HIROSHIMA MATH. J.
48 (2018), 223–251

On a good reduction criterion for proper polycurves with sections

Ippei NAGAMACHI

(Received Aug. 20, 2017)
(Revised Mar. 28, 2018)

ABSTRACT. We give a good reduction criterion for proper polycurves with sections, i.e., successive extensions of family of curves with section, under a mild assumption. This criterion is a higher dimensional version of the good reduction criterion for hyperbolic curves given by Oda and Tamagawa.

1. Introduction

Let K be a discrete valuation field with valuation ring O_K and residue field k of characteristic $p \geq 0$. Let K^{sep} be the separable closure of K, $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ the absolute Galois group of K, and I_K its inertia subgroup. (Note that I_K, as a subgroup of G_K, depends on the choice of a prime ideal in the integral closure of O_K in K^{sep} over the maximal ideal of O_K, but it is independent of this choice up to conjugation.)

When we are given a variety X proper and smooth over K, it is an interesting problem to find a criterion for X of admitting good reduction, that is, to have a scheme $\mathfrak{X}$ proper and smooth over O_K with generic fiber X. (Such an $\mathfrak{X}$ is called a smooth model of X.)

Generalizing results of Néron, Ogg, and Shafarevich for elliptic curves, Serre and Tate [ST] proves that, when X is an abelian variety over K, X has good reduction if and only if the action of I_K on the first l-adic etale cohomology $H^1(X \otimes K^{\mathrm{sep}}, \mathbb{Q}_l)$ is trivial, where l is a prime not equal to p.

When X is a proper hyperbolic curve (a geometrically connected proper smooth curve with genus ≥ 2), it is not always true that X has good reduction even if the action of I_K on $H^1(X \otimes K^{\mathrm{sep}}, \mathbb{Q}_l)$ is trivial, namely, the first l-adic etale cohomology does not have enough information to know whether X has good reduction or not.

If we consider the pro-l completion $\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l$ of the etale fundamental group $\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})$ ($\bar{t}$ is a geometric point of $X \otimes K^{\mathrm{sep}}$), it admits an

2010 *Mathematics Subject Classification*. Primary 11G20; Secondary 11G30.
Key words and phrases. good reduction, anabelian geometry, Tannakian funadamental group.

outer action of G_K (a continuous homomorphism $\rho : G_K \to \mathrm{Out}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l) := \mathrm{Aut}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l)/\mathrm{Inn}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l)$), thus the outer action $\rho|_{I_K}$ of I_K. This is expected to have a finer information than the action of I_K on $H^1(X \otimes K^{\mathrm{sep}}, \mathbb{Q}_l)$ in certain cases. Actually, Oda [Oda1] [Oda2] proved that, for a proper hyperbolic curve X, X has good reduction if and only if the outer action $\rho|_{I_K}$ is trivial. (More strongly, he proved that X has good reduction if and only if the outer action of I_K on $\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l/\Gamma_n\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l$ are trivial for any n, where $\{\Gamma_n\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l\}_n$ is the lower central filtration of $\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l$.)

Note that Oda's result is natural in the framework of anabelian geometry: In anabelian geometry, a hyperbolic curve is considered as a typical anabelian variety, that is, a variety which is determined by its outer Galois representation $G_K \to \mathrm{Out}\,\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})$ (under suitable assumption on K).

The fact that a hyperbolic curve is anabelian in this sense, which is called the Grothendieck conjecture, is proven by Tamagawa [Tama] and Mochizuki [Moch1], [Moch2]. Therefore it would be natural to expect that, for an anabelian variety, a similar good reduction criterion to that of Oda will hold.

Another class of varieties which are considered as anabelian is the class of proper hyperbolic polycurves, that is, varieties X which admit a strucure of succesive smooth fibrations

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} \mathrm{Spec}\ K \tag{1}$$

whose fibers are proper hyperbolic curves (we call such a structure a sequence of parameterizing morphisms): Indeed, the Grothendieck conjecture is known to hold for proper hyperbolic polycurves of dimension up to 4 under suitable assumption on K, by Mochizuki [Moch1] and Hoshi [Ho]. Therefore it would be natural to consider good reduction criterion for hyperbolic polycurves, which is the main interest in this paper. For this good reduction criterion, we can also treat the case of genus 1 thanks to the criterion of Néron, Ogg, and Shafarevich.

If we allow the genera of the curves in the definition of proper hyperbolic polycurves to be 1, we say the resulting variety as a proper polycurve. We call X a proper polycurve with sections if it admits a sequence of parameterizing morphisms (1) such that each f_i admits a section (we call such a structure a sequence of parameterizing morphisms with sections). When we fix a sequence of parameterizing morphisms with sections (1) of X, we call the maximum of the genera of fibers of f_i's the maximal genus of (1), and when only X is given, we call the minimum of the maximal genera of sequences of parameterizing morphisms with sections of X the maximal genus of X.

Also, for such X and any closed point x of X, let $K(x)$ be the residue field of x and consider the pro-p' completion $\pi_1(X \otimes K(x)^{\mathrm{sep}}, \bar{x})^{p'}$ of the etale

fundamental group, where $\bar{x}$ is a geometric point of $X \otimes K(x)^{\mathrm{sep}}$ above x. Because x is $K(x)$-rational, $\pi_1(X \otimes K(x)^{\mathrm{sep}}, \bar{x})^{p'}$ admits an action (not just an outer action) of the absolute Galois group $G_{K(x)}$ of $K(x)$. Thus if we take a valuation ring $O_{K(x)}$ of $K(x)$ which contains O_K, we have the action of the inertia subgroup $I_{K(x)}$ on $\pi_1(X \otimes K(x)^{\mathrm{sep}}, \bar{x})^{p'}$. Then the main theorem is described as follows:

THEOREM 1. *Let K be as above and let X be a proper polycurve with sections over K. Let g be the maximal genus of X. Consider the following conditions:*

(A): *X has good reduction.*

(B): *For any closed point x of X, and for any choice of valuation ring $O_{K(x)}$ of $K(x)$ as above, the action of $I_{K(x)}$ on $\pi_1(X \otimes K(x)^{\mathrm{sep}}, \bar{x})^{p'}$ is trivial.*

Then (A) *implies* (B). *If $p = 0$ or $p > 2g + 1$,* (B) *implies* (A).

Since the implication (A) $\Rightarrow$ (B) is rather easy, we explain the strategy of the proof of the implication (B) $\Rightarrow$ (A) (assuming $p > 2g + 1$). Our proof heavily depends on the machinery of Tannakian categories.

For a prime number l different from p and a geometrically connected scheme Y over a field L, let $\mathrm{Et}_l(Y \otimes L(y)^{\mathrm{sep}})$ be the category of smooth $\mathbb{Q}_l$-sheaves on $Y \otimes L(y)^{\mathrm{sep}}$, which is a Tannakian category over $\mathbb{Q}_l$. Here, y is a closed point of Y and $L(y)$ is the residue field of y. For $r \in \mathbb{N}$, we define its Tannakian subcategories $\mathrm{Et}_l^{\leq r}(Y \otimes L(y)^{\mathrm{sep}})$ (resp. $U\,\mathrm{Et}_l(Y \otimes L(y)^{\mathrm{sep}})$) as the minimal one which contains all the smooth $\mathbb{Q}_l$-sheaves of rank $\leq r$ (resp. the trivial smooth $\mathbb{Q}_l$-sheaf $\mathbb{Q}_l$) and which is closed under taking subquotients, tensor products, duals, and extensions. Also, for a geometrically connected morphism $f : Y \to Z$ of geometrically connected schemes over L, we define the Tannakian subcategory $U_f\,\mathrm{Et}_l^{\leq r}(Y \otimes L(y)^{\mathrm{sep}})$ of $\mathrm{Et}_l(Y \otimes L(y)^{\mathrm{sep}})$ as the minimal one which contains the essential image of $f^* : \mathrm{Et}_l^{\leq r}(Z \otimes L(y)^{\mathrm{sep}}) \to \mathrm{Et}_l^{\leq r}(Y \otimes L(y)^{\mathrm{sep}})$ and which is closed under taking subquotients, tensor products, duals, and extensions. We denote the Tannaka dual of $\mathrm{Et}_l^{\leq r}(Y \otimes L(y)^{\mathrm{sep}})$ (resp. $U\,\mathrm{Et}_l(Y \otimes L(y)^{\mathrm{sep}})$, $U_f\,\mathrm{Et}_l^{\leq r}(Y \otimes L(y)^{\mathrm{sep}})$) with respect to the fiber functor defined by a geometric point over y by $\pi_1(Y \otimes L(y)^{\mathrm{sep}})^{l\text{-alg}, r}$ (resp. $\pi_1(Y \otimes L(y)^{\mathrm{sep}})^{l\text{-unip}}$, $\pi_1(Y \otimes L(y)^{\mathrm{sep}})^{l\text{-rel-unip}, r}$). (In the introduction, we omit to write the base point. Note that the definition of the affine group scheme $\pi_1(Y \otimes L(y)^{\mathrm{sep}})^{l\text{-rel-unip}, r}$ depends on f.) Note that these group schemes are equipped with actions of the absolute Galois group $G_{L(y)}$ of $L(y)$.

We take a sequence of parametrizing morphisms with sections (1) of X whose maximal genus is equal to that of X, and prove the implication (B) $\Rightarrow$ (A) by induction on n. So we assume that X_{n-1} has a good model $\mathfrak{X}_{n-1} \to \mathrm{Spec}\, O_K$. The key ingredient of the proof is the homotopy exact

sequence of Tannaka duals associated to the morphism $X \to X_{n-1}$

$$1 \to \pi_1((X \times_{X_{n-1}} x) \otimes K(x)^{\mathrm{sep}})^{l\text{-unip}} \to \pi_1(X \otimes K(x)^{\mathrm{sep}})^{l\text{-rel-unip},\, r} \to \pi_1(X_{n-1} \otimes K(x)^{\mathrm{sep}})^{l\text{-alg},\, r} \to 1, \quad (2)$$

where x is any closed point of X_{n-1}, which is regarded also as a closed point of X_n via the section of $f_n : X_n \to X_{n-1}$. This is an l-adic analogue of the homotopy exact sequences of de Rham and rigid fundamental groups of Lazda [Laz]. Lazda's proof is motivic in some sense, and so his proof works also in our case without so much changes.

We make a suitable choice of l and prove by using the exact sequence (2) that the action of $I_{K(x)}$ on $\pi_1((X \times_{X_{n-1}} x) \otimes K(x)^{\mathrm{sep}})^{l\text{-unip}}$ is trivial for any closed point x of X. (Here we use the assumption $p > 2g + 1$.) On the other hand, we see from the relative theory of Tannakian category and (2) that $\pi_1((X \times_{X_{n-1}} x) \otimes K(x)^{\mathrm{sep}})^{l\text{-unip}}$'s naturally form a group scheme $\mathscr{E}$ over the category of smooth $\mathbb{Q}_l$-sheaves on X_{n-1}. The triviality of actions of the Galois groups $I_{K(x)}$ on the groups $\pi_1((X \times_{X_{n-1}} x) \otimes K(x)^{\mathrm{sep}})^{l\text{-unip}}$ implies that the restriction of $\mathscr{E}$ to each $x (\in X_{n-1})$ is extendable to a group scheme over the category of smooth $\mathbb{Q}_l$-sheaves on the $O_{K(x)}$-valued point of $\mathfrak{X}_{n-1}$ which extends x. This kind of property and a result of Drinfeld [Dri] imply that $\mathscr{E}$ is extendable to a group scheme over the category of smooth $\mathbb{Q}_l$-sheaves on $\mathfrak{X}_{n-1}$. In particular, $\mathscr{E}$ is unramified at the generic point ξ of the special fiber of $\mathfrak{X}_{n-1}$. This and a variant of Oda's result imply that $X_n \to X_{n-1}$ has good reduction at the local ring $O_{\mathfrak{X}_{n-1},\xi}$ at ξ, and then a result of Moret-Bailly [Mor] implies that the morphism $X_n \to X_{n-1}$ lifts to a smooth morphism $\mathfrak{X}_n \to \mathfrak{X}_{n-1}$, which implies (A).

The content of each section is as follows: In Section 2, we give a review and a preliminary result on l-unipotent envelope of profinite groups which we need in this paper. In Section 3, we give a review on Oda's good reduction criterion for proper hyperbolic curves and prove its variant, which uses l-unipotent envelope of etale fundamental groups. In Section 4, we prove a homotopy exact sequence of Tannaka duals of certain categories of smooth $\mathbb{Q}_l$-sheaves of the form (2). In Section 5, we give a review of Drinfeld's result on extension of smooth $\mathbb{Q}_l$-sheaves. We check that it is applicable in our situation, because the situation of Drinfeld is slightly more restrictive. In Section 6, we give a proof of the main theorem, using the results proved up to previous sections. In Section 7, we give a proof of Oda's good reduction criterion in [Oda1] and [Oda2], which is not proved for a general discrete valuation field and is stated for general discrete valuation field without proof in [Tama] Remark (5.4).

We need to mention the following question.

Question. Let K be a discrete valuation field, I_K the inertia group of K, p the residue characteristic of K, X a proper hyperbolic polycurve not necessarily with section over K. Are the following equivalent?

: (A) X has good reduction.

: (B) The outer action $I_K \to \mathrm{Out}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^{p'})$ is trivial.

The implication (A) $\Rightarrow$ (B) is easy. We want to prove the implication (B) $\Rightarrow$ (A), but we can not use the standard induction because it seems that appropriate homotopy exact sequences associated to fibrations of curves do not exist. In fact, the pro-p' completion (of profinite groups) is not an exact functor. Moreover, if the characteristic of K is positive, then we do not have fibration exact sequence of (full) etale fundamental groups. This fact follows from the existence of specialization homomorphisms which are not isomorphisms. To overcome this problem, we assume the existence of sections and use Tannakian fundamental groups in this paper. Using sections, we can obtain the above homotopy exact sequences. Another difference between the above question and the main theorem of this paper is the assumption about base points. For a family of proper smooth curves over a proper polycurve over K, this assumption is necessary to obtain informations of reduction of curves over the function field of the polycurve from informations of reduction of the closed fibers.

2. Review of l-unipotent envelope of profinite groups

In this section, we recall basic facts on l-unipotent envelope of profinite groups.

We start with a review of [Del] §9. For an abstract group G, we denote the lower central series of G as $\{\Gamma_n G\}_{n\geq 1}$. We write the profinite (resp. pro-p', resp. pro-l) completion of G by $\hat{G}$ (resp. $G^{p'}$, resp. G^l), where l is a prime number and p is a prime number or 0. Here, the pro-p' completion of G is the limit of the projective system of quotient groups of G which are finite groups of order not divisible by p. (Note that the pro-p' completion depends only on p. We do not consider a prime number p' here.) We also denote the lower central series of a profinite group G as $\{\Gamma_n G\}_{n\geq 1}$, where $\Gamma_n G$ is the closure of $\Gamma_n G$ (as abstract group) in G. For an abstract group G and a prime number l, the natural morphism from G to its pro-l completion G^l induces the isomorphism

$$G^l/\Gamma_n(G^l) \cong (G/\Gamma_n G)^l \tag{3}$$

for all $n \geq 1$, since both sides of this isomorphism are the limit of the projective system of quotient groups of G which are finite l groups and have nilpotent length $\leq n$.

DEFINITION 1. The embedding functor

$$(\text{uniquely divisible nilpotent groups}) \to (\text{nilpotent groups}) \tag{4}$$

has a left adjoint functor ([Bo] II §4 ex15 and §ex 6), which we denote by $G \mapsto G_{\mathbb{Q}}$. We refer to $G_{\mathbb{Q}}$ as the *divisible closure* of G.

It is known that, when G is a finitely generated torsion free nilpotent group, the adjunction morphism $G \to G_{\mathbb{Q}}$ is injective.

DEFINITION 2. (1) The *unipotent envelope* of an abstract finitely generated group G is defined to be the Tannaka dual of the category of finite dimensional unipotent representations of G over $\mathbb{Q}$. This is written by G^{unip}.

(2) The *l-unipotent envelope* of a finitely generated group (resp. a profinite group) G is defined to be the Tannaka dual of the category of finite dimensional unipotent representations (resp. finite dimensional continuous unipotent representations) of G over $\mathbb{Q}_l$. This is written by $G^{l\text{-unip}}$.

Let N be a finitely generated torsion free nilpotent group. Then it is known that we have the diagram

$$N \hookrightarrow N_{\mathbb{Q}} \cong N^{\text{unip}}(\mathbb{Q}), \tag{5}$$

where the first map is the adjunction morphism defined by Proposition 1.

On the other hand, for N as above, the profinite completion $\hat{N}$ of N is known to be isomorphic to the closure of N in $N^{\text{unip}}(\mathbb{A}_f) := \prod_l{}' N^{\text{unip}}(\mathbb{Q}_l)$. Here, $\prod_l{}' N^{\text{unip}}(\mathbb{Q}_l)$ is the restricted direct product of the topological groups $N^{\text{unip}}(\mathbb{Z}_l) \subset N^{\text{unip}}(\mathbb{Q}_l)$. Since any finite nilpotent group is the product of their l-Sylow subgroups, we have

$$\hat{N} \cong \prod_l N^l, \tag{6}$$

where l runs over all prime numbers. By looking at the l-component of the inclusion

$$\prod_l N^l \cong \hat{N} \hookrightarrow N^{\text{unip}}(\mathbb{A}_f) = \prod_l{}' N^{\text{unip}}(\mathbb{Q}_l), \tag{7}$$

we obtain the inclusion

$$N^l \hookrightarrow N^{\text{unip}}(\mathbb{Q}_l). \tag{8}$$

Next, we recall the following fact on l-unipotent envelope of profinite groups. For more detailed explanation for l-unipotent envelope, see [Wil].

PROPOSITION 1 ([Wil] Proposition 2.3). *Let G be a finitely generated group, and l be a prime number. Then, we have the isomorphism*

$$G^{l\text{-}unip} \cong G^{\mathrm{unip}} \otimes_{\mathbb{Q}} \mathbb{Q}_l. \tag{9}$$

Moreover, since all the unipotent representations of G over $\mathbb{Q}_l$ factor through G^l, we have

$$(G^l)^{l\text{-}unip} \xleftarrow{\sim} G^{l\text{-}unip} \cong G^{\mathrm{unip}} \otimes_{\mathbb{Q}} \mathbb{Q}_l. \tag{10}$$

Let Σ_g be the closed surface group of genus $g \geq 2$. Then, by the main theorem of [Lab], $\Gamma_n\Sigma_g/\Gamma_{n+1}\Sigma_g$ is a free abelian group for all $n \geq 1$. It implies that $\Sigma_g/\Gamma_n\Sigma_g$ is a finitely generated torsion free nilpotent group. Therefore, if we denote $(\Sigma_g)^l$ by π, we obtain the inclusion

$$\begin{aligned}\pi/\Gamma_n\pi \cong (\Sigma_g/\Gamma_n\Sigma_g)^l \hookrightarrow (\Sigma_g/\Gamma_n\Sigma_g)^{\mathrm{unip}}(\mathbb{Q}_l) &\cong ((\Sigma_g/\Gamma_n\Sigma_g)^{\mathrm{unip}} \otimes_{\mathbb{Q}} \mathbb{Q}_l)(\mathbb{Q}_l) \\ &\cong ((\Sigma_g/\Gamma_n\Sigma_g)^l)^{l\text{-}\mathrm{unip}}(\mathbb{Q}_l) \\ &\cong (\pi/\Gamma_n\pi)^{l\text{-}\mathrm{unip}}(\mathbb{Q}_l). \end{aligned} \tag{11}$$

We will use the inclusion (11) in the next section.

3. Good reduction criterion for proper hyperbolic curves with sections

In this section, we recall the good reduction criterion for proper hyperbolic curves proven by Oda and Tamagawa. Then, we give a modified form of it, when a given hyperbolic curve has a section.

DEFINITION 3. Let S be a scheme and X a scheme over S.

(1) We shall say that X is a *proper hyperbolic curve* (resp. *proper curve*) over S if the structure morphism $X \to S$ is smooth, proper, and of relative dimension one over S, each of whose geometric fiber is connected and of genus ≥ 2 (resp. ≥ 1).

(2) We shall say that X is a *proper hyperbolic curve with a section* (resp. *proper curve with a section*) over S if X is a proper hyperbolic curve (resp. a proper curve) over S, and if the structure morphism has a section.

Let S be the spectrum of a discrete valuation ring O_K, η the generic point of S, σ the closed point of S, $K = \kappa(\eta)$ the field of fractions of O_K, $k = \kappa(\sigma)$ the residue field of O_K, and p the characteristic of k.

DEFINITION 4. Let $X \to \operatorname{Spec} K$ be a proper smooth morphism of schemes. We say that X has good reduction if there exists a proper smooth S-scheme $\mathfrak{X}$ whose generic fiber $\mathfrak{X}_\eta$ is isomorphic to X over K. We refer to $\mathfrak{X}$ as a *smooth model* of X.

Let $X \to \operatorname{Spec} K$ be a proper hyperbolic curve. Take a geometric point $\bar{t}$ of $X \otimes K^{\mathrm{sep}}$. Then we have the exact sequence of profinite groups

$$1 \to \pi_1(X \otimes K^{\mathrm{sep}}, \bar{t}) \to \pi_1(X, \bar{t}) \to G_K \to 1. \tag{12}$$

This exact sequence yields the outer Galois action

$$G_K \to \operatorname{Out}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})), \tag{13}$$

where, for a topological group G, $\operatorname{Out}(G)$ means the quotient group of the group $\operatorname{Aut}(G)$ of continuous group automorphisms of G divided by the group $\operatorname{Inn}(G)$ of inner automorphisms of G. Then, we have natural homomorphisms

$$\begin{aligned} I_K \to G_K &\to \operatorname{Out}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})) \\ &\to \operatorname{Out}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^{p'}) \to \operatorname{Out}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^{l}) \end{aligned} \tag{14}$$

for any prime number $l \neq p$.

Oda and Tamagawa gave the following criterion.

PROPOSITION 2 ([Oda1], [Oda2], [Tama] section 5). *The following are equivalent.*

(1) *X has good reduction.*
(2) *The outer action $I_K \to \operatorname{Out}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^{p'})$ defined by (14) is trivial.*
(3) *There exists a prime number $l \neq p$ such that the outer action $I_K \to \operatorname{Out}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^{l})$ defined by (14) is trivial.*
(4) *There exists a prime number $l \neq p$ such that the outer action of I_K on $\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^{l}/\Gamma_n\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^{l}$ induced by (14) is trivial for all natural numbers n.*

In fact, this proposition is proved in [Oda1] and [Oda2] when the residue field of K is of characteristic 0 and K is a number field or a completion of a number field. In [Tama] Remark 5.4, this proposition is stated for all discrete valuation field K without proof. Since, at the time of writing, a proof of this proposition does not seem to be published, we give a proof in Section 7.

Assume that the scheme X is a proper curve over Spec K and has a section $s : \operatorname{Spec} K \to X$, and take a geometric point $\bar{s}$ over s. Since we have the natural morphism from $\bar{s}$ to $X \otimes K^{\mathrm{sep}}$, we have the homotopy exact sequence (12) with respect to the base point $\bar{s}$. The section s gives a section of the map

$\pi_1(X, \bar{s}) \to G_K$ in the homotopy exact sequence. This induces a homomorphism $G_K \to \mathrm{Aut}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s}))$, whose composition to $\mathrm{Out}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s}))$ is the same as the above homomorphism $G_K \to \mathrm{Out}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s}))$ in (14). Therefore, for a prime number $l \neq p$, we obtain the following morphisms

$$\begin{aligned} I_K \hookrightarrow G_K &\to \mathrm{Aut}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})) \\ &\to \mathrm{Aut}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{p'}) \\ &\to \mathrm{Aut}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{l}) \to \mathrm{Aut}((\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{l})^{l\text{-unip}}) \end{aligned} \tag{15}$$

by universal property of l-unipotent envelope. Here, the composition $\mathrm{Aut}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})) \to \mathrm{Aut}((\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{l})^{l\text{-unip}})$ can be identified with the morphism

$$\mathrm{Aut}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})) \to \mathrm{Aut}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{l\text{-unip}}), \tag{16}$$

which is also induced by universal property of l-unipotent envelope, via the isomorphism in Proposition 1.

PROPOSITION 3. *The following are equivalent.*

(1) *X has good reduction.*
(2) *The action of I_K on $\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{p'}$ defined by (15) is trivial.*
(3) *The action of I_K on $\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{l\text{-unip}}$ defined by (15) and (16) is trivial.*

PROOF. Assume that X has good reduction, and let $\mathfrak{X}$ be a smooth model of X. Fix a separable closure k^{sep} of k, the henselization O_K^{h}, and the strict henselization O_K^{sh} of O_K relative to $\mathrm{Spec}\, k^{\mathrm{sep}} \to \mathrm{Spec}\, k$. Let K^{sep} be a separable closure of the fraction field of O_K^{sh}. Then we have the following diagram:

$$\begin{array}{ccccccc} X \otimes_K K^{\mathrm{sep}} & \longrightarrow & X \otimes_K (\mathrm{Frac}\, O_K^{\mathrm{h}}) & \rightleftarrows & \mathrm{Spec}(\mathrm{Frac}\, O_K^{\mathrm{h}}) & \longleftarrow & \mathrm{Spec}\, K^{\mathrm{sep}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathfrak{X} \otimes_{O_K} O_K^{\mathrm{sh}} & \longrightarrow & \mathfrak{X} \otimes_{O_K} O_K^{\mathrm{h}} & \dashrightarrow & \mathrm{Spec}\, O_K^{\mathrm{h}} & \longrightarrow & \mathrm{Spec}\, O_K^{\mathrm{sh}}. \end{array} \tag{17}$$

Since the morphism $\mathfrak{X} \otimes_{O_K} O_K^{\mathrm{h}} \to \mathrm{Spec}\, O_K^{\mathrm{h}}$ is proper, the unique section s' of this morphism is induced by valuative criterion, which is compatible with vertical arrows of the above diagram and the base change of the section s by the morphism $\mathrm{Spec}(\mathrm{Frac}\, O_K^{\mathrm{h}}) \to \mathrm{Spec}\, K$.

Consider the etale fundamental groups of the schemes in the above diagram with the geometric points from the scheme $\mathrm{Spec}\, K^{\mathrm{sep}}$ (denoted by $\bar{\eta}$). Then, we have the following commutative diagram of homotopy exact

sequences of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{\eta}) & \longrightarrow & \pi_1(X \otimes_K (\mathrm{Frac}\, O_K^{\mathrm{h}}), \bar{\eta}) & \rightleftarrows & G_{\mathrm{Frac}\, O_K^{\mathrm{h}}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(\mathfrak{X} \otimes_{O_K} O_K^{\mathrm{sh}}, \bar{\eta}) & \longrightarrow & \pi_1(\mathfrak{X} \otimes_{O_K} O_K^{\mathrm{h}}, \bar{\eta}) & \rightleftarrows & \pi_1(\mathrm{Spec}\, O_K^{\mathrm{h}}, \bar{\eta}) & \longrightarrow & 1. \end{array}$$

It holds that the first row is an exact sequence by [SGA1] IX Theorem 6.1, and using the same argument in the proof of [SGA1] IX Theorem 6.1, we can show that the second row is an exact sequence. This diagram induces the commutative diagram of exact sequences of profinite groups

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{\eta})^{p'} & \to & \pi_1(X \otimes_K (\mathrm{Frac}\, O_K^{\mathrm{h}}), \bar{\eta})^{(p')} & \rightleftarrows & G_{\mathrm{Frac}\, O_K^{\mathrm{h}}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \pi_1(\mathfrak{X} \otimes_{O_K} O_K^{\mathrm{sh}}, \bar{\eta})^{p'} & \longrightarrow & \pi_1(\mathfrak{X} \otimes_{O_K} O_K^{\mathrm{h}}, \bar{\eta})^{(p')} & \rightleftarrows & \pi_1(\mathrm{Spec}\, O_K^{\mathrm{h}}, \bar{\eta}) & \to & 1. \end{array}$$

Here, the profinite group $\pi_1(X \otimes_K (\mathrm{Frac}\, O_K^{\mathrm{h}}), \bar{\eta})^{(p')}$ is the quotient group $\pi_1(X \otimes_K (\mathrm{Frac}\, O_K^{\mathrm{h}}), \bar{\eta})/\mathrm{Ker}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{\eta}) \to \pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{\eta})^{p'})$, and the profinite group $\pi_1(\mathfrak{X} \otimes_{O_K} O_K^{\mathrm{h}}, \bar{\eta})^{(p')}$ is the quotient group $\pi_1(\mathfrak{X} \otimes_{O_K} O_K^{\mathrm{h}}, \bar{\eta})/\mathrm{Ker}(\pi_1(\mathfrak{X} \otimes_{O_K} O_K^{\mathrm{sh}}, \bar{\eta}) \to \pi_1(\mathfrak{X} \otimes_{O_K} O_K^{\mathrm{sh}}, \bar{\eta})^{p'})$.

Since the left vertical arrow is an isomorphism and the action of I_K on $\pi_1(\mathfrak{X} \otimes_{O_K} O_K^{\mathrm{sh}}, \bar{\eta})^{p'}$ is trivial by the above diagram, the action of I_K on $\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{\eta})^{p'}$ is also trivial.

Assume that the action of I_K on $\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{p'}$ is trivial. Then, the action of I_K on $\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{l\text{-unip}}$ is trivial by (15) and (16).

Assume that the action of I_K on $\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{l\text{-unip}}$ is trivial. By Proposition 2 or Néron-Ogg-Shafarevich criterion, it is sufficient to show that the action of I_K on $\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{l}/\Gamma_n \pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{l}$ is trivial for all natural number n in order to prove that X has good reduction.

The action of I_K on $\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{l\text{-unip}}$ is trivial, and we have the surjective morphism of affine group schemes

$$\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{l\text{-unip}} \twoheadrightarrow (\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})/\Gamma_n \pi_1(X \otimes K^{\mathrm{sep}}, \bar{s}))^{l\text{-unip}} \qquad (18)$$

over $\mathbb{Q}_l$. It follows that the homomorphism of their affine rings is injective, so the action of I_K on $(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})/\Gamma_n \pi_1(X \otimes K^{\mathrm{sep}}, \bar{s}))^{l\text{-unip}}$ is trivial.

We have a natural injection (11) in the previous section, by which the action of I_K on $\pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{l}/\Gamma_n \pi_1(X \otimes K^{\mathrm{sep}}, \bar{s})^{l}$ is trivial for all natural number n. Therefore, X has good reduction by Proposition 2.

REMARK 1. *In the above proposition, we only used the hypothesis that $X \to \mathrm{Spec}\, K$ is proper, smooth, geometrically connected and has a rational point, to prove $1 \Rightarrow 2$. In particular, we can show the following claim.*

CLAIM. *Let X be a proper smooth K-scheme with geometrically connected fibers, and x be a closed point of X. Consider $K(x)$-scheme $X \otimes_K K(x)$ and the associated Galois action $I_{K(x)} \to \mathrm{Aut}(\pi_1((X \otimes_K K(x)) \otimes_{K(x)} K(x)^{\mathrm{sep}}, \bar{x})^{p'})$. Here, $\bar{x}$ is a geometric point over $\mathrm{Spec}\, K(x)$. If X has good reduction, this action is trivial.*

4. Homotopy exact sequence of affine group schemes

In this section, we prove the existence of the homotopy exact sequence of affine group schemes which is similar to [Wil] Corollary 3.2. Wildeshaus showed it in the case of characteristic zero by using transcendental method, but we give an algebraic proof which works also in positive characteristic case. This exact sequence of affine group schemes plays a crucial role to prove the main theorem in this paper. We obtain this exact sequence by applying the argument in [Laz] 1.2 to smooth $\mathbb{Q}_l$-sheaves instead of regular integrable connections.

DEFINITION 5. Let r be a positive integer.

(1) Let X be a connected Noetherian scheme and l be a prime number invertible on X. We denote the category of smooth $\mathbb{Q}_l$-sheaves on X by $\mathrm{Et}_l(X)$, which is a Tannakian category over $\mathbb{Q}_l$. Then, we define the its Tannakian subcategory $\mathrm{Et}_l^{\leq r}(X)$ (resp. $U\,\mathrm{Et}_l(X)$) as the minimal one which contains all the smooth $\mathbb{Q}_l$-sheaves of rank $\leq r$ (resp. the trivial smooth $\mathbb{Q}_l$-sheaf $\mathbb{Q}_l$) and which is closed under taking subquotients, tensor products, duals, and extensions.

(2) Let $f : X \to S$ be a proper smooth morphism between connected Noetherian schemes and l be a prime number invertible on S. We define the Tannakian subcategory $U_f\,\mathrm{Et}_l^{\leq r}(X)$ of $\mathrm{Et}_l^{\leq r}(X)$ as the minimal one which contains the essential image of $f^* : \mathrm{Et}_l^{\leq r}(S) \to \mathrm{Et}_l^{\leq r}(X)$ and which is closed under taking subquotients, tensor products, duals, and extensions.

(3) Let $f : X \to S$ be a proper smooth morphism between connected Noetherian schemes, l be a prime number invertible on S, and $s \to X$ be a geometric point. We write the Tannaka dual of $\mathrm{Et}_l^{\leq r}(X)$, (resp. $U\,\mathrm{Et}_l(X)$, $U_f\,\mathrm{Et}_l^{\leq r}(X)$) with respect to the fiber functor defined by s as $\pi_1(X,s)^{l\text{-alg},r}$ (resp. $\pi_1(X,s)^{l\text{-unip}}$, $\pi_1(X,s)^{l\text{-rel-unip},r}$).

When X is a proper smooth variety over a separably closed field, the category $U\,\mathrm{Et}_l(X)$ is the same as $U_f\,\mathrm{Et}_l^{\leq r}(X)$, where f is the structure morphism. Thus, in this case the category $U\,\mathrm{Et}_l(X)$ is a special case of the category $U_f\,\mathrm{Et}_l^{\leq r}(X)$.

Let us recall some notions in the theory of Tannakian category. We will denote the fundamental group of a Tannakian category $\mathscr{T}$ over a field k by $\pi(\mathscr{T})$ (see [Del] §6). This is an affine group scheme over $\mathscr{T}$, that is, a group object in the opposite category of the category of rings of Ind $\mathscr{T}$. Moreover, for $f : X \to S$ and $s \to X$ as in Definition 5, $s^*\pi(\mathrm{Et}_l^{\leq r}(X)) \cong \pi_1(X,s)^{l\text{-alg},r}$ (resp. $s^*\pi(U\ \mathrm{Et}_l(X)) \cong \pi_1(X,s)^{l\text{-unip}}$, $s^*\pi(U_f\ \mathrm{Et}_l^{\leq r}(X)) \cong \pi_1(X,s)^{l\text{-rel-unip},r}$).

Let $f : X \to S$ be a proper, smooth, and geometrically connected morphism with section p between connected Noetherian schemes. We fix a geometric point $s \to S$, and let X_s be the base change of X by $s \to S$. We write the morphism $X_s \to X$ by i_s, and the base changes of f and p by $s \to S$ as f' and s'.

We have functors of Tannakian categories

$$\mathrm{Et}_l^{\leq r}(S) \underset{p^*}{\overset{f^*}{\rightleftarrows}} U_f\ \mathrm{Et}_l^{\leq r}(X) \overset{i_s^*}{\to} U\ \mathrm{Et}_l(X_s), \tag{19}$$

which induce homomorphisms

$$\pi_1(X_s,s)^{l\text{-unip}} \overset{i_{s*}}{\to} \pi_1(X,s)^{l\text{-rel-unip},r} \underset{p_*}{\overset{f_*}{\rightleftarrows}} \pi_1(S,s)^{l\text{-alg},r} \tag{20}$$

between their Tannaka duals.

Thanks to [Del], it can be seen that these morphisms of affine group schemes come from homomorphisms between the fundamental groups

$$\pi(U_f\ \mathrm{Et}_l^{\leq r}(X)) \overset{f_*}{\to} f^*\pi(\mathrm{Et}_l^{\leq r}(S)), \tag{21}$$

$$p^*\pi(U_f\ \mathrm{Et}_l^{\leq r}(X)) \overset{p_*}{\leftarrow} \pi(\mathrm{Et}_l^{\leq r}(S)), \tag{22}$$

and

$$\pi(U\ \mathrm{Et}_l(X_s)) \overset{i_{s*}}{\to} i_s^*\pi(U_f\ \mathrm{Et}_l^{\leq r}(X)). \tag{23}$$

DEFINITION 6. The *relatively l-unipotent fundamental group* of X/S with respect to (f,r) at p is defined to be the kernel of the morphism (21). This is an affine group scheme over $\mathrm{Et}_l^{\leq r}(S)$. We denote it by $\pi_1(X/S,r,p)^{\text{rel-}l\text{-unip}}$.

The morphisms of schemes $X_s \overset{i_s}{\to} X \overset{f}{\to} S$ induce homomorphisms

$$\pi(U\ \mathrm{Et}_l(X_s)) \overset{i_{s*}}{\longrightarrow} i_s^*\pi(U_f\ \mathrm{Et}_l^{\leq r}(X)) \overset{i_s^*f_*}{\longrightarrow} i_s^*f^*\pi(\mathrm{Et}_l^{\leq r}(S)) \tag{24}$$

of affine group schemes over $U\ \mathrm{Et}_l(X_s)$. Taking fibers at s, we get

$$\pi_1(X_s,s)^{l\text{-unip}} \overset{i_{s*}}{\to} \pi_1(X,s)^{l\text{-rel-unip},r} \overset{f_*}{\to} \pi_1(S,s)^{l\text{-alg},r}. \tag{25}$$

Since inverse image of objects in $\mathrm{Et}_l^{\leq r}(S)$ by $f \circ i_s = f' \circ s$ is trivial, the composition of the above morphisms is trivial. Thus we have the unique

morphism

$$\pi_1(X_s, s)^{l\text{-unip}} = s^*\pi(U\ \mathrm{Et}_l(X_s)) \to s^*\pi_1(X/S, r, p)^{\text{rel-}l\text{-unip}}. \tag{26}$$

The following theorem is an l-adic etale version of [Laz] Theorem 1.6.

THEOREM 2. *Let us suppose that the rank of $R^1f_*\mathbb{Q}_l$ is $\leq r$. Then, the morphism* (26) *is an isomorphism.*

Since $\pi_1(X, s)^{l\text{-rel-unip}, r} \to \pi_1(S, s)^{l\text{-alg}, r}$ is surjective, this is equivalent to saying that

$$1 \to \pi_1(X_s, s)^{l\text{-unip}} \xrightarrow{i_{s*}} \pi_1(X, s)^{l\text{-rel-unip}, r} \xrightarrow{f_*} \pi_1(S, s)^{l\text{-alg}, r} \to 1. \tag{27}$$

is an exact sequence of affine group schemes over $\mathbb{Q}_l$.

We start the proof of Theorem 2, following the proof of Lazda given in [Laz] 1.2. As in the proof of Lazda, it is sufficient to prove the following by [Wil] and [EHS]:

: (A) If $\mathscr{E} \in U_f\ \mathrm{Et}_l^{\leq r}(X)$ satisfies that $i_s^*\mathscr{E}$ is trivial, then there exists $\mathscr{F} \in \mathrm{Et}_l^{\leq r}(S)$ such that $\mathscr{E} \cong f^*\mathscr{F}$.

: (B) Let $\mathscr{E} \in U_f\ \mathrm{Et}_l^{\leq r}(X)$, and let $\mathscr{F}_0 \subset i_s^*\mathscr{E}$ denote the largest trivial sub-object. Then there exists $\mathscr{E}_0 \subset \mathscr{E}$ such that $\mathscr{F}_0 \cong i_s^*\mathscr{E}_0$ as a sub-object of $i_s^*\mathscr{E}$.

: (C) For each $\mathscr{E} \in U\ \mathrm{Et}_l(X_s)$, there exists $\mathscr{F} \in U_f\ \mathrm{Et}_l^{\leq r}(X)$ and a surjective homomorphism $i_s^*\mathscr{F} \to \mathscr{E}$.

Before proving these assertions, we check that the restrictions of functors $f_*, R^1f_* : U_f\ \mathrm{Et}_l(X) \to \mathrm{Et}_l(S)$ to $U_f\ \mathrm{Et}_l^{\leq r}(X) \to \mathrm{Et}_l^{\leq r}(S)$ are well-defined.

DEFINITION 7. Let $g : Z \to W$ be a proper smooth and geometrically connected morphism between connected Noetherian schemes, and t be a natural number. For objects $\mathscr{E} \in U_g\ \mathrm{Et}_l^{\leq t}(Z)$, we define the notion of "having unipotent class $\leq m$ with respect to (g, t)" inductively as follows. If $\mathscr{E}$ belongs to the essential image of $g^* : \mathrm{Et}_l^{\leq t}(W) \to U_g\ \mathrm{Et}_l^{\leq t}(Z)$, then we say $\mathscr{E}$ has unipotent class ≤ 1 with respect to (g, t). If there exists an extension

$$0 \to \mathscr{V} \to \mathscr{E} \to \mathscr{E}' \to 0 \tag{28}$$

with $\mathscr{E}'$ of unipotent class $\leq m-1$ and $\mathscr{V}$ of unipotent class ≤ 1, then we say that $\mathscr{E}$ has unipotent class $\leq m$.

LEMMA 1. *The functors $f_*, R^1f_* : U_f\ \mathrm{Et}_l(X) \to \mathrm{Et}_l(S)$ induce the functors $f_*, R^1f_* : U_f\ \mathrm{Et}_l^{\leq r}(X) \to \mathrm{Et}_l^{\leq r}(S)$.*

PROOF. Let $\mathscr{E}$ be an element of $U_f\ \mathrm{Et}_l^{\leq r}(X)$ whose unipotent class is $\leq m$. We use induction on m. For the case $m = 1$, there exists $\mathscr{F} \in \mathrm{Et}_l^{\leq r}(S)$ such that $f^*\mathscr{F} \cong \mathscr{E}$. Then, $f_*f^*\mathscr{F} \cong \mathscr{F} \in \mathrm{Et}_l^{\leq r}(S)$ and $R^1f_*f^*\mathscr{F} \cong R^1f_*\mathbb{Q}_l \otimes \mathscr{F} \in$

$\mathrm{Et}_l^{\leq r}(S)$ by projection formula. For the case $m \geq 2$, we have an exact sequence

$$0 \to \mathscr{V} \to \mathscr{E} \to \mathscr{E}' \to 0 \tag{29}$$

with $\mathscr{E}'$ of unipotent class $\leq m-1$ and $\mathscr{V}$ of unipotent class ≤ 1. Taking the long exact sequence

$$0 \to f_*\mathscr{V} \to f_*\mathscr{E} \to f_*\mathscr{E}' \to R^1f_*\mathscr{V} \to R^1f_*\mathscr{E} \to R^1f_*\mathscr{E}', \tag{30}$$

it follows that $f_*\mathscr{E}, R^1f_*\mathscr{E} \in \mathrm{Et}_l^{\leq r}(S)$ by induction hypothesis.

For $\mathscr{E} \in \mathrm{Et}_l^{\leq r}(X)$ (resp. $\mathscr{E}' \in \mathrm{Et}_l^{\leq r}(X_s)$) we denote the counit of the adjunction between f^* and f_* (resp. f'^* and f'_*) as $c_{\mathscr{E}} : f^*f_*\mathscr{E} \to \mathscr{E}$ (resp. $c'_{\mathscr{E}'} : f'^*f'_*\mathscr{E}' \to \mathscr{E}'$)

We first verify assertion (A).

PROPOSITION 4. *If $i_s^*\mathscr{E}$ is trivial, then $c_{\mathscr{E}} : f^*f_*\mathscr{E} \to \mathscr{E}$ is an isomorphism.*

PROOF. It is sufficient to show that the homomorphism:

$$i_s^*c_{\mathscr{E}} : i_s^*f^*f_*\mathscr{E} \to i_s^*\mathscr{E}, \tag{31}$$

which we get by pulling back $c_{\mathscr{E}}$ by i_s, is an isomorphism. By proper base change theorem,

$$i_s^*f^*f_*\mathscr{E} \cong f'^*s^*f_*\mathscr{E} \cong f'^*f'_*i_s^*\mathscr{E}, \tag{32}$$

so what we should show is that $c'_{\mathscr{E}'}$ is an isomorphism for any trivial $\mathscr{E}' \in \mathrm{Et}(X_s)$. This follows from the assumption that f is geometrically connected.

We next show assertion (B).

PROPOSITION 5. *Let $\mathscr{E} \in U_f\ \mathrm{Et}_l^{\leq r}(X)$, and let $\mathscr{F}_0 \subset i_s^*\mathscr{E}$ denote the largest trivial sub-object. Then there exists $\mathscr{E}_0 \subset \mathscr{E}$ such that $\mathscr{F}_0 \cong i_s^*\mathscr{E}_0$ as a sub-object of $i_s^*\mathscr{E}$.*

PROOF. Let us denote $i_s^*\mathscr{E}$ as $\mathscr{F}$. We have the following commutative diagram

$$\begin{array}{ccc} \mathscr{F}_0 & \longrightarrow & \mathscr{F} \\ \wr\uparrow c'_{\mathscr{F}_0} & \nearrow & \uparrow c'_{\mathscr{F}} \\ f'^*f'_*\mathscr{F}_0 & \longrightarrow & f'^*f'_*\mathscr{F}. \end{array}$$

Since $\mathscr{F}_0$ is trivial, $c'_{\mathscr{F}_0}$ is an isomorphism, which we have proved in the proof of proposition 4. Since $f'^*f'_*\mathscr{F}$ is trivial, so is the image of $c'_{\mathscr{F}} : f'^*f'_*\mathscr{F} \to \mathscr{F}$,

and then we get the unique homomorphism $f'^{*}f'_{*}\mathscr{F} \to \mathscr{F}_0$. Hence $\mathscr{F}_0$ is the image of $f'^{*}f'_{*}\mathscr{F} \to \mathscr{F}$.

By the proof of the previous proposition, $f'^{*}f'_{*}\mathscr{F} \to \mathscr{F}$ is obtained by pulling back $c_{\mathscr{E}} : f^{*}f_{*}\mathscr{E} \to \mathscr{E}$ by i_s. Thanks to exactness of i_s^{*}, $\mathscr{F}_0$ is the inverse image of the image of $c_{\mathscr{E}}$.

Finally, we start the proof of assertion (C).

For $n \in \mathbb{N}$, we define an object $\mathscr{U}_n \in U\ \mathrm{Et}_l(X_s)$ inductively as follows. Let $\mathscr{U}_1$ be the trivial smooth $\mathbb{Q}_l$-sheaf of rank 1 (denoted by $\mathbb{Q}_l$). For $n \geq 1$, we will define $\mathscr{U}_n$ to be the extension of $\mathscr{U}_{n-1}$ by $f'^{*}(R^1f'_{*}(\mathscr{U}_{n-1}^{\vee}))^{\vee}$ corresponding to the identity under the isomorphisms:

$$\begin{aligned}
\mathrm{Ext}(\mathscr{U}_{n-1}, f'^{*}(R^1f'_{*}(\mathscr{U}_{n-1}^{\vee}))^{\vee}) &\cong \mathrm{Ext}(\mathbb{Q}_l, \mathscr{U}_{n-1}^{\vee} \otimes f'^{*}(R^1f'_{*}(\mathscr{U}_{n-1}^{\vee}))^{\vee}) \\
&\cong H^1(X_s, \mathscr{U}_{n-1}^{\vee} \otimes f'^{*}(R^1f'_{*}(\mathscr{U}_{n-1}^{\vee})^{\vee})) \\
&\cong R^1f'_{*}(\mathscr{U}_{n-1}^{\vee} \otimes f'^{*}(R^1f'_{*}(\mathscr{U}_{n-1}^{\vee})^{\vee})) \\
&\cong R^1f'_{*}(\mathscr{U}_{n-1}^{\vee}) \otimes (R^1f'_{*}(\mathscr{U}_{n-1}^{\vee})^{\vee}) \\
&\cong \mathrm{End}(R^1f'_{*}(\mathscr{U}_{n-1}^{\vee})). \qquad (33)
\end{aligned}$$

Taking higher direct images of the dual of the short exact sequence

$$0 \to f'^{*}(R^1f'_{*}(\mathscr{U}_{n-1}^{\vee}))^{\vee} \to \mathscr{U}_n \to \mathscr{U}_{n-1} \to 0, \qquad (34)$$

we get the following exact sequence

$$0 \to f'_{*}(\mathscr{U}_{n-1}^{\vee}) \to f'_{*}(\mathscr{U}_n^{\vee}) \to R^1f'_{*}(\mathscr{U}_{n-1}^{\vee}) \xrightarrow{\delta} R^1f'_{*}(\mathscr{U}_{n-1}^{\vee}) \to R^1f'_{*}(\mathscr{U}_n^{\vee}). \qquad (35)$$

Lemma 2. *The connecting homomorphism δ is the identity.*

Proof. The element of

$$\mathrm{Ext}(f'^{*}(R^1f'_{*}(\mathscr{U}_{n-1}^{\vee})), \mathscr{U}_{n-1}^{\vee}) \cong \mathrm{End}(R^1f'_{*}(\mathscr{U}_{n-1}^{\vee})) \qquad (36)$$

defined by the extension

$$0 \to \mathscr{U}_{n-1}^{\vee} \to \mathscr{U}_n^{\vee} \to f'^{*}(R^1f'_{*}(\mathscr{U}_{n-1}^{\vee})) \to 0 \qquad (37)$$

is the identity.

From the fact that, for an extension $0 \to \mathscr{E} \to \mathscr{F} \to f'^{*}\mathscr{V} \to 0$ of a trivial smooth $\mathbb{Q}_l$-sheaf $f'^{*}\mathscr{V}$ by $\mathscr{E}$, the extension class under the isomorphism

$$\mathrm{Ext}(f'^{*}\mathscr{V}, \mathscr{E}) \cong \mathscr{V}^{\vee} \otimes R^1f'_{*}(\mathscr{E}) \cong \mathrm{Hom}(\mathscr{V}, R^1f'_{*}(\mathscr{E})) \qquad (38)$$

is nothing but the connecting homomorphism for the long exact sequence

$$0 \to f'_*(\mathscr{E}) \to f'_*(\mathscr{F}) \to \mathscr{V} \to R^1 f'_*(\mathscr{E}), \tag{39}$$

the lemma follows.

In particular, any extension of $\mathscr{U}_{n-1}$ by a trivial smooth $\mathbb{Q}_l$-sheaf $\mathscr{V}$ is split after pulling back to $\mathscr{U}_n$, and $f'_*(\mathscr{U}_{n-1}^\vee) \cong f'_*(\mathscr{U}_n^\vee)$. We get inductively the isomorphisms $\mathbb{Q}_l \cong f'_*(\mathscr{U}_1^\vee) \cong f'_*(\mathscr{U}_n^\vee)$ for all n.

Let $x = p'(s)$, $u_1 = 1 \in (\mathscr{U}_1)_x \cong \mathbb{Q}_l$, and choose an element $u_n \in (\mathscr{U}_n)_x$ for n inductively so that $(\mathscr{U}_n)_x \to (\mathscr{U}_{n-1})_x$ sends u_n to u_{n-1}.

PROPOSITION 6. *Let $\mathscr{F} \in U\,\mathrm{Et}_l(X_s)$ be an object of unipotent class $\leq m$ with respect to (f', r) and $n \geq m$. Then for any $v \in \mathscr{F}_x$, there exists a morphism $\alpha : \mathscr{U}_n \to \mathscr{F}$ which send u_n to v.*

PROOF. We copy the proof of [Laz] Proposition 1.17 and [HJ] Proposition 2.1.6. Let $\mathscr{F}$ be of unipotent class $\leq m$. To show the proposition, we use induction on m. The case $m = 1$ is trivial. For the case $m \geq 2$, choose an exact sequence

$$0 \to \mathscr{E} \xrightarrow{\psi} \mathscr{F} \xrightarrow{\phi} \mathscr{G} \to 0, \tag{40}$$

with $\mathscr{G}$ of unipotent class $\leq m-1$ and $\mathscr{E}$ of unipotent class ≤ 1. By induction hypothesis, there exists a morphism $\beta : \mathscr{U}_{n-1} \to \mathscr{G}$ such that $\phi_x(v) = \beta_x(u_{n-1})$. Consider the following pull-back exact sequences of the above extension with respect to $\mathscr{U}_n \xrightarrow{\text{nat}} \mathscr{U}_{n-1} \xrightarrow{\beta} \mathscr{G}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{F}'' & \rightleftarrows & \mathscr{U}_n & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{F}' & \longrightarrow & \mathscr{U}_{n-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{G} & \longrightarrow & 0.
\end{array}$$

As explained above, the extension of $\mathscr{U}_n$ by $\mathscr{E}$ splits. Fix a section $\mathscr{U}_n \to \mathscr{F}''$ and let us denote the induced morphism by $\gamma : \mathscr{U}_n \to \mathscr{F}$. Then $\phi_x(\gamma_x(u_n) - v) = 0$. By induction hypothesis, there exists $\gamma' : \mathscr{U}_n \to \mathscr{E}$ such that $\gamma'_x(u_n) = \gamma_x(u_n) - v$. Then, $\gamma - \psi \circ \gamma'$ satisfies the condition required for α.

COROLLARY 1. *For all $\mathscr{E} \in U\,\mathrm{Et}_l(X_s)$, there exists a surjective homomorphism $\mathscr{U}_m^{\oplus N} \to \mathscr{E}$ for some $m, N \in \mathbb{N}$.*

PROOF. The assertion follows immediately from the proposition if we take a basis for $\mathscr{E}_x$.

We will define $\mathscr{W}_n \in U_f \operatorname{Et}_l^{\leqq r}(X)$ whose restriction to X_s is isomorphic to $\mathscr{U}_n$ inductively. Moreover, we will construct an isomorphism $f_*(\mathscr{W}_n^\vee) \cong \mathbb{Q}_l$ and a homomorphism $\varepsilon_n : p^*\mathscr{W}_n^\vee \to \mathbb{Q}_l$ such that the composite $\mathbb{Q}_l \cong f_*(\mathscr{W}_n^\vee) \cong p^*f^*f_*(\mathscr{W}_n^\vee) \to p^*\mathscr{W}_n^\vee \xrightarrow{\varepsilon_n} \mathbb{Q}_l$ is an isomorphism.

We start the induction with $\mathscr{W}_1 = \mathbb{Q}_l$. Let us assume that $\mathscr{W}_n$ is defined. Then, we will define $\mathscr{W}_{n+1}$ to be an extension of $\mathscr{W}_n$ by the sheaf $f^*R^1f_*(\mathscr{W}_n^\vee)^\vee$ so that the inverse image of the exact sequence

$$0 \to f^*(R^1f_*(\mathscr{W}_n^\vee))^\vee \to \mathscr{W}_{n+1} \to \mathscr{W}_n \to 0 \tag{41}$$

to X_s is isomorphic to

$$0 \to f'^*(R^1f'_*(\mathscr{U}_n^\vee))^\vee \to \mathscr{U}_{n+1} \to \mathscr{U}_n \to 0. \tag{42}$$

Now consider the extension group

$$\operatorname{Ext}(\mathscr{W}_n, f^*(R^1f_*(\mathscr{W}_n^\vee))^\vee) \cong H^1(X, \mathscr{W}_n^\vee \otimes f^*(R^1f_*(\mathscr{W}_n^\vee))^\vee). \tag{43}$$

Let us denote $\mathscr{W}_n^\vee \otimes f^*(R^1f_*(\mathscr{W}_n^\vee))^\vee$ as $\mathscr{E}$. The Leray spectral sequence for $\mathscr{E}$ associated to $f : X \to S$ gives us the 5-term exact sequence

$$\begin{aligned} 0 \to H^1(S, f_*(\mathscr{E})) \to H^1(X, \mathscr{E}) \to H^0(S, R^1f_*(\mathscr{E})) \\ \to H^2(S, f_*(\mathscr{E})) \to H^2(X, \mathscr{E}). \end{aligned} \tag{44}$$

After rewriting the objects in the above exact sequence by projection formulas, the isomorphism (43) and induction hypothesis, we obtain the following exact sequence

$$\begin{aligned} 0 \to H^1(S, (R^1f_*(\mathscr{W}_n^\vee))^\vee) \to \operatorname{Ext}(\mathscr{W}_n, f^*(R^1f_*(\mathscr{W}_n^\vee))^\vee) \\ \to \operatorname{End}(R^1f_*(\mathscr{W}_n^\vee)) \to H^2(S, R^1f_*(\mathscr{W}_n^\vee)^\vee) \\ \to H^2(X, \mathscr{W}_n^\vee \otimes (R^1f_*(\mathscr{W}_n^\vee))^\vee). \end{aligned} \tag{45}$$

The isomorphism $\mathbb{Q}_l \cong f_*(\mathscr{W}_n^\vee) \cong p^*f^*f_*(\mathscr{W}_n^\vee) \to p^*\mathscr{W}_n^\vee \xrightarrow{\varepsilon_n} \mathbb{Q}_l$ induces an isomorphism

$$\begin{aligned} H^i(S, (R^1f_*(\mathscr{W}_n^\vee))^\vee) &\cong H^i(S, f_*(\mathscr{W}_n^\vee) \otimes (R^1f_*(\mathscr{W}_n^\vee))^\vee) \\ &\to H^i(S, p^*\mathscr{W}_n^\vee \otimes (R^1f_*(\mathscr{W}_n^\vee))^\vee) \\ &\to H^i(S, (R^1f_*(\mathscr{W}_n^\vee))^\vee). \end{aligned} \tag{46}$$

Since this is the same as the following isomorphism

$$\begin{aligned} H^i(S,(R^1f_*(\mathscr{W}_n^\vee))^\vee) &\to H^i(X,\mathscr{W}_n^\vee\otimes f^*(R^1f_*(\mathscr{W}_n^\vee))^\vee)\\ &\to H^i(S,p^*\mathscr{W}_n^\vee\otimes(R^1f_*(\mathscr{W}_n^\vee))^\vee)\\ &\to H^i(S,(R^1f_*(\mathscr{W}_n^\vee))^\vee), \end{aligned} \tag{47}$$

the homomorphisms

$$H^1(S,(R^1f_*(\mathscr{W}_n^\vee))^\vee)\to \mathrm{Ext}(\mathscr{W}_n,f^*(R^1f_*(\mathscr{W}_n^\vee))^\vee)$$
$$H^2(S,(R^1f_*(\mathscr{W}_n^\vee))^\vee)\to H^2(X,\mathscr{W}_n^\vee\otimes(R^1f_*(\mathscr{W}_n^\vee))^\vee)$$

in the exact sequence (45) split. Therefore the morphism $\mathrm{Ext}(\mathscr{W}_n, f^*(R^1f_*(\mathscr{W}_n^\vee))^\vee)\to \mathrm{End}(R^1f_*(\mathscr{W}_n^\vee))$ have a unique section corresponding to the retraction, so in the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}(\mathscr{W}_n,f^*(R^1f_*(\mathscr{W}_n^\vee))^\vee) & \longrightarrow & \mathrm{Ext}(\mathscr{U}_n,f'^*(R^1f'_*(\mathscr{U}_n^\vee))^\vee)\\ \downarrow & & \| \\ \mathrm{End}(R^1f_*(\mathscr{W}_n^\vee)) & \longrightarrow & \mathrm{End}(R^1f'_*(\mathscr{U}_n^\vee)), \end{array}$$

the element $\mathrm{id}\in\mathrm{End}(R^1f_*(\mathscr{W}_n^\vee))$ canonically lifts to $\mathrm{Ext}(\mathscr{W}_n,f^*(R^1f_*(\mathscr{W}_n^\vee))^\vee)$ by the section. Then, $\mathscr{W}_{n+1}$ is defined to be the extension of $\mathscr{W}_n$ by $f^*(R^1f_*(\mathscr{W}_n^\vee))^\vee$ corresponding to this element. Since this is sent to $\mathrm{id}\in\mathrm{End}(R^1f'_*(\mathscr{U}_n^\vee))$, which corresponds to the extension class of $\mathscr{U}_{n+1}$, we have natural isomorphism $i_s^*\mathscr{W}_{n+1}\cong\mathscr{U}_{n+1}$.

To complete the induction, it is sufficient to show that $f_*(\mathscr{W}_{n+1}^\vee)\cong f_*(\mathscr{W}_n^\vee)$ and that there exists a morphism $p^*\mathscr{W}_{n+1}^\vee\to\mathscr{Q}_l$ as in the induction hypothesis. By taking fibers at s and applying proper base change theorem, we can prove the first claim. For the second, we consider the following exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*\mathscr{W}_n^\vee & \longrightarrow & p^*\mathscr{W}_{n+1}^\vee & \longrightarrow & R^1f_*(\mathscr{W}_n^\vee) & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathscr{Q}_l & \dashrightarrow & (\text{pushout}) & \longrightarrow & R^1f_*(\mathscr{W}_n^\vee) & \longrightarrow & 0, \end{array}$$

where the left vertical arrow is ε_n. Then, the lower exact sequence splits since the following diagram

$$\begin{array}{ccc} \mathrm{Ext}((R^1f_*(\mathscr{W}_n^\vee)),p^*\mathscr{W}_n^\vee) & \longleftarrow & \mathrm{Ext}(\mathscr{W}_n,f^*(R^1f_*(\mathscr{W}_n^\vee)^\vee))\\ \downarrow & & \downarrow \\ \mathrm{Ext}((R^1f_*(\mathscr{W}_n^\vee)),\mathscr{Q}_l) & = & H^1(S,R^1f_*(\mathscr{W}_n^\vee)) \end{array}$$

commutes and the right vertical arrow sends the extension class defined by $\mathscr{W}_{n+1}$ to 0. Fixing its retraction, we have a homomorphism $p^*\mathscr{W}_{n+1}^{\vee} \to \mathbb{Q}_l$ such that the composition $p^*\mathscr{W}_n^{\vee} \to p^*\mathscr{W}_{n+1}^{\vee} \to \mathbb{Q}_l$ is equal to ε_n. Now the second assertion follows immediately from the fact that the diagram

$$\begin{array}{ccccccccc}
 & & f_*\mathscr{W}_{n+1}^{\vee} & \longrightarrow & p^*f^*f_*\mathscr{W}_{n+1}^{\vee} & \longrightarrow & p^*\mathscr{W}_{n+1}^{\vee} & & \\
 & \nearrow & \uparrow & & \uparrow & & \uparrow & \searrow & \\
\mathbb{Q}_l & \longrightarrow & f_*\mathscr{W}_n^{\vee} & \longrightarrow & p^*f^*f_*\mathscr{W}_n^{\vee} & \longrightarrow & p^*\mathscr{W}_n^{\vee} & \longrightarrow & \mathbb{Q}_l
\end{array}$$

commutes.

These arguments and Corollary 1 show the following proposition, which is assertion (C).

PROPOSITION 7. *For all $\mathscr{F} \in U\ \mathrm{Et}_l(X_s)$, there exists $\mathscr{E} \in U_f\ \mathrm{Et}_l^{\leq r}(X)$ and a surjective homomorphism $i_s^*\mathscr{E} \to \mathscr{F}$.*

5. Extension of smooth $\mathbb{Q}_l$-sheaves

In this section, we prove the results similar to [Dri] 5.1 when the base scheme is a discrete valuation ring. In [Dri], they are proved when the base scheme is $\mathbb{Z}$ and we check that they remain valid also in our situation.

Throughout this section, K is a discrete valuation field with valuation ring O_K and residue field k of characteristic $p \geq 0$.

LEMMA 3. *Let X be a regular scheme of finite type and flat over O_K, $D \subset X$ the special fiber, and let us suppose that the closed subset D is an irreducible reduced divisor of X. Let G be a finite group, and $\phi : Y \to X\backslash D$ a G-torsor ramified at D. Then there exists a closed point $x \in D$ and a 1-dimensional subspace L of the tangent space $T_xX \overset{\mathrm{def}}{=} (m_x/m_x^2)^*$ with the following property:*

(C) *if $C \subset X_{\bar{x}}$ is any regular 1-dimensional closed subscheme tangent to L such that $C \not\subset D_{\bar{x}}$ then the pullback of $\phi : Y \to X\backslash D$ to $C\backslash\{\bar{x}\}$ is ramified at $\bar{x}$.*

Here $\bar{x}$ is a geometric point corresponding to x and $X_{\bar{x}}$, $D_{\bar{x}}$ are the strict Henselizations.

PROOF. Let $\overline{Y}$ be the normalization of X in the ring of fractions of Y, and $\bar{\pi} : \overline{Y} \to X$ be the canonical morphism. Let us denote the generic point of D by ξ_X and choose a ramified point ξ_Y in $\overline{Y}$ over ξ_X.

Let us consider the quotient scheme $\overline{Y}/I$ of $\overline{Y}$ by the inertia subgroup I of the decomposition group at ξ_Y and the open subscheme X' of $\overline{Y}/I$ obtained by removing divisors over ξ_X except for the image of ξ_Y.

First we prove the lemma in the case $X = X'$, so we can assume that the fiber $\bar{\pi}^{-1}(\xi_X)$ is equal to $\{\xi_Y\}$. Then G is solvable and so we can assume that $|G|$ is a prime number q by replacing Y by Y/H, where $H \subset G$ is a normal subgroup of prime index.

The extension of the rings of fractions of Y and X is finite separable, so $\bar{\pi}$ is a finite morphism. Since Y is finite type over O_K, its regular locus is open by [EGA] Corollaire 6.12.6 and it contains ξ_Y. To prove the lemma we can replace X by any open subscheme of X which contains ξ_X. Thus we can assume that Y is regular by shrinking X because $\bar{\pi}$ is a closed map.

Set $\tilde{D} := (\bar{\pi}^{-1}(D))_{\mathrm{red}}$. Since $\tilde{D}$ is integral and finite type over O_K, the regular locus of $\tilde{D}$ is open and we may assume that $\tilde{D}$ and D are regular. Then, we can prove the theorem for any closed point $x \in D$. The assumption $I = G$ means that the action of G on $\tilde{D}$ is trivial and the morphism $\bar{\pi}_D : \tilde{D} \to D$ is purely inseparable. Let e_1 be its degree and let e_2 be the multiplicity of $\tilde{D}$ in the divisor $\bar{\pi}^{-1}(D)$. Then $e_1 e_2 = |G| = q$, so e_1 equals 1 or q.

Case 1: $e_1 = 1$, $e_2 = q$. Since $e_1 = 1$, the morphism $\bar{\pi}_D : \tilde{D} \to D$ is an isomorphism. If $L \not\subset T_x D$ and $C \subset X_{\bar{x}}$ is any regular 1-dimensional closed subscheme tangent to L, $T_{\bar{x}} C$ is transversal to the image of the tangent map $T_{\bar{\pi}^{-1}(x)} \overline{Y} \to T_x X$. Since the scheme $C \times_Y \overline{Y}$ is regular and the set $\bar{\pi}^{-1}(\bar{x})$ is a point, the pullback of $\pi : Y \to X \backslash D$ to $C \backslash \bar{x}$ is indeed ramified at $\bar{x}$.

Case 2: $e_1 = q$, $e_2 = 1$. Fix any closed point $y \in \tilde{D}$, and let x be $\bar{\pi}_D(y)$. If the extension of their residue fields $k(y) \supset k(x)$ is nontrivial, it is purely inseparable, so any L satisfies the condition (C). Let us assume that $k(y) \cong k(x)$. Let us denote the maximal ideal of $O_{X,x}$, $O_{D,x}$, $O_{Y,y}$, $O_{\tilde{D},y}$ as m_x, n_x, m_y, n_y and choose a local defining equation $f \in O_{X,x}$ of D. Since $e_2 = 1$, $O_{Y,y}/(f) \cong O_{\tilde{D},y}$. Then, we have the following commutative diagram with horizontal lines exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ((f) + m_x^2)/m_x^2 & \longrightarrow & m_x/m_x^2 & \longrightarrow & n_x/n_x^2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & ((f) + m_y^2)/m_y^2 & \longrightarrow & m_y/m_y^2 & \longrightarrow & n_y/n_y^2 & \longrightarrow & 0. \end{array}$$

The left vertical arrow is an isomorphism, but $m_x/m_x^2 \to m_y/m_y^2$ is not an isomorphism because $\bar{\pi}$ is not etale at y, so the right vertical arrow is not an isomorphism. If we choose $L \subset T_x X$ with $L \subset T_x D$ and $L \not\subset \mathrm{Im}(T_y \tilde{D} \to T_x D)$, we can finish the argument just as in Case 1 since the subspace $\mathrm{Im}(T_y \tilde{D} \to T_x D)$ of $T_x D$ is of codimension 1.

So we finished the proof in the case $X = X'$. Finally we prove the lemma in general case. By the lemma for X', we can take a closed point x' in the topological closure of the image of ξ_Y in X' and a 1-dimensional subspace L' of $T_{x'} X'$ which satisfy the condition (C) for X'. Since a closed point of the

special fiber of X' is a closed point of the special fiber of $\overline{Y}/I$, if we define x to be the image of x' by the morphism $\overline{Y}/I \to X$, it is a closed point of D. Then, also, since $X' \to X$ is etale by the Zariski-Nagata purity theorem [SGA1] X Theorem 3.4, $T_{x'}X' = T_xX \otimes_{k(x)} k(x')$. By the argument of taking L' up to the previous paragraph, we see that one can take the 1-dimensional subspace L' of $T_{x'}X'$ in order that it comes from a 1-dimensional subspace L of T_xX. Then we see that the condition (C) is satisfied for this x and L. So we finished the proof of the lemma in general case.

COROLLARY 2. *Let X be a scheme smooth over O_K and $U \subset X$ the generic fiber. Let $\mathscr{E}$ a smooth $\mathbb{Q}_l$-sheaf on U, where l is a prime number not equal to the characteristic of k. Suppose that $\mathscr{E}$ does not extend to a smooth $\mathbb{Q}_l$-sheaf on X. Then there exists a closed point $x \in X \setminus U$ and a 1-dimensional subspace $L \subset T_xX$ with the following property:*

(*) *if C is a Dedekind scheme of finite type over O_K, $c \in C$ a closed point such that the extension $k(c) \supset k(x)$ is separable, and $\varphi : (C, c) \to (X, x)$ a morphism with $\varphi^{-1}(U) \neq \varnothing$ such that the image of the tangent map $T_cC \to T_xX \otimes_{k(x)} k(c)$ equals $L \otimes_{k(x)} k(c)$ then the pullback of $\mathscr{E}$ to $\varphi^{-1}(U)$ is ramified at c.*

PROOF. By the Zariski-Nagata purity theorem [SGA1] X Theorem 3.4, $\mathscr{E}$ is ramified along some irreducible divisor $D \subset X$, $D \cap U = \varnothing$. Now use the above lemma.

We can also show the above corollary for an ind-smooth $\mathbb{Q}_l$-sheaf $\mathscr{E}$ on U. In fact, assume that $\mathscr{E}$ does not extend to an ind-smooth $\mathbb{Q}_l$-sheaf on X. We can write $\mathscr{E}$ as inductive system $(\mathscr{E}_i)_{i \in I}$, where each of its structure morphisms is injective. Then, there exists $i_0 \in I$ such that $\mathscr{E}_{i_0}$ does not extend to a smooth $\mathbb{Q}_l$-sheaf on X.

6. Proof of the main theorem

DEFINITION 8. Let S be a scheme and X a scheme over S.

(1) We shall say that X is a *proper polycurve* (of relative dimension n) over S if there exists a positive integer n and a (not necessarily unique) factorization of the structure morphism $X \to S$

$$X = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = S \tag{48}$$

such that, for each $i \in \{1, \ldots, n\}$, $X_i \to X_{i-1}$ is a proper curve (cf. Definition 3; recall that the genus of the curve is $\neq 0$). We refer to the above factorization of $X \to S$ as a *sequence of parametrizing morphisms*.

(2) We shall say that X is a *proper polycurve with sections* (of relative dimension n) over S if X is proper polycurve (of relative dimension n) over S, whose parametrizing morphisms can be chosen so that for all $i \in \{1, \ldots, n\}$, $X_i \to X_{i-1}$ is a proper curve with section (cf. Definition 3). We refer to such a sequence of parametrizing morphisms as a *sequence of parametrizing morphisms with sections*.

(3) Let X be a proper polycurve with sections of relative dimension n over S. For a sequence of parametrizing morphisms with sections of this,

$$\mathscr{S} : X = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = S, \tag{49}$$

we call the maximum of the genera of fibers of $X_i \to X_{i-1}$ the maximal genus $g_{\mathscr{S}}$ of $\mathscr{S}$. We call the minimum of the maximal genera of sequences of parametrizing morphisms with sections of X the maximal genus g_X of X.

Before proving the main theorem, we prove a lemma.

DEFINITION 9. For a profinite group G, let us consider the smallest Tannakian subcategory of the category of finite dimensional continuous G-representations over $\mathbb{Q}_l$ which contains all the finite dimensional continuous G-representations over $\mathbb{Q}_l$ of dimension $\leq r$ and which is closed under taking subquotients, tensor products, duals, and extensions. We write the Tannaka dual of the Tannakian subcategory with respect to the forgetful functor as $G^{l\text{-alg},r}$. This definition is compatible with Definition 5.3 if G is the etale fundamental group of X.

LEMMA 4. *Let S be a connected Noetherian scheme and take a geometric point s over S. Let r be a natural number. Then, for a prime number l which is invertible on S, the morphism $\pi_1(S,s)^{l\text{-}alg,r} \to (\pi_1(S,s)^{p'})^{l\text{-}alg,r}$ is an isomorphism, if p is 0 or a prime number which does not divide any of $l^h - 1$ $(1 \leq h \leq r)$.*

PROOF. For any continuous representation of $\pi_1(S,s)$ over $\mathbb{Q}_l$ whose rank is $\leq r$, the action of $\pi_1(S,s)$ factors through $\pi_1(S,s)^{p'}$ since $GL(r,\mathbb{Z}_l) \cong \varprojlim GL(r,\mathbb{Z}/l^n\mathbb{Z})$ and the order of $GL(r,\mathbb{Z}/l^n\mathbb{Z})$ is $l^{(n-1)r^2}(l^r-1)(l^r-l)\ldots(l^r-l^{r-1})$.

Let V_1, V_2 be finite dimensional continuous representations of $\pi_1(S,s)$ over $\mathbb{Q}_l$ whose actions of $\pi_1(S,s)$ factor through $\pi_1(S,s)^{p'}$. It is easy to see that any subquotient of V_1, $V_1 \otimes V_2$, and $V_1^\vee$ are $\pi_1(S,s)^{p'}$-representations. What we should show is for any extension V of $\pi_1(S,s)$-representation

$$0 \to V_1 \to V \to V_2 \to 0, \tag{50}$$

the action factors through $\pi_1(S,s)^{p'}$. Thus it is sufficient to show the image $\mathrm{Im}(\pi_1(S,s) \to \mathrm{Aut}\ V)$ is a pro-p' group. It follows from the fact that the image of the homomorphism

$$\mathrm{Im}(\pi_1(S,s) \to \mathrm{Aut}\ V) \to \mathrm{Aut}\ V_1 \times \mathrm{Aut}\ V_2 \tag{51}$$

is pro-p' and its kernel is pro-l. Therefore $\pi_1(S,s)^{l\text{-alg},r}$ and $(\pi_1(S,s)^{p'})^{l\text{-alg},r}$ are isomorphic.

Let K be a discrete valuation field with valuation ring O_K and residue field k of characteristic $p \geq 0$. Let X be a scheme of finite type geometrically connected over K. For any closed point $x \in X$, let $K(x)$ be the residue field of x and $G_{K(x)}$ the absolute Galois group of $K(x)$. Choose a valuation ring $O_{K(x)}$ of $K(x)$ over O_K and let $I_{K(x)}$ be the inertia subgroup of $O_{K(x)}$ (which is well-defined up to conjugation). Because x is $K(x)$-rational, $\pi_1(X \otimes K(x)^{\mathrm{sep}}, \bar{x})^{p'}$ admits an action (not an outer action) of $G_{K(x)}$. Note that the triviality of the action of inertia is independent of its choice.

THEOREM 3. *Assume that X is a proper polycurve with sections over K and $g = g_X$ is the maximal genus of X. Consider the following conditions.*

: (A) *X has good reduction.*

: (B) *The action of $I_{K(x)}$ on $\pi_1(X \otimes K(x)^{\mathrm{sep}}, \bar{x})^{p'}$ is trivial for any closed point $x \in X$, valuation ring $O_{K(x)}$ of $K(x)$ over O_K, and geometric point $\bar{x}$ over* Spec $K(x)$.

Then, we have (A) $\Rightarrow$ (B). *If we assume that $p = 0$ or $p > 2g + 1$,* (B) $\Rightarrow$ (A) *follows.*

PROOF. From Remark 1, (A) $\Rightarrow$ (B) follows. Let us prove (B) $\Rightarrow$ (A). Fix a sequence of parameterizing morphisms with sections

$$X = X_n \underset{s_n}{\overset{f_n}{\rightleftarrows}} X_{n-1} \underset{s_{n-1}}{\overset{f_{n-1}}{\rightleftarrows}} \cdots \underset{s_{i+1}}{\overset{f_{i+1}}{\rightleftarrows}} X_i \underset{s_i}{\overset{f_i}{\rightleftarrows}} X_{i-1} \underset{s_{i-1}}{\overset{f_{i-1}}{\rightleftarrows}} \cdots \underset{s_2}{\overset{f_2}{\rightleftarrows}} X_1 \underset{s_1}{\overset{f_1}{\rightleftarrows}} X_0 = \mathrm{Spec}\ K$$

of $X \to \mathrm{Spec}\ K$, whose maximal genus is g.

We will show that X_n has good reduction by induction on n. For $n = 1$, it is proved in Proposition 3. Now we assume that $n \geq 2$. For any closed point $y \in X_{n-1}$, the natural surjection $\pi_1(X_n \otimes K(x)^{\mathrm{sep}}, \bar{y})^{p'} \to \pi_1(X_{n-1} \otimes K(x)^{\mathrm{sep}}, \bar{y})^{p'}$ makes the action of $I_{K(y)}$ on $\pi_1(X_{n-1} \otimes K(x)^{\mathrm{sep}}, \bar{y})^{p'}$ trivial, where $\bar{y}$ is a geometric point above y. Moreover, the above sequence of parameterizing morphism gives the condition of $X_{n-1} \to \mathrm{Spec}\ K$ about maximal genus. Therefore, we may assume that we have a smooth model $\mathfrak{X}_{n-1}$ of X_{n-1}.

Fix a prime number l such that p does not divide any of $l^h - 1$ $(1 \leq h \leq 2g)$. Note that there exists such a prime number by the theorem on arithmetic progressions and the assumption $p > 2g + 1$. For any closed point $x \in X_{n-1}$ and any geometric point $\bar{x}$ over x, we have an exact sequence

of affine group schemes

$$1 \to \pi_1(X_n \times_{X_{n-1}} \bar{x}, \bar{x})^{l\text{-unip}} \to \pi_1(X_n \times_{\operatorname{Spec} K} \bar{x}, \bar{x})^{l\text{-rel-unip}, 2g} \to \pi_1(X_{n-1} \times_{\operatorname{Spec} K} \bar{x}, \bar{x})^{l\text{-alg}, 2g} \to 1 \tag{52}$$

by Theorem 2.

Now we consider the diagram

$$\begin{array}{ccccccc} X_n \times_{X_{n-1}} K(X_{n-1}) & \longrightarrow & X_n & \cdots\cdots\twoheadrightarrow & \mathfrak{X}_n & \twoheadleftarrow\cdots\cdots & \mathfrak{X} \\ {\scriptstyle s_n'}\uparrow\downarrow{\scriptstyle f_n'} & & {\scriptstyle s_n}\uparrow\downarrow{\scriptstyle f_n} & & \vdots\downarrow & & \vdots\downarrow \\ \operatorname{Spec} K(X_{n-1}) & \longrightarrow & X_{n-1} & \longrightarrow & \mathfrak{X}_{n-1} & \longleftarrow & \operatorname{Spec} O_{\mathfrak{X}_{n-1},\xi} \end{array}$$

where $K(X_{n-1})$ is the function field of X_{n-1}, ξ is the generic point of the special fiber $\mathfrak{X}_{n-1} \backslash X_{n-1}$, f_n', s_n' are the base change of f_n, s_n respectively, and $O_{\mathfrak{X}_{n-1},\xi}$ is the local ring at ξ. What we should show is that there exist a proper hyperbolic curve $\mathfrak{X}_n$ over $\mathfrak{X}_{n-1}$ whose base change by $X_{n-1} \to \mathfrak{X}_{n-1}$ is isomorphic to X_n. Thanks to [Mor], it is sufficient for this to show that there exists a proper hyperbolic curve $\mathfrak{X}$ over $\operatorname{Spec} O_{\mathfrak{X}_{n-1},\xi}$ whose base change by $\operatorname{Spec} K(X_{n-1}) \to \operatorname{Spec} O_{\mathfrak{X}_{n-1},\xi}$ is isomorphic to $X_n \times_{X_{n-1}} K(X_{n-1})$. Fix an inertia subgroup $I \subset G_{K(X_{n-1})}$ at ξ and a geometric point $\bar{t}$ of $X_n \times_{X_{n-1}} K(X_{n-1})$ above $\operatorname{Spec} K(X_{n-1})$. To complete the proof, it comes down to show that the action of I on $\pi_1(X_n \times_{X_{n-1}} \bar{t}, \bar{t})^{l\text{-unip}}$ is trivial by Proposition 3.

Let us denote the morphism $\operatorname{Spec} K(X_{n-1}) \to X_{n-1}$ as i. Then, we have the following exact sequences of affine group schemes over $\operatorname{Et}_l^{\le 2g}(K(X_{n-1}))$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker} f_{n*}' & \longrightarrow & s_n'^{*}\pi(U_{f_n'} \operatorname{Et}_l^{\le 2g}(X_n \times_{X_{n-1}} K(X_{n-1}))) & \xrightarrow{f_{n*}'} & \pi(\operatorname{Et}_l^{\le 2g}(\operatorname{Spec} K(X_{n-1}))) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & i^* \operatorname{Ker} f_{n*} & \longrightarrow & i^* s_n^* \pi(U_{f_n} \operatorname{Et}_l^{\le 2g}(X_n)) & \xrightarrow{i^*(f_{n*})} & i^* \pi(\operatorname{Et}_l^{\le 2g}(X_{n-1})) & \longrightarrow & 0. \end{array}$$

By Theorem 2, $\operatorname{Ker} f_{n*}'$ and $i^* \operatorname{Ker} f_{n*}$ are isomorphic and their fibers at $\bar{t}$ are isomorphic to $\pi_1(X_n \times_{X_{n-1}} \bar{t}, \bar{t})^{l\text{-unip}}$. Let us denote the ind-smooth $\mathbb{Q}_l$-sheaf on X_{n-1} corresponding to $\operatorname{Ker} f_{n*}$ by $\mathscr{E}$. To show the triviality of the action of I on $\pi_1(X_n \times_{X_{n-1}} \bar{t}, \bar{t})^{l\text{-unip}}$, it is sufficient to prove that $\mathscr{E}$ extends to $\mathfrak{X}_{n-1}$. To show this, we consider the following claim.

CLAIM. Let x be a closed point in $\mathfrak{X}_{n-1} \backslash X_{n-1}$ and O_L be any discrete valuation ring over O_K whose field of fraction L is a finite extension over K. Then, for every morphism $\operatorname{Spec} O_L \to \mathfrak{X}_{n-1}$ over O_K sending the closed point $y \in \operatorname{Spec} O_L$ to x, $\mathscr{E}|_{\operatorname{Spec} L}$ is unramified at y.

We prove the claim. Let us define the valuation ring $O_{K(x')}$ of the residue field $K(x')$ at the image x' of the generic point of O_L by $O_{K(x')} = O_L \cap K(x')$.

By condition (B) and Lemma 4, the action of I on $\pi_1(X_n \times_{\mathrm{Spec}\, K} \bar{x}', \bar{x}')^{l\text{-rel-unip}, 2g}$ is trivial. Therefore for $O_{K(x')}$, this claim follows from (52), and so is for O_L.

Finally, we prove that $\mathscr{E}$ extends to $\mathfrak{X}_{n-1}$. By the above claim and Corollary 2, it suffices to show that for all closed point x in $\mathfrak{X}_{n-1} \backslash X_{n-1}$ and for all 1 dimensional linear subspace $M \subset T_x \mathfrak{X}_{n-1}$ there exists a discrete valuation ring O_L over O_K whose field of fraction L is a finite extension over K and O_K-morphism $\mathrm{Spec}\, O_L \to \mathfrak{X}_{n-1}$ which induces the isomorphism from the tangent space of the closed point y of $\mathrm{Spec}\, O_L$ to M. Let us denote the maximal ideal of the local ring $O_{\mathfrak{X}_{n-1}, x}$ as m_x and the $n-1$ dimensional linear subspace of m_x/m_x^2 annihilated by M as N. Let us choose a regular system of parameter $\{t_1, \ldots, t_n\}$ of m_x so that the image of $\{t_1, \ldots, t_{n-1}\}$ becomes a basis of N.

Case 1: If the image of the maximal ideal of O_K to m_x/m_x^2 is not contained in N, the quotient ring $O_{\mathfrak{X}_{n-1}, x}/(t_1, \ldots, t_{n-1})$ works as O_L.

Case 2: If the image of the maximal ideal of O_K to m_x/m_x^2 is contained in N, we may assume that $t_1 = \varpi - t_n^2$. Here, ϖ is a generator of the maximal ideal of O_K. Then, the quotient ring $O_{\mathfrak{X}_{n-1}, x}/(t_1, \ldots, t_{n-1})$ works as O_L.

7. Appendix

In this section, we give a proof of Proposition 2 which we could not find in literature. Let us restate the proposition.

Let S be the spectrum of a discrete valuation ring O_K, η the generic point of S, s the closed point of S, $K = \kappa(\eta)$ the fractional field of O_K, $k = \kappa(s)$ the residue field of O_K, and p the characteristic of k.

Consider a proper hyperbolic curve $X \to \mathrm{Spec}\, K$ and take a geometric point $\bar{t}$ over $X \otimes K^{\mathrm{sep}}$.

Proposition 8 ([Oda1], [Oda2], [Tama] section 5). *The following are equivalent.*

(1) *X has good reduction.*
(2) *The outer action $I_K \to \mathrm{Out}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^{p'})$ defined by (14) is trivial.*
(3) *There exists a prime number $l \neq p$ such that the outer action $I_K \to \mathrm{Out}(\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^{l})$ defined by (14) is trivial.*
(4) *There exists a prime number $l \neq p$ such that the outer action of I_K on $\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^{l}/\Gamma_n \pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^{l}$ induced by (14) is trivial for all natural numbers n.*

We can show $1 \Rightarrow 2$ as in the proof of Proposition 3, and the assertion that $2 \Rightarrow 3 \Rightarrow 4$ is trivial. To show $4 \Rightarrow 1$, we may assume that the Jacobian of X has good reduction and that X has bad reduction as in [Oda2]. Then, we have

the outer action of the absolute Galois group:

$$\rho_l(\mathrm{mod}\ m) : I_K \to \mathrm{Out}\ \pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l / \Gamma_{m+1}\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l.$$

THEOREM 4. *Under the above assumption, the homomorphism*

$$\rho_l(\mathrm{mod}\ 2) : I_K \to \mathrm{Out}\ \pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l / \Gamma_3\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l$$

is trivial, and the homomorphism

$$\rho_l(\mathrm{mod}\ 3) : I_K \to \mathrm{Out}\ \pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l / \Gamma_4\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l$$

by factoring through $I_K \to I_K^l \to \mathrm{Out}\ \pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l / \Gamma_4\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l$, *defines the injective homomorphism,*

$$\rho_l'(\mathrm{mod}\ 3) : I_K^l \to \mathrm{Out}\ \pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l / \Gamma_4\pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})^l.$$

Let us denote the stable model of X by $\mathfrak{X}$ and suppose that the special fiber has n nodes $x_1, \ldots, x_n$. To show Theorem 4, we may assume that O_K is a complete discrete valuation ring and k is an algebraically closed field. Let us construct a two-dimensional family of stable curves. Consider the classifying morphism of $\mathfrak{X}$

$$\mathrm{cl}_{\mathfrak{X}} : \mathrm{Spec}(O_K) \to \mathfrak{M}_g.$$

Here, $\mathfrak{M}_g$ is the moduli stack of stable curves of genus g over $\mathbb{Z}_p$. We write the induced morphism $\mathrm{Spec}(k) \to \mathfrak{M}_g$ as κ. As a regular system of parameters of the strict henselian local ring $\mathrm{Spec}\ O^{\mathrm{sh}}_{\mathfrak{M}_g, \kappa}$ at κ, we can choose $3g-2$ elements $p, T_1, \ldots, T_{3g-3}$. We can assume that the local equation of the singularity $\tilde{x}_i$ of the universal family $\mathfrak{C}_g$ over x_i is given by

$$X_i Y_i = T_i \quad (1 \le i \le n).$$

If we write the local equation of x_i as

$$X_i' Y_i' = a_i \quad (0 \ne a_i \in \pi O_K),$$

where π is a uniformizer of O_K, the local homomorphism $O^{\mathrm{sh}}_{\mathfrak{C}_g, \tilde{x}_i} \to O^{\mathrm{sh}}_{\mathfrak{X}, x_i}$ sends X_i, Y_i and T_i to $u_i X_i'$, $u_i^{-1} Y_i'$ and a_i by [Moch3] §3.7, §3.8 and [Kato] Lemma 2.1, Lemma 2.2. Here, u_i is a unit in $O^{\mathrm{sh}}_{\mathfrak{X}, x_i}$.

Let us consider the ring

$$R = \begin{cases} O_K[[t]] & (\mathrm{char}\ K = 0) \\ W(k)[[t]] & (\mathrm{char}\ K = p) \end{cases}$$

and the natural morphism $R \to O_K$ which sends t to π. Here, $W(k)$ is the Witt ring for k.

We write the local homomorphism $O^{\mathrm{sh}}_{\tilde{\mathfrak{M}}_g,\tilde{\kappa}} \to O_K$ induced by the natural morphism $\tilde{\kappa} : \operatorname{Spec} k \to \operatorname{Spec} O_K \to \mathfrak{M}_g \times_{\mathbb{Z}_p} \operatorname{Spec} R =: \tilde{\mathfrak{M}}_g$ as $\tilde{\phi}$. The elements of $O^{\mathrm{sh}}_{\tilde{\mathfrak{M}}_g,\tilde{\kappa}}$

$$\begin{cases} \pi, t, T_1, \ldots, T_{3g-3} & (\operatorname{char} K = 0) \\ p, t, T_1, \ldots, T_{3g-3} & (\operatorname{char} K = p) \end{cases}$$

become a regular system of parameters.

Choose an element $\tilde{a}_i \in R$ so that its image in O_K is equal to a_i if $1 \leq i \leq n$ and the image of T_i in O_K if $n+1 \leq i$, and set $U_i = T_i - \tilde{a}_i$. The subset of $O^{\mathrm{sh}}_{\tilde{\mathfrak{M}}_g,\tilde{\kappa}}$

$$\begin{cases} \pi, t, U_1, \ldots, U_{3g-3} & (\operatorname{char} K = 0) \\ p, t, U_1, \ldots, U_{3g-3} & (\operatorname{char} K = p) \end{cases}$$

becomes a regular system of parameters again. Then, it holds that $U_i \in \operatorname{Ker}(\tilde{\phi})$. We write the quotient ring $O^{\mathrm{sh}}_{\tilde{\mathfrak{M}}_g,\tilde{\kappa}}/(U_1, \ldots, U_{3g-3})$ as A and the induced homomorphism $A \to O_K$ as ψ. If we denote the field of fraction of the strict henselization $A^{\mathrm{sh}}_{(t)}$ of $A_{(t)}$ by L, then we get the following commutative diagram.

$$\begin{array}{ccccc} \operatorname{Spec} L & \longrightarrow & \operatorname{Spec} A[1/t] & \longleftarrow & \operatorname{Spec} K \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} A^{\mathrm{sh}}_{(t)} & \longrightarrow & \operatorname{Spec} A & \longleftarrow & \operatorname{Spec} O_K \end{array}$$

Let us recall Abhyankar's lemma.

PROPOSITION 9. *Let $\overline{K}$, $\overline{L}$ be an separable closure of K, L. Then we have the natural outer isomorphisms*

$$\operatorname{Gal}(\overline{L}/L)^{p'} \cong \pi_1(\operatorname{Spec} A[1/t])^{p'} \cong \operatorname{Gal}(\overline{K}/K)^{p'}.$$

Let us start the proof of Theorem 4. Let us write the local monodromy homomorphism as

$$\rho : \operatorname{Gal}(\overline{K}/K) \to \operatorname{Out} \pi_1(X \otimes \overline{K}, \bar{t})^{l}.$$

Since $\operatorname{Gal}(\overline{K}/K)$ acts trivially on the abelianization $\pi_1(X \otimes \overline{K}, \bar{t})^{l,\mathrm{ab}}$, ρ induces the homomorphism $\operatorname{Gal}(\overline{K}/K)^{l} \to \operatorname{Out} \pi_1(X \otimes \overline{K}, \bar{t})^{l}$. By Proposition 9, we have the morphism

$$\pi_1(\operatorname{Spec} A[1/t]) \to \pi_1(\operatorname{Spec} A[1/t])^{l} \to \operatorname{Out} \pi_1(X \otimes \overline{K}, \bar{t})^{l},$$

which is compatible with the isomorphisms of Proposition 9 and the monodromy homomorphism

$$\rho_0 : \operatorname{Gal}(\overline{L}/L) \to \operatorname{Out} \pi_1(\mathfrak{Y} \otimes \overline{L}, \xi).$$

Here, we write the restriction of the curve $\mathfrak{C}_g$ to the scheme Spec A as $\mathfrak{Y}$ and a geometric point of the curve $\mathfrak{Y} \otimes \overline{L}$ as ξ. It hold that $\tilde{a}_i$ is not equal to 0 in L, so $\mathfrak{Y} \otimes L$ is smooth over Spec L. Since the residue characteristic of Spec $A^{\mathrm{sh}}_{(t)}$ is 0, this theorem follows from the transcendental method in [Oda2].

Acknowledgement

The author thanks his supervisor Atsushi Shiho for useful discussions and helpful advice.

References

[Bo] N. Bourbaki, Éléments de mathématique. Groupes et algèbres de Lie. Chapitre II: Algèbres de Lie libres. Chapitre III: Groupes de Lie, Actualités Scientifiques et Industrielles, No. 1349. Hermann, Paris, 1972.

[Del] P. Deligne, Le groupe fondamental de la droite projective moins trois points, Galois Groups over $\mathbb{Q}$, Math. Sci. Res. Inst. Pub., 1989, pp. 79–297.

[Dri] V. Drinfeld, On a conjecture of Deligne, Mosc. Math. J. 12 (2012), no. 3, 515–542, 668.

[EGA] A. Grothendieck, Eléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II, Inst. Hautes Études Sci. Publ. Math. No. 24 (1965).

[EHS] H. Esnault; P. H. Hai; X. Sun; On Nori's fundamental group scheme, Geometry and Dynamics of Groups and Spaces, Progress in Mathematics, vol. 265, Birkhäuser, 2007, pp. 377–398.

[HJ] M. Hadian-Jazi, Motivic fundamental groups and integral points, Ph.D. thesis, Universität Bonn, 2010.

[Ho] Y. Hoshi, The Grothendieck conjecture for hyperbolic polycurves of lower dimension, to appear in J. Math. Sci. Univ. Tokyo.

[Kato] F. Kato, Log smooth deformation and moduli of log smooth curves., Internat. J. Math. 11 (2000), no. 2, 215–232.

[Lab] J.-P. Labute, On the descending central series of groups with a single defining relation, J. Algebra 14 (1970), 16–23.

[Laz] C. Lazda, Relative fundamental groups and rational points, Rend. Semin. Mat. Univ. Padova 134 (2015), 1–45.

[Moch1] S. Mochizuki, The local pro-p anabelian geometry of curves, Invent. Math. 138 (1999), no. 2, 319–423.

[Moch2] S. Mochizuki, Absolute anabelian cuspidalizations of proper hyperbolic curves, J. Math. Kyoto Univ. 47 (2007), no. 3, 451–539.

[Moch3] S. Mochizuki, The geometry of the compactification of the Hurwitz scheme, Publ. of RIMS. 31 (1995), 355–411.

[Mor] L. Moret-Bailly, Un theoreme de purete pour les familles de courbes lisses, C. R. Acad. Sci. Paris, 300 no. 14, (1985), 489–492.

[Oda1] T. Oda, A note on ramification of the Galois representation on the fundamental group of an algebraic curve, J. Number Theory, 34 (1990), 225–228.

[Oda2] T. Oda, A note on ramification of the Galois representation on the fundamental group of an algebraic curve II, J. Number Theory, 53 (1995), 342–355.

[SGA1] A. Grothendieck and Mme. Raynaud. M, Séminaire de Géometrie Algébrique du Bois Marie 1960/61, Revétements Etales et Groupe Fondamental (SGA 1), Lecture Notes in Mathematics, 224, Springer-Verlag, Berlin/Heidelberg/New York, 1971.

[ST] J.-P. Serre; J. Tate, Good reduction of abelian varieties, Ann. of Math. (2) 88 (1968), 492–517.

[Tama] A. Tamagawa, The Grothendieck conjecture for affine curves, Compositio Math. 109 (1997), no. 2, 135–194.

[Wil] J. Wildeshaus, Realizations of polylogarithms, Lecture Notes in Mathematics, 1650. Springer-Verlag, Berlin, 1997.

Ippei Nagamachi
Depertment of Mathematics
Graduate School of Science
The University of Tokyo
Meguroku-Komaba 153-8914 *Japan*
E-mail: nagachi@ms.u-tokyo.ac.jp