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Two existence results between an affine resolvable SRGD design and a difference scheme

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Abstract. The existence of affine resolvable block designs has been discussed since 1942 in the literature (cf. Bose (1942), Clatworthy (1973), Raghavarao (1988)). Kadowaki and Kageyama (2009, 2010, 2012) obtained a number of results on combinatorics for the existence of an affine resolvable SRGD design. In this paper, a new existence result is shown as a generalization of Theorem 3.3.3 given in Kadowaki and Kageyama (2009, 2010). Furthermore, another existence result is shown as a conditional converse of Theorem 3.3.3 and also a generalization of Theorem 3.3.4, both theorems given in Kadowaki and Kageyama (2009, 2010).

1. Introduction

A block design BD(v, b, r, k) with v points is said to be resolvable if the b blocks of size k each can be grouped into r resolution sets of b/r blocks each such that in each resolution set every point occurs exactly once. A resolvable BD is said to be affine resolvable if every two blocks belonging to different resolution sets intersect in the same number, say q, of points. It is known that for an affine resolvable BD(v, b, r, k), $q = k^2/v$ holds.

A BD(v, b, r, k) is called a group divisible (GD) design with parameters $v = mn, b, r, k, \lambda_1, \lambda_2$ if the mn points are divided into m groups of n points each such that any two points in the same group occur together in exactly λ_1 blocks, whereas any two points from different groups occur together in exactly λ_2 blocks. The GD designs are further classified into three subclasses: Singular if $r - \lambda_1 = 0$; Semi-Regular (SR) if $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$; Regular if $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$.

Furthermore, a special type of a difference scheme is utilized. An $sx \times sx$ matrix A with entries from an abelian group S of order $s(\geq 2)$ is called a difference scheme, denoted by $DS(sx, s; x)$, if in a vector difference on any two columns of A every entry of S occurs x times. $DS(sx, s; x)$ is also called a

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generalized Hadamard matrix, usually denoted by $GH(s,x)$, or a difference matrix, usually denoted by $D(m,m,s)$ in literature. It is seen that (i) all entries in the first row and first column of a $DS(sx,s;x)$ can be set 0, and further (ii) in each of columns except for the first, every entry of S occurs x times. Furthermore, the following properties can be derived.

(iii) In each of rows except for the first one of the $DS(sx,s;x)$, every entry of S occurs x times.

(iv) In a vector difference on any two rows of a $DS(sx,s;x)$, every entry of S occurs x times.

It is clear that a $DS(2x,2;x)$ exists iff a Hadamard matrix of order $2x$ exists. The following results are also available.

THEOREM 1 (Theorem 3.3.3 corrected in [3]). *For a prime s, the existence of a $DS(sx,s;x)$ implies the existence of an affine resolvable SRGD design with parameters $v=b=xs^2$, $r=k=sx$, $\lambda_1=0$, $\lambda_2=x$, $q=x$; $m=sx$, $n=s$ for $s\geq 2$.*

THEOREM 2 (Theorem 3.3.4 in [3]). *The existence of a Hadamard matrix of order $2x$ is equivalent to the existence of an affine resolvable SRGD design with parameters $v=b=4x$, $r=k=2x$, $\lambda_1=0$, $\lambda_2=x$, $q=x$; $m=2x$, $n=2$.*

In this paper, we derive a new existence result which provides a generalization of Theorem 1. Furthermore, we show another existence result which reveals a conditional converse of Theorem 1 and also a generalization of Theorem 2.

2. Statement

The following result will be shown as a generalization of Theorem 1.

THEOREM 3. *Let s be a prime or a prime power. Then the existence of a $DS(sx,s;x)$ implies the existence of an affine resolvable SRGD design with parameters $v=b=xs^2$, $r=k=sx$, $\lambda_1=0$, $\lambda_2=x$, $q=x$; $m=sx$, $n=s$.*

Before the proof of Theorem 3, some preliminaries are made. Let $s=p^n$, where p is a prime and n is a positive integer, and $S=\{\alpha_0,\alpha_1,\dots,\alpha_{s-1}\}$. Consider πI_p with a row-permutation π and the identity matrix I_p of order p. Also take the following $s\times s$ matrix as

$$\pi^{L_i}I_s=(\pi^{a_{i0}}I_p)\otimes(\pi^{a_{i1}}I_p)\otimes\cdots\otimes(\pi^{a_{i,n-1}}I_p),$$

where $L_i=a_{i0}+a_{i1}x+\cdots+a_{i,n-1}x^{n-1}$ for $a_{i0},a_{i1},\dots,a_{i,n-1}\in Z_p$, $i=0,1,\dots,s-1$, $\otimes$ denotes the Kronecker product of matrices, and also L_i's constitute $GF(s)$ $(=S,\text{say})$.

An illustration of Theorem 3 is given for $s = 4 = 2^2$ $(p = n = 2)$ and $x = 1$, i.e., $S = GF(4) = \{0, 1, x, 1 + x\}$ with $x^2 = 1 + x$.

Consider a $DS(4, 2^2; 1)$ given by, for example,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & x & 1+x \\ 0 & 1+x & 1 & x \\ 0 & x & 1+x & 1 \end{bmatrix}.$$

Then take the following four matrices as

$$\pi^{L_0} I_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \pi^{L_1} I_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\pi^{L_x} I_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \pi^{L_{1+x}} I_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

with $L_0 = 0 + 0 \cdot x$, $L_1 = 1 + 0 \cdot x$, $L_x = 0 + 1 \cdot x$ and $L_{1+x} = 1 + 1 \cdot x$. By replacing elements $0, 1, x, 1 + x$ $(\in S)$ in the above $DS(4, 2^2; 1)$ with $\pi^{L_0} I_4$, $\pi^{L_1} I_4$, $\pi^{L_x} I_4$, $\pi^{L_{1+x}} I_4$, respectively, we get the following 16×16 matrix D, which can be checked to be the usual incidence matrix of an affine resolvable SRGD design with parameters $v = b = 16$, $r = k = 4$, $\lambda_1 = 0$, $\lambda_2 = 1$, $q = 1$; $m = n = 4$:

$$D = \left[\begin{array}{cccc|cccc|cccc|cccc}
1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0\\
0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\
\hline
1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1\\
0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&0\\
0&0&1&0&1&0&0&0&0&0&0&1&0&1&0&0\\
0&0&0&1&0&1&0&0&0&0&1&0&1&0&0&0\\
\hline
1&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0\\
0&1&0&0&0&0&1&0&0&0&0&1&1&0&0&0\\
0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1\\
0&0&0&1&1&0&0&0&0&1&0&0&0&0&1&0\\
\hline
1&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0\\
0&1&0&0&1&0&0&0&0&0&1&0&0&0&0&1\\
0&0&1&0&0&0&0&1&0&1&0&0&1&0&0&0\\
0&0&0&1&0&0&1&0&1&0&0&0&0&1&0&0
\end{array}\right].$$

This illustrates Theorem 3 for $s = 4$ and $x = 1$. The illustration can be generalized as the following proof shows.

PROOF. By replacing elements $\alpha_0, \alpha_1, \dots, \alpha_{s-1}$ ($\in S$) in the $sx \times sx$ matrix as the existent $DS(sx, s; x)$ with $\pi^{L_0} I_s, \pi^{L_1} I_s, \dots, \pi^{L_{s-1}} I_s$, respectively, we get an $xs^2 \times xs^2$ matrix D. Now it will be shown that the matrix D itself is the incidence matrix of the required affine resolvable SRGD design. In fact, $v = b = xs^2$ is obvious and the resolvability is introduced as usual. The other design parameters can be obtained as follows. At first a GD association scheme of xs^2 points is here given by the $sx \times s$ array as

$$\begin{bmatrix} 1 & 2 & \cdots & s \\ s+1 & s+2 & \cdots & 2s \\ \vdots & \vdots & \ddots & \vdots \\ s(xs-1)+1 & s(xs-1)+2 & \cdots & xs^2 \end{bmatrix}.$$

Here let for each column $m = sx$ (i.e., the number of groups in the GD association scheme) and for each row $n = s$ (i.e., the number of points in each group in the GD association scheme). Since there is exactly one '1' in every row of the matrix $\pi^{L_i} I_s$, it is clear that $r = sx$. Similarly, since there is exactly one '1' in every column of the matrix $\pi^{L_i} I_s$, it is seen that $k = sx$ and $\lambda_1 = 0$. Furthermore it follows that $\lambda_2 = x$, because each element of S in row vector differences of $DS(sx, s; x)$ occurs x times and on the matrices $\pi^{L_0} I_s, \pi^{L_1} I_s, \dots, \pi^{L_{s-1}} I_s$, by definition, the $\{L_i\}$ coincides with $GF(s) = S$. Similarly, it can be seen that $q = x$ (showing the affine resolvability).

Next, we consider a converse of Theorem 1 under some assumption.

By the definition, an affine resolvable SRGD design with parameters $v = b = xs^2$, $r = k = sx$, $\lambda_1 = 0$, $\lambda_2 = x$, $q = x$; $m = sx$, $n = s$ has $m(= sx)$ groups in the GD association scheme and $r(= sx)$ resolution sets of s blocks each. In the incidence matrix, $(xs)^2$ submatrices C_{ij} of order s are newly introduced such that (i) C_{ij}'s are $(0, 1)$-matrices corresponding to the i-th group of the GD association scheme and the j-th resolution set of the design for $i, j = 1, 2, \dots, sx$, and (ii) $C_{ij} = I_s$ for $i = 1$ or $j = 1$. For example, the incidence matrix D in the illustration of Theorem 3 is expressed by

$$D = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}$$

with $C_{ij} = I_4$ for $i = 1$ or $j = 1$. Some conditions on these C_{ij} are newly assumed in the following theorem.

THEOREM 4. *Let s be a prime. Then the existence of an affine resolvable SRGD design with parameters* $v = b = xs^2$, $r = k = sx$, $\lambda_1 = 0$, $\lambda_2 = x$, $q = x$; $m = sx$, $n = s$ *implies the existence of a* $DS(sx, s; x)$, *if all* C_{ij}*'s have a structure formed by some cyclic row-permutations of* I_s.

Before the proof of Theorem 4, we will give an illustration of Theorem 4 for $s = 3$ and $x = 1$, along with new three procedures, T_1, T_2, T_3, of transformation.

Now let D be the incidence matrix of an affine resolvable SRGD design with parameters $v = b = 9$, $r = k = 3$, $\lambda_1 = 0$, $\lambda_2 = 1$, $q = 1$; $m = n = 3$, whose solution can be found in Table VI of [2], with the following incidence matrix

$$D = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

The matrix D can be transformed into the following D^*, whose first three rows and columns are the juxtaposition of I_3, without loss of generality, by some permutation of rows and/or columns in D (let this type of transformation be called T_1):

$$D \overset{T_1}{\mapsto} D^* = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

with $C_{ij} = I_3$ for $i = 1$ or $j = 1$.

Here it should be noted that C_{ij}'s of D^* have a structure formed by some cyclic row-permutations of I_3 (i.e., the assumption on C_{ij} is satisfied in the illustration), and each of 3 columns displayed above corresponds to each of 3 resolution sets in the starting affine resolvable design.

Next form a new 9×3 matrix D^{**}, as a submatrix of the matrix D^*, of consisting only of the first column in each of 3 resolution sets in D^* (let this type of transformation be called T_2):

$$D^* \overset{T_2}{\mapsto} D^{**} = \left[\begin{array}{c|c|c} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right].$$

The matrix D^{**} is now partitioned into 3 groups of 3 rows each and then let $\boldsymbol{d}_{ij}$ be the j-th column vector of size 3 in the i-th group for $1 \le i \le 3$ and $1 \le j \le 3$. For example, $(\boldsymbol{d}_{1j}^T, \boldsymbol{d}_{2j}^T, \boldsymbol{d}_{3j}^T)^T$ is the j-th column of D^{**}. Here, since in the starting SRGD design every block contains only one point from each group (by $k/m = 1$), $\boldsymbol{d}_{ij}$'s have only one '1' and other 2 '0's for all i and j. In this stage, the following procedure is now taken (this type of replacement procedure will be called T_3): For $1 \le l \le 3$ when the l-th component of $\boldsymbol{d}_{ij}$ is a '1', the $\boldsymbol{d}_{ij}$ is replaced with a value $l-1$, that is, $(1,0,0)^T$ is replaced by 0, $(0,1,0)^T$ by 1 and $(0,0,1)^T$ by 2. It is obvious that each column, except for the first column, contains all the distinct elements of $Z_3 = \{0,1,2\}$ once. Hence the resulting matrix D^{***} of order 3 is clearly a $DS(3,3;1)$ based on the additive group $S = Z_3$:

$$D^{**} \overset{T_3}{\mapsto} D^{***} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

This illustrates Theorem 4 for $s = 3$ and $x = 1$. The illustration can be generalized as the following proof shows.

PROOF. Take an affine resolvable SRGD design with the given parameters, having the incidence matrix D. The matrix D can be transformed into the following D_1^*, whose first s rows and columns are the juxtaposition of I_s, of order sx^2 without loss of generality, by some permutation of rows and/or columns in D (this is done by transformation T_1):

$$D \overset{T_1}{\mapsto} D_1^* = \left.\begin{bmatrix} I_s & I_s & \cdots & I_s \\ \hline I_s & & & \\ \hline \vdots & & & \\ \hline I_s & & & \end{bmatrix}\right\} xs \text{ times}$$

Next form a new $xs^2 \times xs$ matrix D_1^{**}, which is a submatrix of the matrix D_1^*, consisting only of the first column in each of xs resolution sets in D_1^* (this is done by T_2). The matrix D_1^{**} is now partitioned into xs groups of s rows each and let $\boldsymbol{d}_{ij}$ be the j-th column vector of size s in the i-th group for $1 \le i \le xs$ and $1 \le j \le xs$. Here, since in the starting SRGD design every block contains only one point from each group (by $k/m = 1$), $\boldsymbol{d}_{ij}$'s have only one '1' and other $s-1$ '0's for all i and j. In this stage, the following replacement procedure (called T_3) is now taken: For $1 \le l \le s$ when the l-th component of $\boldsymbol{d}_{ij}$ is a '1', the $\boldsymbol{d}_{ij}$ is replaced with a value $l-1$ which will become possible elements of the required DS. Then the resulting matrix D_1^{***} of order xs can be shown to be the required $DS(xs, s; x)$ on $Z_s = \{0, 1, \ldots, s-1\}$ as follows.

Let $S = Z_s$. In D_1^*, any column in the first resolution set has an inner product q $(= x)$ as vectors with the first column in other resolution sets. This means that in D_1^{***} formed from D_1^* by both T_2 and T_3, any j-th column for $2 \le j \le xs$ contains each of elements of Z_s x times. That is, in the vector differences between the first column and any j-th column of D_1^{***} for $2 \le j \le xs$ each element of Z_s appears x times.

On the other hand, since, by the assumption, C_{ij}'s of D_1^* have a structure formed by some cyclic row-permutations of I_s for $i, j = 1, 2, \ldots, xs$, the matrix D_1^* can be transformed equivalently into the following D_2^*, whose first s rows and s columns in the second resolution set are the juxtaposition of I_s, of order xs^2 by some cyclic row-permutations in each group (let this type of transformation be called T_4):

$$D_1^* \overset{T_4}{\leftrightarrow} D_2^* = \left.\begin{bmatrix} I_s & I_s & \cdots & I_s \\ \hline & I_s & & \\ \hline & \vdots & & \\ \hline & I_s & & \end{bmatrix}\right\} xs \text{ times}$$

As before, let D_2^{**} be an $xs^2 \times xs$ matrix formed from D_2^* by T_2 and further let D_2^{***} be formed from the matrix D_2^{**} by T_3. Then D_2^{***} is of the form:

$$D_2^* \overset{T_2}{\mapsto} D_2^{**} \overset{T_3}{\mapsto} D_2^{***} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ * & 0 & * & \cdots & * \\ * & 0 & * & \cdots & * \\ & \vdots & & \vdots & \\ * & 0 & * & \cdots & * \end{bmatrix}.$$

Under the procedures of transforming D_2^* to D_2^{***}, it follows that in the matrix D_2^{***}, any j-th column for $1 \le j\ (\ne 2) \le xs$ contains each of elements of Z_s x times, because any column in the secound resolution set of D_2^* has an inner product $q\ (= x)$ as vectors with the first column in other resolution sets. It is further shown that in the vector differences between the second column and any j-th column of D_2^{***} for $1 \le j\ (\ne 2) \le xs$ each element of Z_s appears x times. In fact, let d_{12} be an element of the i-th row and the second column of D_1^{***} and d_{13} be an element of the i-th row and the third column of D_1^{***}. Similarly, let d_{22} be an element of the i-th row and the second column of D_2^{***} and d_{23} be an element of the i-th row and the third column of D_2^{***}. Furthermore in the i-th group of D_1^* (and D_2^*), let μ_i be the frequency of cyclic row-permutations depending on T_4. Then, it holds that $d_{12} + \mu_i \equiv d_{22} (\mathrm{mod}\ s)$ and $d_{13} + \mu_i \equiv d_{23} (\mathrm{mod}\ s)$. Thus, it follows that $d_{12} - d_{13} \equiv (d_{12} + \mu_i) - (d_{13} + \mu_i)$ $\equiv d_{22} - d_{23} (\mathrm{mod}\ s)$. Furthermore, it is remembered that in the vector differences between the first column and any j-th column of D_1^{***} for $2 \le j \le xs$ each element of Z_s appears x times, and in the vector differences between the second column and any j-th column of D_2^{***} for $1 \le j\ (\ne 2) \le xs$ each element of Z_s appears x times. Therefore these mean that in the vector differences between the second and third columns of D_1^{***} each element of Z_s appears equally in the vector differences between the second and third columns of D_2^{***}.

Thus, similarly to the transformation $D_1^* \leftrightarrow D_2^*$, if we consider the transformation $D_1^* \leftrightarrow D_j^*$ for $3 \le j \le xs$, it can be seen that in the vector differences between "any two columns" of D_1^{***} each element of Z_s appears x times. This means that the matrix D_1^{***} is a $DS(xs, s; x)$ on Z_s.

Note that Theorem 4 shows a generalization of Theorem 2.

REMARK. The affine resolvability in the proof of Theorem 3 is also shown by use of the property (Corollary 8.5.10.1 in [5]) such that a resolvable SRGD design is affine resolvable if and only if (a) $b = v - m + r$ and (b) k^2/v is an integer, which can be easily checked in the present case.

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