak{A})=\{(e)_r;\ e\in I(\mathfrak{A})\}$ and $\mathcal{L}_I(\mathfrak{A})=\{(e)_l;\ e\in I(\mathfrak{A})\}$. If $\mathfrak{A}$ has 1, since $(e)^r=(1-e)_r$ and $(e)^l=(1-e)_l$ we have $\mathcal{R}_I(\mathfrak{A})=\{(e)^r;\ e\in I(\mathfrak{A})\}$ and $\mathcal{L}_I(\mathfrak{A})=\{(e)^l;\ e\in I(\mathfrak{A})\}$. $\mathcal{R}_I(\mathfrak{A})$ and $\mathcal{L}_I(\mathfrak{A})$ is each a partially ordered set with 0, 1, ordered by set-inclusion, and there is a dual-isomorphism between them by $(e)_r\leftrightarrow(1-e)_l$. Now, we shall show that if $\mathfrak{A}$ is a Rickart ring then $\mathcal{R}_I(\mathfrak{A})$ and $\mathcal{L}_I(\mathfrak{A})$ are lattices. We remark that if $e, f\in I(\mathfrak{A})$ and $ef=0$ then $(1-f)(1-e)\in I(\mathfrak{A})$.

Lemma 1. 1. *Let $\mathfrak{A}$ be a ring with unity. The following two statements are equivalent.*

(α) *For every $e, f\in I(\mathfrak{A})$, ef has a right idempotent, i. e. $(ef)^r\in\mathcal{R}_I(\mathfrak{A})$.*

(β) *For every $e, f\in I(\mathfrak{A})$, $(e, f)^r\in\mathcal{R}_I(\mathfrak{A})$.* ($(e, f)^r$ is the right annihilator of the set $\{e, f\}$.)

Similar property holds for left annihilators.

Proof. $(\alpha)\Rightarrow(\beta)$. For $e, f\in I(\mathfrak{A})$, there is $g\in I(\mathfrak{A})$ with $(e(1-f))^r=(g)^r$ by (α). Since $f\in(e(1-f))^r$ we have $gf=0$, and hence $(1-f)(1-g)\in I(\mathfrak{A})$. We have $f(1-f)(1-g)=0$, and since $1-g\in(g)^r=(e(1-f))^r$ we have $e(1-f)(1-g)=0$. Hence $(e, f)^r\supset((1-f)(1-g))_r$. On the other hand, if $ex=fx=0$, then it follows from $e(1-f)x=0$ that $gx=0$, and hence $(1-f)(1-g)x=(1-f)x=x$. Therefore $(e, f)^r=((1-f)(1-g))_r\in\mathcal{R}_I(\mathfrak{A})$. $(\beta)\Rightarrow(\alpha)$. For $e, f\in I(\mathfrak{A})$, there is $g\in I(\mathfrak{A})$ with $(e, 1-f)^r=(g)^r$ by (β). Since $(1-f)(1-g)=0$ we have $gf\in I(\mathfrak{A})$. It follows from $(e)^r\supset(g)^r$ that $(ef)^r\supset(gf)^r$. If $efx=0$, then $fx\in(e, 1-f)^r=(g)^r$, and hence $gfx=0$. Therefore $(ef)^r=(gf)^r\in\mathcal{R}_I(\mathfrak{A})$.

Lemma 1. 2. *Let $\mathfrak{A}$ be a ring with unity. The following two statements are equivalent.*

(α) *For every $e, f\in I(\mathfrak{A})$, ef has a right idempotent and a left idempotent.*

(β) *$\mathcal{R}_I(\mathfrak{A})$ and $\mathcal{L}_I(\mathfrak{A})$ are lattices where $(e)_r\frown(f)_r$ (resp. $(e)_l\frown(f)_l$) is the intersection of $(e)_r$ and $(f)_r$ (resp. $(e)_l$ and $(f)_l$).*

Proof. Since $(1-e, 1-f)^r$ is the intersection of $(e)_r$ and $(f)_r$, $(e)_r\frown(f)_r$ in $\mathcal{R}_I(\mathfrak{A})$ exists and is equal to the intersection of $(e)_r$ and $(f)_r$ if and only if $(1-e, 1-f)^r\in\mathcal{R}_I(\mathfrak{A})$. And then, by the duality of $\mathcal{R}_I(\mathfrak{A})$ and $\mathcal{L}_I(\mathfrak{A})$, $(1-e)_l\smile(1-f)_l$ in $\mathcal{L}_I(\mathfrak{A})$ exists. Similar properties for $(e)_l\frown(f)_l$ and $(1-e)_r\smile(1-f)_r$ also hold. Hence, this lemma is implied from Lemma 1. 1.

In a lattice with 0, the semi-orthogonal relation "$\perp$" is defined by the following axioms ([7, § 1]): ($\perp$1) $a\perp a$ implies $a=0$; ($\perp$2) $a\perp b$ implies $b\perp a$; ($\perp$3) $a\perp b$, $a_1\leqq a$ imply $a_1\perp b$; ($\perp$4) $a\perp b$, $a\smile b\perp c$ imply $a\perp b\smile c$. A lattice L with 0, 1 is called to be relatively semi-orthocomplemented if it has a semi-orthogonal relation "$\perp$" and for $a, b\in L$ with $a\leqq b$ there is $c\in L$ with $a\perp c$,

$a\cup c=b$. In [7, § 3], the canonical semi-orthogonal relation in $\mathcal{R}_I(\mathfrak{A})$ (resp. $\mathcal{L}_I(\mathfrak{A})$) is defined as follows: $(e)_r\perp(f)_r$ (resp. $(e)_l\perp(f)_l$) if there are $e_0, f_0 \in I(\mathfrak{A})$ with $(e_0)_r=(e)_r$, $(f_0)_r=(f)_r$ (resp. $(e_0)_l=(e)_l$, $(f_0)_l=(f)_l$) and $e_0f_0=f_0e_0=0$. The following theorem is a consequence of Lemma 1.2 and [7], Theorem 4.

THEOREM 1. 1.[1)] *If a ring $\mathfrak{A}$ with unity has the property (α) of Lemma 1.2, especially if $\mathfrak{A}$ is a Rickart ring, then $\mathcal{R}_I(\mathfrak{A})$ and $\mathcal{L}_I(\mathfrak{A})$ form relatively semi-orthocomplemented lattices, ordered by set-inclusion.*

LEMMA. 1. 3. *If $\mathfrak{A}$ is a Rickart ring and the lattice $\mathcal{R}_I(\mathfrak{A})$ is complete, then $\mathfrak{A}$ is a Baer ring (the converse is obvious). And then, the meet $\bigwedge_\alpha(e_\alpha)_r$ in $\mathcal{R}_I(\mathfrak{A})$ is equal to the intersection of the ideals $(e_\alpha)_r$.*

PROOF. First, putting $(e)_r=\bigwedge_\alpha(e_\alpha)_r$ in $\mathcal{R}_I(\mathfrak{A})$, we shall prove that $(e)_r$ is equal to the intersection $\mathfrak{J}$ of $(e_\alpha)_r$. Since $(e)_r\subset(e_\alpha)_r$ for every α, we have $(e)_r\subset\mathfrak{J}$. Let $x\in\mathfrak{J}$ and f be a left idempotent of x. Since $x\in(g)_r\Leftrightarrow(1-g)x=0\Leftrightarrow(1-g)f=0\Leftrightarrow(f)_r\leqq(g)_r$ for any $g\in I(\mathfrak{A})$, it follows from $x\in\mathfrak{J}$ that $(f)_r\leqq(e_\alpha)_r$ for every α, and hence $(f)_r\leqq(e)_r$, which implies $x\in(e)_r$. Therefore $(e)_r=\mathfrak{J}$.

Next, let $\{x_\alpha\}$ be a subset of $\mathfrak{A}$, e_α be a right idempotent of x_α and $\bigwedge_\alpha(1-e_\alpha)_r=(e)_r$ in $\mathcal{R}_I(\mathfrak{A})$. Then, the right annihilator of $\{x_\alpha\}$ is equal to the intersection of $(x_\alpha)^r=(e_\alpha)^r=(1-e_\alpha)_r$ and hence equal to $(e)_r$ by the above result. Therefore $\mathfrak{A}$ is a Baer ring (Kaplansky [4], Chap. I, Theorem 1 and Definition 1).

LEMMA 1. 4. *If $\mathfrak{A}$ is a Rickart ring and $e\in I(\mathfrak{A})$, then $e\mathfrak{A}e$ is also a Rickart ring and $\mathcal{R}_I(e\mathfrak{A}e)$ is lattice-isomorphic to the sublattice $\{(f)_r\in\mathcal{R}_I(\mathfrak{A});\ (f)_r\leqq(e)_r\}$ of $\mathcal{R}_I(\mathfrak{A})$.*

PROOF. Let $x\in e\mathfrak{A}e$, and f be a right idempotent of x. Since $xe=x$, we have $1-e\in(x)^r=(f)^r$. Then, since $f=fe$, putting $f_0=ef=efe$ we have $f_0\in I(e\mathfrak{A}e)$ and $(f_0)^r=(f)^r$, which implies $(x)^r\cap e\mathfrak{A}e=(f_0)^r\cap e\mathfrak{A}e$. Hence f_0 is a right idempotent of x in $e\mathfrak{A}e$. Similarly, x has a left idempotent in $e\mathfrak{A}e$. Therefore $e\mathfrak{A}e$ is a Rickart ring. The last statement of the lemma is obvious.

LEMMA 1. 5. *Let $\mathfrak{A}$ be the direct product of rings $\{\mathfrak{A}_\alpha\}$. $\mathfrak{A}$ is a Rickart ring if and only if every $\mathfrak{A}_\alpha$ is a Rickart ring, and then $\mathcal{R}_I(\mathfrak{A})$ is the direct product of the lattices $\{\mathcal{R}_I(\mathfrak{A}_\alpha)\}$.*

PROOF. For $x=(x_\alpha)\in\mathfrak{A}$, the annihilator of x is equal to the product set of

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1) The author was informed by Professor Amemiya, after the submission of the manuscript, that, in the paper:

I. Amemiya and I. Halperin, *Complemented modular lattices derived from non-associative rings*, Acta Sci. Math. Szeged., 20 (1959), 181-201,

it was proved (§ 3) that if a ring $\mathfrak{A}$ (not necessarily associative) is idempotent-associative and semi-regular then $\mathcal{R}_I(\mathfrak{A})$ and $\mathcal{L}_I(\mathfrak{A})$ form relatively complemented lattices. In our paper, rings are always associative. It is easily seen that in a ring with unity if e is a right (resp. left) idempotent of x in our sense then e is a left (resp. right) idempotent of x in the sense of Amemiya-Halperin and hence Rickart rings are semi-regular.

the annihilators of x_α in $\mathfrak{A}_\alpha$, and $x \in I(\mathfrak{A}) \Leftrightarrow x_\alpha \in I(\mathfrak{A}_\alpha)$ for every α. Under this consideration it is easy to prove the lemma.

REMARK. If $\mathfrak{A}$ is a commutative Rickart ring, then $I(\mathfrak{A})$ forms a lattice which is isomorphic to $\mathcal{R}_I(\mathfrak{A}) = \mathcal{L}_I(\mathfrak{A})$.

§ 2. The center of a Rickart ring

LEMMA 2.1. *Let $\mathfrak{A}$ be a ring. The following statements for $e \in I(\mathfrak{A})$ are equivalent.*

(α) *e is in the center of $\mathfrak{A}$ (a central idempotent).*

(β) *e commutes with every idempotent of $\mathfrak{A}$.*

(γ) *When $f \in I(\mathfrak{A})$, each of the equations $(e)_r = (f)_r$ and $(e)_l = (f)_l$ implies $e = f$.*

PROOF. It is trivial that (α) implies (β). (β) implies (γ), because if $f \in I(\mathfrak{A})$, $(e)_r = (f)_r$ (or $(e)_l = (f)_l$) and e commutes with f then $e = f$. Putting $f = e + ex - exe$ and $g = e + xe - exe$, we have $f, g \in I(\mathfrak{A})$, $(e)_r = (f)_r$ and $(e)_l = (g)_l$. If (γ) holds, then since $e = f = g$ we have $ex = exe = xe$, which implies (α).

THEOREM 2.1. *Let $\mathfrak{A}$ be a Rickart ring. The following statements for $e \in I(\mathfrak{A})$ are equivalent.*

(α) *e is in the center of $\mathfrak{A}$.*

(β) *$(e)_r$ is in the center of $\mathcal{R}_I(\mathfrak{A})$.*

(γ) *$(e)_l$ is in the center of $\mathcal{L}_I(\mathfrak{A})$.*

PROOF. Since $\mathcal{R}_I(\mathfrak{A})$ is a relatively semi-orthocomplemented lattice by Theorem 1.1, it follows from [7], Lemma 2 that (β) holds if and only if $(e)_r$ has a unique complement. If (α) holds and $(f)_r$ is a complement of $(e)_r$, then since $ef \in I(\mathfrak{A})$ and $(ef)_r = (e)_r \cap (f)_r = (0)_r$, we have $ef = 0$, and then $e + f \in I(\mathfrak{A})$ and $(e+f)_r = (e)_r \cup (f)_r = (1)_r$, which implies $e + f = 1$ (see [7], Lemma 4 (ii)). Therefore $f = 1 - e$ is uniquely determined, and hence, (β) holds. Conversely, let $(e)_r$ be in the center of $\mathcal{R}_I(\mathfrak{A})$. If $(e)_r = (f)_r$, then since both $(1-e)_r$ and $(1-f)_r$ are complements of $(e)_r$, we have $(1-e)_r = (1-f)_r$, and then $e = f$. Since $(1-e)_r$ is also in the center of $\mathcal{R}_I(\mathfrak{A})$, it is proved similarly that $(1-e)_r = (1-f)_r$ implies $1 - e = 1 - f$, which means that $(e)_l = (f)_l$ implies $e = f$. Therefore (γ) of Lemma 2.1 holds, and hence (α) holds. The equivalence of (α) and (γ) is proved similarly.

COROLLARY 1. *Let $\mathfrak{Z}$ be the center of a Rickart ring $\mathfrak{A}$. $\mathfrak{Z}$ is also a Rickart ring and the lattice $I(\mathfrak{Z})$ (isomorphic to $\mathcal{R}_I(\mathfrak{Z})$) is isomorphic to the center of $\mathcal{R}_I(\mathfrak{A})$ and to that of $\mathcal{L}_I(\mathfrak{A})$, by the mapping $e \to (e)_r = (e)_l$.*

PROOF. To prove that $\mathfrak{Z}$ is a Rickart ring it suffices to show that if $x \in \mathfrak{Z}$ and e and f are respectively a right idempotent and a left idempotent of x, then $e, f \in \mathfrak{Z}$. It follows from $(x)^r = (x)^l$ that $(e)^r = (f)^l$. Since $(1-e)y \in (e)^r$ and

$y(1-f) \in (f)^l$ for any $y \in \mathfrak{A}$, we have $(1-e)yf=0$ and $ey(1-f)=0$, which implies $ey=eyf=yf$ for any $y \in \mathfrak{A}$. Hence we have $e=f \in \mathfrak{Z}$. The last statement follows from the above theorem.

COROLLARY 2. *Let $\mathfrak{A}$ be a Rickart ring. If $\mathcal{R}_I(\mathfrak{A})$ is a direct product of sublattices L_1 and L_2, then there exist Rickart subrings $\mathfrak{A}_1$ and $\mathfrak{A}_2$ such that $\mathfrak{A}=\mathfrak{A}_1 \times \mathfrak{A}_2$ and that $\mathcal{R}_I(\mathfrak{A}_i)$ is lattice-isomorphic to L_i $(i=1, 2)$.*

PROOF. By the assumption there are $e_i \in I(\mathfrak{A})$ $(i=1, 2)$ such that $(e_i)_r$ are in the center of $\mathcal{R}_I(\mathfrak{A})$ and $L_i=\{(f)_r \in \mathcal{R}_I(\mathfrak{A}); (f)_r \leqq (e_i)_r\}$. Since e_i are in the center of $\mathfrak{A}$ by the above theorem and since $(e_1)_r \frown (e_2)_r=(0)_r$ and $(e_1)_r \smile (e_2)_r=(1)_r$, we have $e_1e_2=0$ and $e_1+e_2=1$. Hence $\mathfrak{A}_i=e_i\mathfrak{A}$ have the desired properties.

§ 3. Quotient lattices of relatively semi-orthocomplemented lattices

In a lattice L, an equivalence relation "$\equiv$" is called a congruence relation if $a_i \equiv b_i$ $(i=1, 2)$ imply $a_1 \smile a_2 \equiv b_1 \smile b_2$ and $a_1 \frown a_2 \equiv b_1 \frown b_2$. It is obvious that $a \equiv b$ if and only if $a \smile b \equiv a \frown b$. A pair of $a, b \in L$ with $a \geqq b$ is called a quotient and denoted by a/b. A set N of quotients is called a quotient-ideal if it has the following four properties (F. Maeda [5], Kap. I, Definition 4.1): (1°) $a/a \in N$ for every $a \in L$; (2°) $a \geqq a_1 \geqq b_1 \geqq b$ and $a/b \in N$ imply $a_1/b_1 \in N$; (3°) $a/b \in N$ and $b/c \in N$ imply $a/c \in N$; (4°) $a \smile b/a \in N$ if and only if $b/a \frown b \in N$. By [5], Kap. I, Satz 4.2, if "$\equiv$" is a congruence relation, then $\{a/b; a \equiv b\}$ is a quotient-ideal, and conversely, for any quotient-ideal N there exists one and only one congruence relation with $\{a/b; a \equiv b\}=N$. For any $a \in L$, the class $\{x \in L; x \equiv a\}$ is denoted by a/N, and the lattice formed by $\{a/N; a \in L\}$ is denoted by L/N.

LEMMA 3.1. *Let L be a lattice with 0 and "$\equiv$" be a congruence relation with the corresponding quotient-ideal N. Moreover, let L have a semi-orthogonal relation "$\perp$".*

(i) *If we define a relation "$\perp$" in L/N as follows: $a/N \perp b/N$ if there exist $a_0, b_0 \in L$ with $a_0 \equiv a$, $b_0 \equiv b$ and $a_0 \perp b_0$, then it is a semi-orthogonal relation in L/N.*

(ii) *If a semi-orthogonal relation in L has the property that $a \perp b$, $a \perp c$ imply $a \perp b \smile c$, then so does the induced semi-orthogonal relation in L/N.*

(iii) *If L is semi-orthocomplemented (resp. relatively semi-orthocomplemented), then so is L/N.*

PROOF. (i) We shall show that the relation "$\perp$" in L/N satisfies the axioms ($\perp$1)—($\perp$4). It is obvious that ($\perp$2) holds. If $a/N \perp a/N$, then since there are $a_1, a_2 \in L$ with $a_1 \equiv a_2 \equiv a$ and $a_1 \perp a_2$ we have $a \equiv a_1 \frown a_2 = 0$, which shows that ($\perp$1) holds. Let $c/N \leqq a/N \perp b/N$. We may assume that $a \perp b$. Since $a \frown c \equiv c$ and $a \frown c \perp b$, we have $c/N \perp b/N$, that is, ($\perp$3) holds. Let $a/N \perp b/N$ and $a/N \smile b/N \perp c/N$. We may assume that $a \perp b$. There exist $d, c_0 \in L$ with $d \equiv a \smile b$, $c_0 \equiv c$ and $d \perp c_0$. Since $d \frown a \equiv a$, $d \frown b \equiv b$ and since $(d \frown a) \dot{\smile} (d \frown b) \perp c_0$[2] implies $d\cap a\perp(d\cap b)\cup c_0$, we have $a/N\perp b/N\cup c/N$, that is, $(\perp 4)$ holds.

(ii) Let $a/N\perp b/N$ and $a/N\perp c/N$. There exist $a_1, a_2, b_0, c_0 \in L$ such that $a_1\equiv a_2\equiv a$, $b_0\equiv b$, $c_0\equiv c$, $a_1\perp b_0$ and $a_2\perp c_0$. Since $a_1\cap a_2\perp b_0$ and $a_1\cap a_2\perp c_0$, it follows from the assumption that $a_1\cap a_2\perp b_0\cup c_0$. Since $a_1\cap a_2\equiv a$, we have $a/N \perp b/N\cup c/N$.

(iii) If L is semi-orthocomplemented and $a^\perp$ is a semi-orthocomplement of $a \in L$, then it is obvious that $a^\perp/N$ is a semi-orthocomplement of a/N. If L is relatively semi-orthocomplemented and $a/N\leqq b/N$, then since $a\cap b\equiv a$ and since there is $c \in L$ with $(a\cap b)\dot{\cup} c=b$, we have $a/N\perp c/N$ and $a/N\cup c/N=b/N$. This completes the proof.

Next, we shall prove Hilfssatz 4.6 of [5], Kap. I without the assumption that L is modular.

Lemma 3.2. *Let L be a relatively complemented lattice with 0 and "$\equiv$" be a congruence relation in L. The following three statements are equivalent.*

(α) *$a\equiv b$.*

(β) *There exists $t \in L$ such that $a\cup b=(a\cap b)\cup t$ and $t\equiv 0$.*

(γ) *There exists $t \in L$ such that $a\cup t=b\cup t$ and $t\equiv 0$.*

Proof. Since L is relatively complemented, there is $t \in L$ with $a\cup b=(a\cap b)\cup t$ and $a\cap b\cap t=0$. If $a\equiv b$, then $t=t\cap(a\cup b)\equiv t\cap(a\cap b)=0$. Hence ($\alpha$) implies ($\beta$). Since $a\cup b=(a\cap b)\cup t$ implies $a\cup t=a\cup b=b\cup t$, (β) implies (γ). If (γ) holds, then $a\equiv a\cup t=b\cup t\equiv b$, that is, ($\alpha$) holds. This completes the proof.

By [5], Kap. I, Satz 4.5, if "$\equiv$" is a congruence relation in a lattice L with 0, then its kernel $J=\{a \in L;\ a\equiv 0\}$ is a neutral ideal. We shall show the following lemma which includes the last part of Satz 4.5 as a special case.

Lemma 3.3. *Let L be a relatively semi-orthocomplemented lattice. If J is a neutral ideal of L, then there exists one and only one congruence relation whose kernel is equal to J.*

Proof. Let N be the set of quotients a/b such that a relatively semi-orthocomplement of b in a is in J. Remark that if $a/b \in N$, every semi-orthocomplement of b in a is in J since J is neutral. We shall show that N is a quotient-ideal. (1°) $a/a \in N$ for every $a \in L$ since $0 \in J$. (2°) If $a\geqq a_1\geqq b_1\geqq b$ and $a/b \in N$, then putting $a=a_1\dot{\cup} d_1$, $a_1=b_1\dot{\cup} d_2$ and $b_1=b\dot{\cup} d_3$, we have $a=b\dot{\cup}(d_3\cup d_2\cup d_1)$ by the axiom $(\perp 4)$. Since $d_3\cup d_2\cup d_1 \in J$, we have $d_2 \in J$, which implies that $a_1/b_1 \in N$. (3°) If a/b, $b/c \in N$, then putting $a=b\dot{\cup} d_1$ and $b=c\dot{\cup} d_2$ we have $a=c\dot{\cup}(d_1\cup d_2)$. Since $d_1, d_2 \in J$, we have $d_1\cup d_2 \in J$, which implies that $a/c \in N$. (4°) Let $a\cup b=a\dot{\cup} d_1$ and $b=(a\cap b)\dot{\cup} d_2$. Since d_1 and d_2 are perspective and J is neutral, we have $d_1 \in J\Leftrightarrow d_2 \in J$, which implies $a\cup b/a \in N\Leftrightarrow b/a\cap b \in N$. Therefore, N is a quotient-ideal, and there exists a congruence relation "$\equiv$" such that $\{a/b;\ a\equiv b\}=N$. Then, since $a\equiv 0\Leftrightarrow a/0 \in N\Leftrightarrow a \in J$, the kernel of the con-

2) $a\cup b$ is denoted by $a\dot{\cup} b$ when $a\perp b$.

gruence relation is equal to J. The uniqueness of the congruence relation follows from Lemma 3.2.

THEOREM 3.1. *Let L be a relatively semi-orthocomplemented lattice and J be a neutral ideal of L. Then, the quotient lattice L/J can be defined and is also relatively semi-orthocomplemented. If L is relatively orthocomplemented (resp. complemented modular), then so is L/J.*

PROOF. The first part of the theorem follows from Lemmas 3.3 and 3.1. If L is relatively orthocomplemented, then so is L/J by Theorem 1 of [7] and Lemma 3.1 (ii). If L is complemented modular, then it is obvious that L/J is modular, and hence is complemented modular.

§ 4. Perspectivity in $\mathcal{R}_I(\mathfrak{A})$

The set of all right (resp. left) idempotents of an element x of a ring is denoted by $RI(x)$ (resp. $LI(x)$). If $\mathfrak{A}$ is a Rickart ring, $RI(x)$ and $LI(x)$ are not empty for every $x \in \mathfrak{A}$. If $e, f \in RI(x)$, then since $(e)^r=(f)^r$ we have $(1-e)_r=(1-f)_r$ and $(e)_l=(f)_l$. If $\mathcal{R}_I(\mathfrak{A})$ is a lattice and $e, f \in RI(x)$, then $(e)_r$ and $(f)_r$ have the same semi-orthocomplement and then they are perspective in $\mathcal{R}_I(\mathfrak{A})$. If $e, f \in LI(x)$ then we have $(e)_r=(f)_r$.

LEMMA 4.1. *Let $\mathfrak{A}$ be a Rickart ring and $e, f \in I(\mathfrak{A})$.*

(i) *If $g \in LI((1-e)f)$, then $e+g-ge \in I(\mathfrak{A})$ and $(e+g-ge)_r=(e)_r\dot{\cup}(g)_r=(e)_r\cup(f)_r$ in $\mathcal{R}_I(\mathfrak{A})$. There exists $g_0 \in LI((1-e)f)$ with $eg_0=g_0e=0$. Then $(e+g_0)_r=(e)_r\cup(f)_r$.*

(ii) *If $h \in RI((1-e)f)$, then $f-fh \in I(\mathfrak{A})$ and $(f-fh)_r=(e)_r\cap(f)_r$ in $\mathcal{R}_I(\mathfrak{A})$. There exists $h_0 \in RI((1-e)f)$ with $fh_0=h_0f=h_0$. Then $(h_0)_r\dot{\cup}((e)_r\cap(f)_r)=(f)_r$.*

PROOF. (i) Since $e \in ((1-e)f)^l=(g)^l$ we have $eg=0$. Putting $g_0=g(1-e)$, we have $g_0 \in I(\mathfrak{A})$, $(g_0)_r=(g)_r$, $eg_0=g_0e=0$, and hence $e+g-ge=e+g_0 \in I(\mathfrak{A})$, $(e+g_0)_r=(e)_r\dot{\cup}(g)_r$. Since $(g_0)^l=(g)^l$ we have $g_0 \in LI((1-e)f)$ and, as in the proof of Lemma 1.1, it is easy to show that $((1-e)(1-g_0))_l=(e, f)^l=(1-e)_l\cap(1-f)_l$. Hence $(e+g_0)_r=(e)_r\cup(f)_r$.

(ii) Since $1-f \in ((1-e)f)^r=(h)^r$ we have $h(1-f)=0$. As in the proof of Lemma 1.1, it is easy to show that $f-fh=f(1-h) \in I(\mathfrak{A})$ and $(f-fh)_r=(1-e, 1-f)^r=(e)_r\cap(f)_r$. On the other hand, putting $h_0=fh$, we have $h_0 \in I(\mathfrak{A})$ and $(h_0)^r=(h)^r$, which imply $h_0 \in RI((1-e)f)$. Since $fh_0=h_0f=h_0$, we have $(f)_r=(h_0)_r\dot{\cup}(f-h_0)_r=(h_0)_r\dot{\cup}((e)_r\cap(f)_r)$.

LEMMA 4.2. *Let $\mathfrak{A}$ be a Rickart ring and $e, f, g \in I(\mathfrak{A})$.*

(i) *$(e)_r\cup(g)_r=(f)_r\cup(g)_r$ in $\mathcal{R}_I(\mathfrak{A})$ if and only if $((1-g)e)^l=((1-g)f)^l$.*

(ii) *$(e)_r\cap(g)_r=(f)_r\cap(g)_r$ in $\mathcal{R}_I(\mathfrak{A})$ if and only if $((1-e)g)^r=((1-f)g)^r$.*

(iii) *$(e)_r \cap (g)_r = (0)_r$ in $\mathcal{R}_I(\mathfrak{A})$ if and only if $((1-g)e)^r = (e)^r$.*

Proof. (i) By Lemma 4.1 (i), there are $h \in LI((1-g)e)$ and $k \in LI((1-g)f)$ with $hg=gh=0$, $kg=gk=0$ and $(g+h)_r=(e)_r \cup (g)_r$, $(g+k)_r=(f)_r \cup (g)_r$. Since $(g+h)(g+k)=g+hk$ and $(g+k)(g+h)=g+kh$, the equation $(g+h)_r=(g+k)_r$ is equivalent to $(h)_r=(k)_r$ and hence to $(h)^l=(k)^l$. Hence we get the statement (i).

(ii) It is proved by the similar method as (i) that $(e)_l \cup (g)_l = (f)_l \cup (g)_l \Leftrightarrow (e(1-g))^r = (f(1-g))^r$. Since $(e)_r \cap (g)_r = (f)_r \cap (g)_r \Leftrightarrow (1-e)_l \cup (1-g)_l = (1-f)_l \cup (1-g)_l$, we get the statement (ii).

(iii) By Lemma 4.1 (ii), there is $h \in RI((1-g)e)$ with $eh=he=h$ and then $(e)_r \cap (g)_r = (e-h)_r$. Hence $(e)_r \cap (g)_r = (0)_r \Leftrightarrow e=h$, and it is easy to show that $e=h \Leftrightarrow (e)^r=(h)^r$.

Theorem 4.1. *Let $\mathfrak{A}$ be a Rickart ring and $I^2(\mathfrak{A}) = \{ef;\ e, f \in I(\mathfrak{A})\}$.*

(i) *If $x \in I^2(\mathfrak{A})$ and $e \in LI(x)$, then there exists $f_0 \in RI(x)$ such that $(f_0)_r$ is perspective to $(e)_r$ in $\mathcal{R}_I(\mathfrak{A})$, and hence for any $f \in RI(x)$, $(e)_r$ and $(f)_r$ are projective in $\mathcal{R}_I(\mathfrak{A})$.*

(ii) *If $(e)_r$ and $(f)_r$ are perspective in $\mathcal{R}_I(\mathfrak{A})$, then there exist $x, y \in I^2(\mathfrak{A})$ such that $e \in RI(x)$, $f \in RI(y)$ and $(x)^l=(y)^l$.*

Proof. (i) Putting $x=(1-g)h$ $(g, h \in I(\mathfrak{A}))$, there is $f_0 \in RI(x)$ with $(f_0)_r \dot{\cup} ((g)_r \cap (h)_r) = (h)_r$ by Lemma 4.1 (ii). If $e \in LI(x)$, then we have $(e)_r \dot{\cup} (g)_r = (g)_r \cup (h)_r$ by Lemma 4.1 (i). Hence $(e)_r$ and $(f_0)_r$ are perspective, and since $(f)_r$ $(f \in RI(x))$ is perspective to $(f_0)_r$, $(e)_r$ and $(f)_r$ are projective.

(ii) If $(e)_r$ and $(f)_r$ are perspective, then there is $g \in I(\mathfrak{A})$ with $(e)_r \cup (g)_r = (f) \cup (g)_r$ and $(e)_r \cap (g)_r = (f)_r \cap (g)_r = (0)_r$. Putting $x=(1-g)e$ and $y=(1-g)f$, we have $e \in RI(x)$, $f \in RI(y)$ and $(x)^l=(y)^l$ by Lemma 4.2 (iii) and (i).

Remark that these lemmas and theorem hold if $\mathfrak{A}$ is a ring with unity having the property: every element of $I^2(\mathfrak{A})$ has a right idempotent and a left idempotent.

Definition 4.1. In a ring $\mathfrak{A}$, two idempotents e, f are called to be *algebraically equivalent*, denoted by $e \overset{a}{\sim} f$, if there exist $x, y \in \mathfrak{A}$ with $xy=e$ and $yx=f$ (Berberlian [1], p. 503). We may assume that $x \in e\mathfrak{A}f$ and $y \in f\mathfrak{A}e$. "$\overset{a}{\sim}$" is an equivalence relation in $I(\mathfrak{A})$. Remark that $(e)_r=(f)_r$ (or $(e)_l=(f)_l$) implies $e \overset{a}{\sim} f$ and that any two right (or left) idempotents of an element of a ring with unity are algebraically equivalent.

Lemma 4.3. *Let $\mathfrak{A}$ be a Rickart ring. If $(e)_r \perp (f)_r$ in $\mathcal{R}_I(\mathfrak{A})$ and $e \overset{a}{\sim} f$, then $(e)_r$ and $(f)_r$ are perspective.*

Proof. We may assume that $ef=fe=0$. There are $x \in e\mathfrak{A}f$ and $y \in f\mathfrak{A}e$ with $xy=e$, $yx=f$. It follows from $(e+y)x=x+f$, $(x+f)y=e+y$ that $(e+y)^l=(x+f)^l$. Since $(e)^l \cap (f)^l \subset (e+fy)^l = (e+y)^l$, $(e+y)^l \cap (e)^l \subset (y)^l \subset (f)^l$ and $(x+f)^l \cap (f)^l \subset (x)^l \subset (e)^l$, we have $(e+y)^l \cap (e)^l = (e+y)^l \cap (f)^l = (e)^l \cap (f)^l$. Hence, putting $g \in LI(e+y)$, we have $(1-g)_l \cap (1-e)_l = (1-g)_l \cap (1-f)_l = (1-e)_l \cap (1-f)_l$ and hence $(g)_r \cup (e)_r = (g)_r \cup (f)_r = (e)_r \cup (f)_r = (e+f)_r$. Since $g=(e+f)g$ and since it follows from $(e-x)(x+f)=(e-xf)(ex+f)=ex-xf=0$ that $(e-x)g=0$, we have $g=(x+f)g=(x+f)fg$. If $u \in (e)_r \cap (g)_r$, then since $u=eu=gu$ we have $u=(x+f)fgu=(x+f)feu=0$. Hence $(e)_r \cap (g)_r=(0)_r$ and similarly $(f)_r \cap (g)_r=(0)$. Therefore $(e)_r$ and $(f)_r$ are perspective.

LEMMA 4.4. *Let $\mathfrak{A}$ be a Rickart ring.*

(i) *The following statements are equivalent.*

(α) *For any $x \in \mathfrak{A}$, its right idempotent and left idempotent are algebraically equivalent.*

(β) *For any $x \in \mathfrak{A}$, there exists a relatively regular element u* (see §1) *such that $(x)^r=(u)^r$ and $(x)^l=(u)^l$.*

(ii) *The statement* (α) *implies the following property: if $(e)_r$ and $(f)_r$ are perspective in $\mathcal{R}_l(\mathfrak{A})$, then $e \overset{a}{\sim} f$.*

PROOF. (i) Let $e \in LI(x)$ and $f \in RI(x)$. If (α) holds, then there are $u \in e\mathfrak{A}f$ and $v \in f\mathfrak{A}e$ with $uv=e$ and $vu=f$. Since $u=eu=uf=uvu$, u is relatively regular and $(u)^r=(f)^r=(x)^r$, $(u)^l=(e)^l=(x)^l$. Conversely, if (β) holds, then since there is $v \in \mathfrak{A}$ with $uvu=u$, we have $uv \in LI(x)$ and $vu \in RI(x)$, which implies that $e \overset{a}{\sim} uv \overset{a}{\sim} vu \overset{a}{\sim} f$.

(ii) If $(e)_r$ and $(f)_r$ are perspective, then by Theorem 4.1 (ii) there are x, y with $e \in RI(x)$, $f \in RI(y)$ and $(x)^l=(y)^l$. Putting $g \in LI(x)=LI(y)$, we have $e \overset{a}{\sim} g \overset{a}{\sim} f$ by (α).

§ 5. Quotient rings of Rickart rings

Let ϕ be a homomorphism of a ring $\mathfrak{A}$ into another ring. Then, it is easy to see that $\phi((x)_r)=(\phi(x))_r \cap \phi(\mathfrak{A})$ and $\phi((x)_l)=(\phi(x))_l \cap \phi(\mathfrak{A})$ for every $x \in \mathfrak{A}$, and that $\phi((x)^r) \subset (\phi(x))^r \cap \phi(\mathfrak{A})$ and $\phi((x)^l) \subset (\phi(x))^l \cap \phi(\mathfrak{A})$.

DEFINITION 5.1. A homomorphism ϕ of a ring $\mathfrak{A}$ into another ring is called to be *annihilator preserving* if $\phi((x)^r)=(\phi(x))^r \cap \phi(\mathfrak{A})$ and $\phi((x)^l)=(\phi(x))^l \cap \phi(\mathfrak{A})$ for every $x \in \mathfrak{A}$, and the kernel of ϕ is called an *AP-ideal*. It is obvious that an ideal (two-sided) $\mathfrak{J}$ of $\mathfrak{A}$ is an *AP*-ideal if and only if the canonical homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{J}$ is annihilator preserving, in notation, $(x/\mathfrak{J})^r=(x)^r/\mathfrak{J}$ and $(x/\mathfrak{J})^l=(x)^l/\mathfrak{J}$ for every $x \in \mathfrak{A}$. If e is a right (resp. left) idempotent of $x \in \mathfrak{A}$ and $\mathfrak{J}$ is an *AP*-ideal of $\mathfrak{A}$ then $e/\mathfrak{J}$ is a right (resp. left) idempotent of $x/\mathfrak{J}$ in $\mathfrak{A}/\mathfrak{J}$. Hence, if $\mathfrak{A}$ is a Rickart ring then so is $\mathfrak{A}/\mathfrak{J}$.

LEMMA 5.1. *Let $\mathfrak{A}$ be a Rickart ring and $\mathfrak{J}$ be an ideal of $\mathfrak{A}$. The following statements are equivalent.*

(α) *$\mathfrak{J}$ is an AP-ideal.*

(β) *If $(x)^r=(y)^r$ or $(x)^l=(y)^l$, then $x \in \mathfrak{J}$ implies $y \in \mathfrak{J}$.*

(γ) *$x \in \mathfrak{J} \Leftrightarrow RI(x) \subset \mathfrak{J} \Leftrightarrow LI(x) \subset \mathfrak{J}$.*

Remark that if $\mathfrak{J}$ is an ideal and $RI(x) \cap \mathfrak{J}$ (resp. $LI(x) \cap \mathfrak{J}$) is not empty then $RI(x) \subset \mathfrak{J}$ (resp. $LI(x) \subset \mathfrak{J}$).

PROOF. $(\alpha)\Rightarrow(\beta)$. If $\mathfrak{J}$ is an AP-ideal, $x\in\mathfrak{J}$ and $(x)^r=(y)^r$ then we have $(y/\mathfrak{J})^r=(x/\mathfrak{J})^r=(0/\mathfrak{J})^r=\mathfrak{A}/\mathfrak{J}$ which implies $y\in\mathfrak{J}$. If $(x)^l=(y)^l$, then it follows from $(y/\mathfrak{J})^l=\mathfrak{A}/\mathfrak{J}$ that $y\in\mathfrak{J}$.

$(\beta)\Rightarrow(\alpha)$. Let $e\in RI(x)$. Since $(x)^r=(e)^r$, we have $(xy)^r=(ey)^r$ for any $y\in\mathfrak{A}$ and it follows from (β) that $xy\in\mathfrak{J}\Leftrightarrow ey\in\mathfrak{J}$, which implies $(x/\mathfrak{J})^r=(e/\mathfrak{J})^r$. It is easy to show that $(e/\mathfrak{J})^r=(e)^r/\mathfrak{J}$ for any $e\in I(\mathfrak{A})$. Hence $(x/\mathfrak{J})^r=(x)^r/\mathfrak{J}$ and similarly we have $(x/\mathfrak{J})^l=(x)^l/\mathfrak{J}$.

(β) implies (γ), since if $e\in RI(x)$ and $f\in LI(x)$ then $(e)^r=(x)^r$ and $(f)^l=(x)^l$. (γ) implies (β), since if $(x)^r=(y)^r$ or $(x)^l=(y)^l$ then $RI(x)=RI(y)$ or $LI(x)=LI(y)$.

DEFINITION 5.2. Let $\mathfrak{A}$ be a Rickart ring. An ideal J of $\mathcal{R}_I(\mathfrak{A})$ is called to be *AP-neutral* if it has the following property: if $e\in RI(x)$ and $f\in LI(x)$ for $x\in\mathfrak{A}$ then $(e)_r\in J\Leftrightarrow(f)_r\in J$. If J is AP-neutral and $(e)_r$ and $(f)_r$ are perspective in $\mathcal{R}_I(\mathfrak{A})$, then since it follows from Theorem 4.1 (ii) that there are x, y with $e\in RI(x)$, $f\in RI(y)$ and $(x)^l=(y)^l$, putting $g\in LI(x)=LI(y)$, we have $(e)_r\in J\Leftrightarrow(g)_r\in J\Leftrightarrow(f)_r\in J$. Hence, any AP-neutral ideal is a neutral ideal.

Since if $f_1, f_2\in LI(x)$ then $(f_1)_r=(f_2)_r$, we sometimes denote $(f_1)_r$ by $(LI(x))_r$. Remark that $e_1, e_2\in RI(x)$ does not imply $(e_1)_r=(e_2)_r$ in general.

LEMMA 5.2. *If an ideal $\mathfrak{J}$ of a Rickart ring $\mathfrak{A}$ has the property: $RI(x)\subset\mathfrak{J}\Leftrightarrow LI(x)\subset\mathfrak{J}$ (especially $\mathfrak{J}$ is an AP-ideal), then $J(\mathfrak{J})=\{(e)_r;\ e\in\mathfrak{J}\cap I(\mathfrak{A})\}$ is an AP-neutral ideal of $\mathcal{R}_I(\mathfrak{A})$.*

PROOF. If $(f)_r\leqq(e)_r\in J(\mathfrak{J})$, then it follows from $e\in\mathfrak{J}$ and $f=ef$ that $f\in\mathfrak{J}$, and hence $(f)_r\in J(\mathfrak{J})$. Let $(e)_r$, $(f)_r\in J(\mathfrak{J})$, that is, $e, f\in\mathfrak{J}\cap I(\mathfrak{A})$. It follows from Lemma 4.1 (i) that there is $g\in LI((1-e)f)$ with $ge=eg=0$ and $(e+g)_r=(e)_r\cup(f)_r$. If $h\in RI((1-e)f)$, then since it follows from $(1-e)f(1-f)=0$ that $h(1-f)=0$, we have $h=hf\in\mathfrak{J}$, which implies $g\in\mathfrak{J}$ by the assumption for $\mathfrak{J}$. Hence, $e+g\in\mathfrak{J}$ and we have $(e)_r\cup(f)_r\in J(\mathfrak{J})$. Therefore $J(\mathfrak{J})$ is an ideal. It is obvious that $J(\mathfrak{J})$ is AP-neutral.

LEMMA 5.3. *Let $\mathfrak{A}$ be a Rickart ring. If J is an ideal of $\mathcal{R}_I(\mathfrak{A})$, then $\mathfrak{J}(J)=\{x\in\mathfrak{A};\ (LI(x))_r\in J\}$ is a right ideal generated by the set $\{e\in I(\mathfrak{A});\ (e)_r\in J\}$. If, moreover, J is AP-neutral, then $\mathfrak{J}(J)$ is an AP-ideal.*

PROOF. If $e\in LI(x)$, $f\in LI(y)$ and $g\in LI(x-y)$, then since $(x-y)^l\supset(x)^l\cap(y)^l$ we have $(1-g)_l\geqq(1-e)_l\cap(1-f)_l$ in $\mathcal{L}_I(\mathfrak{A})$ and hence $(g)_r\leqq(e)_r\cup(f)_r$ in $\mathcal{R}_I(\mathfrak{A})$. If $x, y\in\mathfrak{J}(J)$, then since $(e)_r$, $(f)_r\in J$ we have $(g)_r\in J$, which implies $x-y\in\mathfrak{J}(J)$. Similarly, it follows from $(xy)^l\supset(x)^l$ that $x\in\mathfrak{J}(J)\Rightarrow xy\in\mathfrak{J}(J)$. Hence $\mathfrak{J}(J)$ is a right ideal. For any $e\in I(\mathfrak{A})$, since $e\in LI(e)$ we have $e\in\mathfrak{J}(J)\Leftrightarrow(e)_r\in J$. Therefore, $\mathfrak{J}(J)$ includes the set $\{e\in I(\mathfrak{A});\ (e)_r\in J\}$ and is generated by this set since if $x\in\mathfrak{J}(J)$ and $e\in LI(x)$ then $(e)_r\in J$ and $x=ex$.

Next, let J be AP-neutral. If $e\in RI(x)$ and $f\in RI(yx)$ then since $(yx)^r\supset(x)^r$ we have $(1-f)_r\geqq(1-e)_r$. Putting $1-e_0=(1-e)(1-f)$, we have $e_0\in RI(x)$ and $1-e_0=(1-e_0)(1-f)=(1-f)(1-e_0)$ since $(1-f)(1-e)=1-e$, and then $(f)_r\leqq(e_0)_r$.

If $x \in \mathfrak{J}(J)$ then since $(LI(x))_r \in J$ and J is AP-neutral, we have $(e_0)_r \in J$ and hence $(f)_r \in J$, which implies $(LI(yx))_r \in J$ and consequently $yx \in \mathfrak{J}(J)$. Hence $\mathfrak{J}(J)$ is an ideal. Since $e \in \mathfrak{J}(J) \Leftrightarrow (e)_r \in J$ for $e \in I(\mathfrak{A})$, we have $x \in \mathfrak{J}(J) \Leftrightarrow (LI(x))_r \in J \Leftrightarrow LI(x) \subset \mathfrak{J}(J)$ and similarly $x \in \mathfrak{J}(J) \Leftrightarrow RI(x) \subset \mathfrak{J}(J)$. Therefore $\mathfrak{J}(J)$ is an AP-ideal by Lemma 5.1.

DEFINITON 5.3. An ideal of a ring is called to be *restricted* if it is generated by idempotents.

LEMMA 5.4. *An ideal $\mathfrak{J}$ of a Rickart ring $\mathfrak{A}$ is an AP-ideal if and only if it is restricted and $RI(x) \subset \mathfrak{J} \Leftrightarrow LI(x) \subset \mathfrak{J}$.*

PROOF. If $\mathfrak{J}$ is an AP-ideal then it follows from (γ) of Lemma 5.1 that $RI(x) \subset \mathfrak{J} \Leftrightarrow LI(x) \subset \mathfrak{J}$ and that $x \in \mathfrak{J}$, $e \in RI(x)$ imply $e \in \mathfrak{J}$. Since $x = xe$, $\mathfrak{J}$ is restricted. Conversely, let $\mathfrak{J}$ be restricted and $RI(x) \subset \mathfrak{J} \Leftrightarrow LI(x) \subset \mathfrak{J}$. By Lemma 5.2, $J(\mathfrak{J})$ is an AP-neutral ideal of $\mathcal{R}_I(\mathfrak{A})$, and hence by Lemma 5.3, $\mathfrak{J}(J(\mathfrak{J}))$ is an AP-ideal. We shall show that $\mathfrak{J}(J(\mathfrak{J})) = \mathfrak{J}$. If $x \in \mathfrak{J}(J(\mathfrak{J}))$ and $e \in LI(x)$, then it follows from $(e)_r \in J(\mathfrak{J})$ that $e \in \mathfrak{J}$, which implies $x = ex \in \mathfrak{J}$. Hence $\mathfrak{J}(J(\mathfrak{J})) \subset \mathfrak{J}$. If $e \in \mathfrak{J} \cap I(\mathfrak{A})$, then it follows from $(LI(e))_r = (e)_r \in J(\mathfrak{J})$ that $e \in \mathfrak{J}(J(\mathfrak{J}))$. Since $\mathfrak{J}$ is restricted, we have $\mathfrak{J} \subset \mathfrak{J}(J(\mathfrak{J}))$. This completes the proof.

THEOREM 5.1. *Let $\mathfrak{A}$ be a Rickart ring. There is a one-to-one correspondence between the set of all AP-ideals $\mathfrak{J}$ of $\mathfrak{A}$ and the set of all AP-neutral ideals J of $\mathcal{R}_I(\mathfrak{A})$. This correspondence is given by $\mathfrak{J} \to J(\mathfrak{J}) = \{(e)_r;\ e \in \mathfrak{J} \cap I(\mathfrak{A})\}$ and $J \to \mathfrak{J}(J) = \{x \in \mathfrak{A};\ (LI(x))_r \in J\}$.*

PROOF. As in the proof of Lemma 5.4, if $\mathfrak{J}$ is an AP-ideal, then $\mathfrak{J}(J(\mathfrak{J}))$ can be defined by Lemmas 5.2 and 5.3 and $\mathfrak{J}(J(\mathfrak{J})) = \mathfrak{J}$. If J is an AP-neutral ideal, then $J(\mathfrak{J}(J))$ can be defined and since $(e)_r \in J(\mathfrak{J}(J)) \Leftrightarrow e \in \mathfrak{J}(J) \Leftrightarrow (e)_r \in J$ we have $J(\mathfrak{J}(J)) = J$. This completes the proof.

THEOREM 5.2. *Let $\mathfrak{A}$ be a Rickart ring. If $\mathfrak{J}$ is an AP-ideal of $\mathfrak{A}$, then the quotient ring $\mathfrak{A}/\mathfrak{J}$ is also a Rickart ring and $\mathcal{R}_I(\mathfrak{A}/\mathfrak{J})$ is lattice-isomorphic to the quotient lattice $\mathcal{R}_I(\mathfrak{A})/J(\mathfrak{J})$, where $J(\mathfrak{J}) = \{(e)_r;\ e \in \mathfrak{J} \cap I(\mathfrak{A})\}$. By this isomorphism, the canonical semi-orthogonality in $\mathcal{R}_I(\mathfrak{A}/\mathfrak{J})$ corresponds to that in $\mathcal{R}_I(\mathfrak{A})/J(\mathfrak{J})$ induced by the canonical semi-orthogonality in $\mathcal{R}_I(\mathfrak{A})$.*

PROOF. It is obvious that $\mathfrak{A}/\mathfrak{J}$ is a Rickart ring. Since $J(\mathfrak{J})$ is a neutral ideal of a relatively semi-orthocomplemented lattice $\mathcal{R}_I(\mathfrak{A})$, the quotient lattice $\mathcal{R}_I(\mathfrak{A})/J(\mathfrak{J})$ can be defined by Theorem 3.1 and is also relatively semi-orthocomplemented. Let $e, f \in I(\mathfrak{A})$ and $g \in LI((1-e)f)$. Since $(g)_r \dot{\cup} (e)_r = (e)_r \cup (f)_r$ by Lemma 4.1 (i), $(e)_r/J(\mathfrak{J}) \geqq (f)_r/J(\mathfrak{J})$ in $\mathcal{R}_I(\mathfrak{A})/J(\mathfrak{J})$ if and only if $(g)_r \in J(\mathfrak{J})$. Since $(g)_r \in J(\mathfrak{J})$ is equivalent to $(1-e)f \in \mathfrak{J}(J(\mathfrak{J})) = \mathfrak{J}$ and hence to $f/\mathfrak{J} = ef/\mathfrak{J}$, we have

$$(e)_r/J(\mathfrak{J}) \geqq (f)_r/J(\mathfrak{J}) \text{ in } \mathcal{R}_I(\mathfrak{A})/J(\mathfrak{J}) \Leftrightarrow (e/\mathfrak{J})_r \geqq (f/\mathfrak{J})_r \text{ in } \mathcal{R}_I(\mathfrak{A}/\mathfrak{J}).$$

Especially, $(e)_r/J(\mathfrak{J})=(f)_r/J(\mathfrak{J})\Leftrightarrow(e/\mathfrak{J})_r=(f/\mathfrak{J})_r$. Therefore the mapping ψ: $(e)_r/J(\mathfrak{J})\rightarrow(e/\mathfrak{J})_r$ is an order-preserving, one-to-one mapping of $\mathcal{R}_I(\mathfrak{A})/J(\mathfrak{J})$ into $\mathcal{R}_I(\mathfrak{A}/\mathfrak{J})$. If $x/\mathfrak{J}\in I(\mathfrak{A}/\mathfrak{J})$ and $e\in LI(x)$ then $(x/\mathfrak{J})_r=(LI(x/\mathfrak{J}))_r=(e/\mathfrak{J})_r$. Hence $\psi((e)_r/J(\mathfrak{J}))=(x/\mathfrak{J})_r$, which means that ψ is onto, and hence $\mathcal{R}_I(\mathfrak{A}/\mathfrak{J})$ is lattice-isomorphic to $\mathcal{R}_I(\mathfrak{A})/J(\mathfrak{J})$.

Next, we shall show that $(e)_r/J(\mathfrak{J})\perp(f)_r/J(\mathfrak{J})\Leftrightarrow(e/\mathfrak{J})_r\perp(f/\mathfrak{J})_r$. Let $(e)_r/J(\mathfrak{J})\perp(f)_r/J(\mathfrak{J})$. Then, we may assume that $(e)_r\perp(f)_r$ in $\mathcal{R}_I(\mathfrak{A})$, and then we may assume that $ef=fe=0$. Hence $e/\mathfrak{J}\cdot f/\mathfrak{J}=f/\mathfrak{J}\cdot e/\mathfrak{J}=0/\mathfrak{J}$ which means that $(e/\mathfrak{J})_r\perp(f/\mathfrak{J})_r$ in $\mathcal{R}_I(\mathfrak{A}/\mathfrak{J})$. Conversely, let $(e/\mathfrak{J})_r\perp(f/\mathfrak{J})_r$. Then, we may assume that $ef\equiv fe\equiv 0$ (mod. $\mathfrak{J}$). Since there is $g\in RI(ef)$ with $(g)_r\dot{\cup}((1-e)_r\cap(f)_r)=(f)_r$ by Lemma 4.1 (ii) and since it follows from $ef\in\mathfrak{J}$ that $(g)_r\in J(\mathfrak{J})$, we have $(f)_r/J(\mathfrak{J})=(1-e)_r\cap(f)_r/J(\mathfrak{J})$. It follows from $(e)_r\perp(1-e)_r\cap(f)_r$ in $\mathcal{R}_I(\mathfrak{A})$ that $(e)_r/J(\mathfrak{J})\perp(f)_r/J(\mathfrak{J})$.

Lemma 5.5. *Let $\mathfrak{A}$ be a Rickart ring satisfying the condition (α) in Lemma 4.4, that is, if $e\in RI(x)$ and $f\in LI(x)$ then $e\overset{a}{\sim}f$. Then, an ideal of $\mathfrak{A}$ is an AP-ideal if and only if it is restricted, and an ideal of $\mathcal{R}_I(\mathfrak{A})$ is AP-neutral if and only if it has the property: $(e)_r\in J$ and $e\overset{a}{\sim}f$ imply $(f)_r\in J$.*

Proof. Let $\mathfrak{J}$ be an ideal of $\mathfrak{A}$. Since $e\overset{a}{\sim}f\in\mathfrak{J}$ implies $e\in\mathfrak{J}$, it follows from (α) that $RI(x)\subset\mathfrak{J}\Leftrightarrow LI(x)\subset\mathfrak{J}$. Hence, by Lemma 5.4, $\mathfrak{J}$ is an *AP*-ideal if and only if it is restricted.

Let J be an *AP*-neutral ideal of $\mathcal{R}_I(\mathfrak{A})$. If $(e)_r\in J$ and $e\overset{a}{\sim}f$, then as in the proof of Lemma 4.4 (i), there is $u\in\mathfrak{A}$ with $e\in LI(u)$ and $f\in RI(u)$, and hence $(f)_r\in J$. Conversely, if J is an ideal having the property: $(e)_r\in J$ and $e\overset{a}{\sim}f$ imply $(f)_r\in J$, then it follows from (α) that J is *AP*-neutral.

Remark. If $\mathfrak{A}$ is a Rickart ring satisfying (α) and $\mathfrak{J}$ is a restricted ideal of $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{J}$ is a Rickart ring since $\mathfrak{J}$ is an *AP*-ideal; and then $\mathfrak{A}/\mathfrak{J}$ also satisfies (α), because if $e\in RI(x)$ and $f\in LI(x)$ then $e/\mathfrak{J}\in RI(x/\mathfrak{J})$ and $f/\mathfrak{J}\in LI(x/\mathfrak{J})$ and it follows from $e\overset{a}{\sim}f$ that $e/\mathfrak{J}\overset{a}{\sim}f/\mathfrak{J}$ in $\mathfrak{A}/\mathfrak{J}$.

Lemma 5.6. *If $\mathfrak{A}$ is a regular ring, then any ideal of $\mathfrak{A}$ is an AP-ideal, and any neutral ideal of $\mathcal{R}_I(\mathfrak{A})$ is AP-neutral.*

Proof. Let $\mathfrak{J}$ be an ideal of $\mathfrak{A}$. Since for any $x\in\mathfrak{J}$ there is $y\in\mathfrak{A}$ with $x=xyx$, we have $xy\in\mathfrak{J}\cap I(\mathfrak{A})$ and $x=(xy)x$, which implies that $\mathfrak{J}$ is restricted. Since a regular ring satisfies the condition (α) by Lemma 4.4 (i), it follows from Lemma 5.5 that $\mathfrak{J}$ is an *AP*-ideal.

Let J be a neutral ideal of $\mathcal{R}_I(\mathfrak{A})$. To prove that J is *AP*-neutral it suffices by Lemma 5.5 to show that $(e)_r\in J$ and $e\overset{a}{\sim}f$ imply $(f)_r\in J$. Let $e=xy$, $f=yx$, $x\in e\mathfrak{A}f$ and $y\in f\mathfrak{A}e$. Since $\mathcal{R}_I(\mathfrak{A})$ is relatively semi-orthocomplemented, there is $g\in I(\mathfrak{A})$ with $(f)_r=((e)_r\cap(f)_r)\dot{\cup}(g)_r$. Then, since $g=fg=yxg$, we have $xgy\in I(\mathfrak{A})$ and $xgy\overset{a}{\sim}g$, and it follows from $(xgy)_r\leqq(e)_r$ that $(xgy)_r\in J$. Since $(xgy)_r\cap(g)_r=(xgy)_r\cap(e)_r\cap(f)_r\cap(g)_r=(0)_r$, it follows from the last remark of [7] that

$(xgy)_r \perp (g)_r$. Hence $(xgy)_r$ and $(g)_r$ are perspective by Lemma 4.3, and since J is neutral, we have $(g)_r \in J$. Since $(e)_r \frown (f)_r \in J$, we conclude that $(f)_r \in J$. This completes the proof.

Remark that any quotient ring of a regular ring $\mathfrak{A}$ is regular and that $(LI(x))_r = (x)_r$ for any $x \in \mathfrak{A}$. The following theorem is a consequence of Lemma 5.6 and Theorems 5.1 and 5.2.

THEOREM 5.3. *Let $\mathfrak{A}$ be a regular ring. There is a one-to-one correspondence between the set of all ideals $\mathfrak{J}$ of $\mathfrak{A}$ and the set of all neutral ideals J of $\mathcal{R}_I(\mathfrak{A})$. This correspondence is given by $\mathfrak{J} \to J(\mathfrak{J}) = \{(e)_r;\, e \in \mathfrak{J} \frown I(\mathfrak{A})\}$ and $J \to \mathfrak{J}(J) = \{x \in \mathfrak{A};\, (x)_r \in J\}$. Any quotient ring $\mathfrak{A}/\mathfrak{J}$ is also regular and $\mathcal{R}_I(\mathfrak{A}/\mathfrak{J})$ is lattice-isomorphic to $\mathcal{R}_I(\mathfrak{A})/J(\mathfrak{J})$.*

§ 6. Rickart *-rings

DEFINITION 6.1. A *-ring is a ring with involution $x \to x^*$. A projection (=self-adjoint idempotent) e of a *-ring $\mathfrak{A}$ is called a *right* (resp. *left*) *projection* of an element $x \in \mathfrak{A}$ if $(x)^r = (e)^r$ (resp. $(x)^l = (e)^l$) (Rickart [8]). If e is a right projection of x then it is a left projection of x^*. If $\mathfrak{A}$ has the unity element, the right (resp. left) projection of $x \in \mathfrak{A}$ is uniquely determined and is denoted by $RP(x)$ (resp. $LP(x)$).

A *-ring $\mathfrak{A}$ with unity is called a *Rickart* *-ring if every element has the right projection (or equivalently, if every element has the left projection). Remark that a *-ring with unity is a Rickart *-ring if and only if it satisfies the condition (R_r^*) in the introduction, which is equal to the axiom (i) of Berberlian [1, § 3]; and that a Rickart *-ring is a Rickart ring where each $RI(x)$ (resp. $LI(x)$) includes one and only one projection $RP(x)$ (resp. $LP(x)$). Baer *-rings of Kaplansky [4], *-regular rings (Kaplansky [3]; F. Maeda [5], Kap. XII) and B_p^*-algebras of Rickart [8] are Rickart *-rings.

The set of all projections of a *-ring $\mathfrak{A}$ is denoted by $P(\mathfrak{A})$. $P(\mathfrak{A})$ is a partially ordered set when $e \leqq f$ is defined by $e = ef(=fe)$.

THEOREM 6.1. *If $\mathfrak{A}$ is a Rickart *-ring, then the set $P(\mathfrak{A})$ of all projections of $\mathfrak{A}$ forms a relatively orthocomplemented lattice, isomorphic to $\mathcal{R}_I(\mathfrak{A})$ and to $\mathcal{L}_I(\mathfrak{A})$.*

PROOF. Since $\mathfrak{A}$ is a Rickart ring, $\mathcal{R}_I(\mathfrak{A})$ and $\mathcal{L}_I(\mathfrak{A})$ are relatively semi-orthocomplemented lattices by Theorem 1.1, and it is easy to show that they are equal to $\{(e)_r;\, e \in P(\mathfrak{A})\}$ and $\{(e)_l;\, e \in P(\mathfrak{A})\}$ respectively. Since the mappings of $P(\mathfrak{A})$ to $\mathcal{R}_I(\mathfrak{A})$ and to $\mathcal{L}_I(\mathfrak{A})$: $e \to (e)_r$, $e \to (e)_l$ are order-preserving and one-to-one, $P(\mathfrak{A})$ is a lattice, isomorphic to $\mathcal{R}_I(\mathfrak{A})$ and to $\mathcal{L}_I(\mathfrak{A})$. The mapping $e \to 1-e$ defines an orthogonal relation in $P(\mathfrak{A})$ and it is easy to show that $P(\mathfrak{A})$ is relatively orthocomplemented.

REMARK. $P(\mathfrak{A})$ has a semi-orthogonal relation induced from $\mathcal{R}_I(\mathfrak{A})$ by the isomorphism $e \to (e)_r$. This relation coincides with that induced from $\mathscr{L}_I(\mathfrak{A})$; because, for $e, f \in P(\mathfrak{A})$, if there are $e_0, f_0 \in I(\mathfrak{A})$ with $(e_0)_r = (e)_r$, $(f_0)_r = (f)_r$ and $e_0 f_0 = f_0 e_0 = 0$, then $e_0^*, f_0^* \in I(\mathfrak{A})$, $(e_0^*)_l = (e)_l$, $(f_0^*)_l = (f)_l$ and $e_0^* f_0^* = f_0^* e_0^* = 0$. If two projections e, f are orthogonal ($ef=0$) then they are semi-orthogonal.

As in the case of Rickart rings, it is easy to show the following lemmas.

Lemma 6.1. *If $\mathfrak{A}$ is a Rickart *-ring and the lattice $P(\mathfrak{A})$ is complete, then $\mathfrak{A}$ is a Baer *-ring.*

Lemma 6.2. *If $\mathfrak{A}$ is a Rickart *-ring and $e \in P(\mathfrak{A})$, then $e\mathfrak{A}e$ is also a Rickart *-ring and $P(e\mathfrak{A}e)$ is lattice-isomorphic to the sublattice $\{f \in P(\mathfrak{A}); f \leqq e\}$ of $P(\mathfrak{A})$.*

Lemma 6.3. *Let $\mathfrak{A}$ be the direct product of *-rings $\{\mathfrak{A}_\alpha\}$. $\mathfrak{A}$ is a Rickart *-ring if and only if every $\mathfrak{A}_\alpha$ is a Rickart *-ring, and then $P(\mathfrak{A})$ is the direct product of lattices $\{P(\mathfrak{A}_\alpha)\}$.*

Remark that if $\mathfrak{A}$ is a commutative Rickart *-ring, then since $e \in I(\mathfrak{A})$ implies $e = RP(e) \in P(\mathfrak{A})$ we have $I(\mathfrak{A}) = P(\mathfrak{A})$. It follows from Theorem 2.1 that

Theorem 6.2. *Let $\mathfrak{A}$ be a Rickart *-ring. The following statements for $e \in P(\mathfrak{A})$ are equivalent.*

(α) *e is in the center of $\mathfrak{A}$.*

(β) *e is in the center of the lattice $P(\mathfrak{A})$.*

Corollary 1. *The center $\mathfrak{Z}$ of a Rickart *-ring $\mathfrak{A}$ is also a Rickart *-ring and $P(\mathfrak{Z}) = I(\mathfrak{Z})$ is equal to the center of the lattice $P(\mathfrak{A})$.*

Corollary 2. *Let $\mathfrak{A}$ be a Rickart *-ring. If $P(\mathfrak{A})$ is a direct product of sublattices L_1 and L_2, then there exist Rickart *-subrings $\mathfrak{A}_1$ and $\mathfrak{A}_2$ such that $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2$ and that $P(\mathfrak{A}_i)$ is equal to $L_i (i=1, 2)$.*

The following statements for a Rickart *-ring $\mathfrak{A}$ are implied from the results of § 4.

Lemma 6.4. *$RP((1-e)f) = f - e \cap f$ and $LP((1-e)f) = e \cup f - e$ for $e, f \in P(\mathfrak{A})$.*

Lemma 6.5. *Let $e, f, g \in P(\mathfrak{A})$.*

(i) $e \cup g = f \cup g \Leftrightarrow ((1-g)e)^l = ((1-g)f)^l \Leftrightarrow (e(1-g))^r = (f(1-g))^r$.

(ii) $e \cap g = 0 \Leftrightarrow ((1-g)e)^r = (e)^r \Leftrightarrow (e(1-g))^l = (e)^l$.

Theorem 6.3. *Let $\mathfrak{A}$ be a Rickart *-ring and $P^2(\mathfrak{A}) = \{ef; e, f \in P(\mathfrak{A})\}$.*

(i) *If $x \in I^2(\mathfrak{A})$ then $RP(x)$ and $LP(x)$ are projective in $P(\mathfrak{A})$. If $x \in P^2(\mathfrak{A})$ then $RP(x)$ and $LP(x)$ are perspective.*

(ii) *If e and f are perspective in $P(\mathfrak{A})$, then there exist $x, y \in P^2(\mathfrak{A})$ such that $e = RP(x)$, $f = RP(y)$ and $(x)^l = (y)^l$.*

(The second statement of (i) follows from Lemma 6.4.)

Lemma 6.6. *If e and f are semi-orthogonal in $P(\mathfrak{A})$ and $e \overset{a}{\sim} f$, then they are perspective.*

REMARK. When $\mathfrak{A}$ is a Rickart *-ring, the conditon (α) in Lemma 4.4 is equivalent to

(α_p): $RP(x)\overset{a}{\sim}LP(x)$ for any $x \in \mathfrak{A}$,

which implies that if e and f are perspective in $P(\mathfrak{A})$ then $e\overset{a}{\sim}f$.

Next, we shall apply the results of § 5 to the case of Rickart *-rings.

LEMMA 6.7. *Let $\mathfrak{J}$ be an ideal of a Rickart *-ring $\mathfrak{A}$.*

(i) *The following statements are equivalent.*

(α) *$\mathfrak{J}$ is an AP-ideal.*

(β) *If $(x)^r=(y)^r$ or $(x)^l=(y)^l$, then $x \in \mathfrak{J}$ implies $y \in \mathfrak{J}$.*

(γ) $x \in \mathfrak{J} \Leftrightarrow RP(x) \in \mathfrak{J} \Leftrightarrow LP(x) \in \mathfrak{J}$.

(ii) *If $\mathfrak{J}$ is an AP-ideal, then it is self-adjoint and $\mathfrak{A}/\mathfrak{J}$ is also a Rickart *-ring, where $RP(x/\mathfrak{J})=RP(x)/\mathfrak{J}$* and *$LP(x/\mathfrak{J})=LP(x)/\mathfrak{J}$.*

PROOF. (i) follows from Lemma 5.1. (ii) Since $RP(x)=LP(x^*)$, if $\mathfrak{J}$ is an AP-ideal then it is self-adjoint by (γ). Hence $\mathfrak{A}/\mathfrak{J}$ is a *-ring, and since $RP(x)/\mathfrak{J} \in P(\mathfrak{A}/\mathfrak{J})$ we have $RP(x/\mathfrak{J})=RP(x)/\mathfrak{J}$ and $\mathfrak{A}/\mathfrak{J}$ is a Rickart *-ring.

DEFINITION 6.2. Let $\mathfrak{A}$ be a Rickart *-ring. An ideal J of the lattice $P(\mathfrak{A})$ is called to be *AP-neutral* if $RP(x) \in J \Leftrightarrow LP(x) \in J$ for $x \in \mathfrak{A}$. Let Ψ be the isomorphism from $P(\mathfrak{A})$ to $\mathcal{R}_l(\mathfrak{A})$. It is obvious that a subset J of $P(\mathfrak{A})$ is an AP-neutral ideal if and only if $\Psi(J)$ is an AP-neutral ideal of $\mathcal{R}_l(\mathfrak{A})$ defined in Definition 5.2.

It follows from Lemmas 5.2 and 5.3 that

LEMMA 6.8. (i) *If an ideal of a Rickart *-ring $\mathfrak{A}$ has the property: $RP(x) \in \mathfrak{J} \Leftrightarrow LP(x) \in \mathfrak{J}$, then $J(\mathfrak{J})=\mathfrak{J}\cap P(\mathfrak{A})$ is an AP-neutral ideal of $P(\mathfrak{A})$.*

(ii) *If J is an ideal of $P(\mathfrak{A})$, then $\mathfrak{J}(J)=\{x \in \mathfrak{A};\ LP(x) \in J\}$ is a right ideal of $\mathfrak{A}$ generated by J. If, moreover, J is AP-neutral, then $\mathfrak{J}(J)$ is an AP-ideal.*

Remark that an ideal of a Rickart *-ring is restricted (Definition 5.3) if and only if it is generated by projections, that is, it is restricted in the sense of Dixmier [2, p. 15]. It follows from Lemma 5.4 that

LEMMA 6.9. *An ideal $\mathfrak{J}$ of a Rickart *-ring $\mathfrak{A}$ is an AP-ideal if and only if it is restricted and $RP(x) \in \mathfrak{J} \Leftrightarrow LP(x) \in \mathfrak{J}$ for $x \in \mathfrak{A}$.*

It follows from Theorems 5.1 and 5.2 that

THEOREM 6.4. *Let $\mathfrak{A}$ be a Rickart *-ring. There is a one-to-one correspondence between the set of all AP-ideals $\mathfrak{J}$ of $\mathfrak{A}$ and the set of all AP-neutral ideals J of $P(\mathfrak{A})$. This correspondence is given by $\mathfrak{J}\rightarrow J(\mathfrak{J})=\mathfrak{J}\cap P(\mathfrak{A})$ and $J\rightarrow\mathfrak{J}(J)=$the ideal generated by J. If $\mathfrak{J}$ is an AP-ideal, then the quotient ring $\mathfrak{A}/\mathfrak{J}$ is also a Rickart *-ring and $P(\mathfrak{A}/\mathfrak{J})$ is lattice-isomorphic to the quotient lattice $P(\mathfrak{A})/J(\mathfrak{J})$. By this isomorphism, the canonical semi-orthogonality (resp. orthogonality) in $P(\mathfrak{A}/\mathfrak{J})$ corresponds to that in $P(\mathfrak{A})/J(\mathfrak{J})$ induced by the canonical semi-orthogonality (resp. orthogonality) in $P(\mathfrak{A})$.*

It follows from Lemma 5.5 that

LEMMA 6.10. *Let $\mathfrak{A}$ be a Rickart *-ring satisfying the condition* (α_p):$RP(x)\overset{a}{\sim} LP(x)$ *for any* $x \in \mathfrak{A}$. *Then an ideal of* $\mathfrak{A}$ *is an AP-ideal if and only if it is restricted, and an ideal* J *of* $P(\mathfrak{A})$ *is AP-neutral if and only if it has the property*: $e \in J$ *and* $e \overset{a}{\sim} f$ *imply* $f \in J$, *that is, a p-ideal in the sense of* [6], *Definition* 3.1. *If* $\mathfrak{J}$ *is a restricted ideal, then* $\mathfrak{A}/\mathfrak{J}$ *is also a Rickart *-ring satisfying* (α_p), *and* $P(\mathfrak{A}/\mathfrak{J}) \cong P(\mathfrak{A})/J(\mathfrak{J})$.

Remark that these results are generalisation and reformation of the lemmas of [6, §3], where the condition (α_p) is denoted by $(\bar{a})$. Baer *-rings satisfying the *EP*-and *SR*-axioms of Kaplansky [4] and Rickart *-rings satisfying the *PD*-axiom of [6, p. 85] (especially AW^*-algebras) satisfy (α_p).

A regular ring is called to be *-regular if it is a *-ring and $x^*x=0$ implies $x=0$ (Kaplansky [3, § 2]). It is easy to show that a regular ring is *-regular if and only if it is a Rickart *-ring (F. Maeda [5], Kap. XII, Satz 2.3). It follows from Lemma 5.6 and Theorem 5.3 that

THEOREM 6.5. *Let* $\mathfrak{A}$ *be a *-regular ring. Any ideal of* $\mathfrak{A}$ *is self-adjoint and there is a one-to-one correspondence between the set of all ideals* $\mathfrak{J}$ *of* $\mathfrak{A}$ *and the set of all neutral ideals* J *of* $P(\mathfrak{A})$. *This correspondence is given by* $\mathfrak{J} \to J(\mathfrak{J}) = \mathfrak{J} \cap P(\mathfrak{A})$ *and* $J \to \mathfrak{J}(J) =$ *the ideal generated by* J. *Any quotient ring* $\mathfrak{A}/\mathfrak{J}$ *is also a *-regular ring and* $P(\mathfrak{A}/\mathfrak{J})$ *is lattice-isomorphic to* $P(\mathfrak{A})/J(\mathfrak{J})$.

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