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Polynomial argument for q-binomial cubic sums*

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Abstract. By means of the polynomial argument, a class of cubic sums of q-binomial coefficients are evaluated in closed forms.

By means of the finite difference method, Chu [3] proved several closed formulae for the following alternating binomial sums

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{k+y}{m+\varepsilon}\binom{k-y+\lambda}{m+\varepsilon}$$

where $m \in \mathbb{N}_0$, $\lambda, \varepsilon \in \mathbb{Z}$ and y is an indeterminate. This has partially been motivated by Gould–Quaintance [4], who obtained a closed formula for the case $m = 2n$ and $\varepsilon = 1 + \lambda$, extending an earlier result found by Vosmansky [5].

Define the q-shifted factorial by $(x; q)_0 \equiv 1$ and

$$(x; q)_n = \prod_{k=0}^{n-1}(1 - xq^k) \qquad \text{for } n \in \mathbb{N}.$$

We have Gauss' q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^{1+n-k}; q)_k}{(q; q)_k} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

The objective of the present paper is to investigate the following q-binomial sums

$$\Omega_n(\varepsilon, \delta|\lambda, y) := \sum_{k=0}^{2n+\delta}(-1)^k \begin{bmatrix} 2n+\delta \\ k \end{bmatrix} \begin{bmatrix} k+y \\ 2n+\varepsilon \end{bmatrix} \begin{bmatrix} k-y+\lambda \\ 2n+\varepsilon \end{bmatrix} q^{\binom{\varepsilon+3n-k}{2}}$$

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where for brevity, the parity of the summation limit is indicated by whether $\delta = 0$ or $\delta = 1$. The main tool is the following polynomial argument: "Two polynomials of degree $\leq m$ are identical if they agree at $m+1$ distinct points".

As a crucial fact, we first show that $\Omega_n(\varepsilon, \delta|\lambda, y)$ is a Laurent polynomial in q^y consisting of the terms $q^{y\ell}$ with $|\ell| \leq \varepsilon + n$ (instead of $|\ell| \leq 2n + \delta$). According to Euler's q-binomial theorem (cf. Bailey [1, §8.1])

$$(x; q)_m = \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} q^{\binom{k}{2}} x^k$$

there exist connection coefficients $\Theta(i, j)$ (independent of k) such that

$$\begin{aligned} \Omega_n(\varepsilon, \delta|\lambda, y) &= \sum_{k=0}^{2n+\delta} (-1)^k \begin{bmatrix} 2n+\delta \\ k \end{bmatrix} q^{\binom{k}{2}} \sum_{i,j=0}^{2n+\varepsilon} \Theta(i, j) q^{y(i-j)+k(1+i+j-3n-\varepsilon)} \\ &= \sum_{|\ell| \leq 2n+\varepsilon} q^{y\ell} \sum_{j=0}^{2n+\varepsilon} \Theta(\ell + j, j) \times (q^{1+\ell+2j-3n-\varepsilon}; q)_{2n+\delta} \end{aligned}$$

where the last line is justified by the substitution $i - j = \ell$ on summation indices. Observe that $\Omega_n(\varepsilon, \delta|\lambda, y)$ is a Laurent polynomial of q^y consisting of the terms $q^{y\ell}$ such that the factorial $(q^{1+\ell+2j-3n-\varepsilon}; q)_{2n+\delta} \neq 0$ for some j with $0 \leq j \leq 2n + \varepsilon$. This can happen only when one of the following two inequalities holds: $\ell + 2j - 3n - \varepsilon \geq 0$ and $\ell + 2j - 3n - \varepsilon < -2n - \delta$, which can be reformulated respectively as $\ell \geq 3n + \varepsilon - 2j \geq -n - \varepsilon$ and $\ell < n + \varepsilon - \delta - 2j \leq n + \varepsilon$. This confirms that $\Omega_n(\varepsilon, \delta|\lambda, y)$ is a Laurent polynomial in q^y consisting of the terms $q^{y\ell}$ with ℓ being restricted between $-\varepsilon - n$ and $\varepsilon + n$.

To evaluate the q-binomial sum $\Omega(y) := \Omega_n(\varepsilon, \delta|\lambda, y)$ for specific ε, δ and λ, the following procedure will be carried out:

- Determine the "degree" of the Laurent polynomial $\Omega(y)$.
- Figure out zeros of $\Omega(y)$ that are identified explicitly by $\omega(y)$.
- Find out the multiplicative constant β such that $\Omega(y) = \beta\omega(y)$.

We shall examine eleven formulae in the rest of the paper. They are remarkable examples of the so–called "almost poised q-series" [2], where further identities can be found. In order to ensure the accuracy, all the formulae displayed in the paper have been verified by appropriately devised Mathematica commands.

§1. Formula for $\Omega_n(\lambda + 1, 0|\lambda, y)$. First, we prove the following q-analogue for the cubic sum of binomial coefficients evaluated by Gould–Quaintance [4].

THEOREM 1 ($n \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$).

$$\sum_{k=0}^{2n}(-1)^k \begin{bmatrix}2n\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+\lambda+1\end{bmatrix}\begin{bmatrix}k-y+\lambda\\2n+\lambda+1\end{bmatrix} q^{\binom{\lambda+3n-k+1}{2}}$$

$$= \frac{\begin{bmatrix}2n\\n\end{bmatrix}}{\begin{bmatrix}2n+\lambda+1\\n\end{bmatrix}}\begin{bmatrix}y\\n+\lambda+1\end{bmatrix}\begin{bmatrix}\lambda-y\\n+\lambda+1\end{bmatrix} q^{((\lambda+n)/2)(1+\lambda+3n)}.$$

PROOF. Define the Laurent polynomial by the q-binomial sum

$$P(y) = \sum_{k=0}^{2n}(-1)^k \begin{bmatrix}2n\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+\lambda+1\end{bmatrix}\begin{bmatrix}k-y+\lambda\\2n+\lambda+1\end{bmatrix} q^{\binom{\lambda+3n-k+1}{2}}$$

which consists of the terms $q^{y\ell}$ with $|\ell| \le n+\lambda+1$.

When $\lambda < 0$, we claim that $P(i) = 0$ for $i = 0, 1, \dots, n+\lambda$. Observing that the non zero summands in $P(y)$ contain both $\begin{bmatrix}k+i\\2n+\lambda+1\end{bmatrix} \neq 0$ and $\begin{bmatrix}k-i+\lambda\\2n+\lambda+1\end{bmatrix} \neq 0$, we have $k+i \ge 2n+\lambda+1$ and $k-i+\lambda < 0$ simultaneously. Rewriting the second inequality $i-k > \lambda$ and then adding it to the first one, we get $2i > 2n+2\lambda+1$, which is equivalent to $i > n+\lambda$. Hence $P(i) = 0$ for $0 \le i \le n+\lambda$. According to the symmetry $P(y) = P(\lambda - y)$, we get $P(y) = 0$ for $y \in \{i, \lambda - i : 0 \le i \le n+\lambda\}$.

Now that the q-binomial product $\begin{bmatrix}y\\n+\lambda+1\end{bmatrix}\begin{bmatrix}\lambda-y\\n+\lambda+1\end{bmatrix}$ has the same zeros as $P(y)$, there is a constant β such that

$$P(y) = \beta\begin{bmatrix}y\\n+\lambda+1\end{bmatrix}\begin{bmatrix}\lambda-y\\n+\lambda+1\end{bmatrix}$$

which can be determined by letting $y = n+\lambda+1$ as

$$\beta = \frac{P(n+\lambda+1)}{\begin{bmatrix}-n-1\\n+\lambda+1\end{bmatrix}} = \frac{\begin{bmatrix}2n\\n\end{bmatrix}}{\begin{bmatrix}2n+\lambda+1\\n\end{bmatrix}} q^{((\lambda+n)/2)(1+\lambda+3n)}$$

because there is only one surviving term corresponding to $k = n$ in $P(n+\lambda+1)$.

When $\lambda \ge 0$, consider the quotient $Q(y) = \frac{P(y)}{\begin{bmatrix}y\\\lambda+1\end{bmatrix}\begin{bmatrix}\lambda-y\\\lambda+1\end{bmatrix}}$. Following the same procedure above for $P(y)$, we can show that $Q(y)$ is a Laurent polynomial of "degree" $|\ell| \le n$ with all the zeros $\{i, \lambda - i : 1+\lambda \le i \le n+\lambda\}$. Then by determining the constant factor at the same point $y = n+\lambda+1$, we confirm the identity displayed in Theorem 1 also when $\lambda \ge 0$. □

§ 2. Formula for $\Omega_n(\lambda, 0|\lambda, y)$.

THEOREM 2 ($n \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$).

$$\sum_{k=0}^{2n}(-1)^k \begin{bmatrix}2n\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+\lambda\end{bmatrix}\begin{bmatrix}k-y+\lambda\\2n+\lambda\end{bmatrix} q^{\binom{\lambda+3n-k}{2}}$$
$$= \frac{\begin{bmatrix}2n\\n\end{bmatrix}}{\begin{bmatrix}2n+\lambda\\n\end{bmatrix}} \begin{bmatrix}y-1\\n+\lambda\end{bmatrix}\begin{bmatrix}\lambda-y-1\\n+\lambda\end{bmatrix} q^{((\lambda+n)/2)(1+\lambda+3n)}.$$

PROOF. Define the Laurent polynomial in q^y by the q-binomial sum

$$P(y) = \sum_{k=0}^{2n}(-1)^k \begin{bmatrix}2n\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+\lambda\end{bmatrix}\begin{bmatrix}k-y+\lambda\\2n+\lambda\end{bmatrix} q^{\binom{\lambda+3n-k}{2}}$$

whose "degree" is limited by $|\ell| \le n+\lambda$.

When $\lambda \le 0$, we have $P(i) = 0$ for $i = 1, 2, \dots, n+\lambda$ because the non zero summands in $P(y)$ contain both $\left[\begin{smallmatrix}k+i\\2n+\lambda\end{smallmatrix}\right] \ne 0$ and $\left[\begin{smallmatrix}k-i+\lambda\\2n+\lambda\end{smallmatrix}\right] \ne 0$ that are equivalent to $i+k \ge 2n+\lambda$ and $i-k > \lambda$ respectively. Combining with the symmetric property $P(y) = P(\lambda - y)$, we get $P(y) = 0$ for $y \in \{i, \lambda - i : 1 \le i \le n+\lambda\}$.

Since the q-binomial product $\left[\begin{smallmatrix}y-1\\n+\lambda\end{smallmatrix}\right]\left[\begin{smallmatrix}\lambda-y-1\\n+\lambda\end{smallmatrix}\right]$ has the same zeros as $P(y)$, there is a constant β such that

$$P(y) = \beta \begin{bmatrix}y-1\\n+\lambda\end{bmatrix}\begin{bmatrix}\lambda-y-1\\n+\lambda\end{bmatrix}$$

which can be determined at $y = 0$ as

$$\beta = \frac{P(0)}{\left[\begin{smallmatrix}-1\\n+\lambda\end{smallmatrix}\right]\left[\begin{smallmatrix}\lambda-1\\n+\lambda\end{smallmatrix}\right]} = \frac{\left[\begin{smallmatrix}2n\\2n+\lambda\end{smallmatrix}\right] q^{\binom{\lambda+n}{2}}}{\left[\begin{smallmatrix}-1\\n+\lambda\end{smallmatrix}\right]\left[\begin{smallmatrix}\lambda-1\\n+\lambda\end{smallmatrix}\right]} = \frac{\left[\begin{smallmatrix}2n\\n\end{smallmatrix}\right]}{\left[\begin{smallmatrix}2n+\lambda\\n\end{smallmatrix}\right]} q^{((\lambda+n)/2)(1+\lambda+3n)}$$

where $P(0)$ contains essentially the only term corresponding to $k = 2n$.

When $\lambda > 0$, observing that $Q(y) = \frac{P(y)}{\left[\begin{smallmatrix}y-1\\\lambda-1\end{smallmatrix}\right]\left[\begin{smallmatrix}\lambda-y-1\\\lambda-1\end{smallmatrix}\right]}$ is, in fact, a Laurent polynomial of "degree" $|\ell| \le n+1$ with all the zeros $\{i, \lambda - i : \lambda \le i \le n+\lambda\}$ and then determining the constant factor at the same point $y = 0$, we prove the identity in Theorem 2 also when $\lambda > 0$. □

§ 3. Formula for $\Omega_n(\lambda+2, 0|\lambda, y)$.

THEOREM 3 ($n \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$).

$$\sum_{k=0}^{2n}(-1)^k \begin{bmatrix}2n\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+\lambda+2\end{bmatrix}\begin{bmatrix}k-y+\lambda\\2n+\lambda+2\end{bmatrix} q^{\binom{\lambda+3n-k+2}{2}}$$
$$= \frac{q\begin{bmatrix}2n\\n\end{bmatrix}}{\begin{bmatrix}2n+\lambda+2\\n\end{bmatrix}} \begin{bmatrix}y\\n+\lambda+2\end{bmatrix}\begin{bmatrix}\lambda-y\\n+\lambda+2\end{bmatrix} q^{((\lambda+n)/2)(3+\lambda+3n)}.$$

Sketch of proof. Define the Laurent polynomial by the q-binomial sum

$$P(y)=\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+\lambda+2\end{bmatrix}\begin{bmatrix}k-y+\lambda\\2n+\lambda+2\end{bmatrix}q^{\binom{\lambda+3n-k+2}{2}}$$

which consists of the terms $q^{y\ell}$ with $|\ell|\le n+\lambda+2$.

When $\lambda<0$, the theorem can be shown by verifying the following statements:

- All the zeros of $P(y)$ are given by $y\in\{i,\lambda-i:0\le i\le n+\lambda+1\}$, that are the same as the zeros of the q-binomial product $\left[\begin{smallmatrix}y\\n+\lambda+2\end{smallmatrix}\right]\left[\begin{smallmatrix}\lambda-y\\n+\lambda+2\end{smallmatrix}\right]$.
- The constant factor is determined at $y=n+\lambda+2$ where $P(y)$ contains only two surviving terms corresponding to $k=n$ and $k=n+1$.

When $\lambda\ge 0$, Theorem 3 can be confirmed by examining the following quotient $Q(y)=\frac{P(y)}{\left[\begin{smallmatrix}y\\\lambda+2\end{smallmatrix}\right]\left[\begin{smallmatrix}\lambda-y\\\lambda+2\end{smallmatrix}\right]}$, which is a Laurent polynomial of "degree" $|\ell|\le n$ with all the zeros $\{i,\lambda-i:\lambda+2\le i\le n+\lambda+1\}$, and then determining the constant factor at the point $y=n+\lambda+2$ analogously. □

§4. Formula for $\Omega_n(\lambda+2,1|\lambda,y)$.

Theorem 4 ($n\in\mathbb{N}_0$ and $\lambda\in\mathbb{Z}$).

$$\sum_{k=0}^{2n+1}(-1)^k\begin{bmatrix}2n+1\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+\lambda+2\end{bmatrix}\begin{bmatrix}k-y+\lambda\\2n+\lambda+2\end{bmatrix}q^{\binom{\lambda+3n-k+2}{2}}$$

$$=\frac{q^{\left[\begin{smallmatrix}2n\\n\end{smallmatrix}\right]}(1-q^{2n+1})}{\left[\begin{smallmatrix}2n+\lambda+2\\n\end{smallmatrix}\right]}\begin{bmatrix}y\\n+\lambda+2\end{bmatrix}\begin{bmatrix}\lambda-y\\n+\lambda+2\end{bmatrix}q^{((\lambda+n)/2)(3+\lambda+3n)}.$$

Sketch of proof. Define the Laurent polynomial in q^y by the q-binomial sum

$$P(y)=\sum_{k=0}^{2n+1}(-1)^k\begin{bmatrix}2n+1\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+\lambda+2\end{bmatrix}\begin{bmatrix}k-y+\lambda\\2n+\lambda+2\end{bmatrix}q^{\binom{\lambda+3n-k+2}{2}}.$$

whose "degree" is restricted to $|\ell|\le n+\lambda+2$.

When $\lambda<0$, the theorem can be proved by verifying the following assertions:

- All the zeros of $P(y)$ are given by $\{i,\lambda-i:0\le i\le n+\lambda+1\}$, that are the same as the zeros of the q-binomial product $\left[\begin{smallmatrix}y\\n+\lambda+2\end{smallmatrix}\right]\left[\begin{smallmatrix}\lambda-y\\n+\lambda+2\end{smallmatrix}\right]$.
- The constant factor is determined at $y=n+\lambda+2$ where $P(y)$ contains only two surviving terms corresponding to $k=n$ and $k=n+1$.

When $\lambda\ge 0$, Theorem 4 can be confirmed by considering the following quotient $Q(y)=\frac{P(y)}{\left[\begin{smallmatrix}y\\\lambda+1\end{smallmatrix}\right]\left[\begin{smallmatrix}\lambda-y\\\lambda+1\end{smallmatrix}\right]}$, which is a Laurent polynomial of "degree" $|\ell|\le n+1$

with all the zeros $\{i, \lambda - i : \lambda + 1 \le i \le n + \lambda + 1\}$, and then determining the constant factor at the point $y = n + \lambda + 2$ similarly. □

For the remaining seven formulae, the identified zeros are not enough to determine Laurent polynomials for the absence of a pair of key zeros. According to the symmetric property $\Omega(y) = \Omega(\lambda - y)$, there is an extra factor $a + b(q^y + q^{\lambda - y})$ to be figured out. By choosing two particular values of y so that $\Omega(y)$ can be easily evaluated, we shall find out a and b by resolving a linear system of two equations concerning both unknowns a and b.

§5. Formula for $\Omega_n(\lambda, 1|\lambda, y)$.

THEOREM 5 ($n \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$).

$$\sum_{k=0}^{2n+1} (-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix} \begin{bmatrix} k+y \\ 2n+\lambda \end{bmatrix} \begin{bmatrix} k-y+\lambda \\ 2n+\lambda \end{bmatrix} q^{\binom{\lambda+3n-k}{2}}$$
$$= \frac{\begin{bmatrix} 2n+1 \\ n \end{bmatrix}}{\begin{bmatrix} 2n+\lambda \\ n+1 \end{bmatrix}} \begin{bmatrix} y-2 \\ n+\lambda-1 \end{bmatrix} \begin{bmatrix} \lambda-y-2 \\ n+\lambda-1 \end{bmatrix} q^{((\lambda+n)/2)(1+\lambda+3n)}$$
$$\times \frac{q^{\lambda} + q^{\lambda-1} + q^{\lambda+1+n} - q^{-n} - q^{y} - q^{\lambda-y}}{q(1-q^{n+\lambda})}.$$

PROOF. Define the Laurent polynomial by the q-binomial sum

$$P(y) = \sum_{k=0}^{2n+1} (-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix} \begin{bmatrix} k+y \\ 2n+\lambda \end{bmatrix} \begin{bmatrix} k-y+\lambda \\ 2n+\lambda \end{bmatrix} q^{\binom{\lambda+3n-k}{2}}$$

which consists of the terms $q^{y\ell}$ with $|\ell| \le n + \lambda$.

When $\lambda \le 3$, it is not hard to check the following statements:

- For the q-binomial product $\begin{bmatrix} y-2 \\ n+\lambda-1 \end{bmatrix} \begin{bmatrix} \lambda-y-2 \\ n+\lambda-1 \end{bmatrix}$, its zeros $\{i, \lambda - i : 2 \le i \le n + \lambda\}$ are also $2n + 2\lambda - 2$ zeros of $P(y)$.
- There is an extra factor $a + b(q^y + q^{\lambda-y})$ such that

$$P(y) = \{a + b(q^y + q^{\lambda-y})\} \begin{bmatrix} y-2 \\ n+\lambda-1 \end{bmatrix} \begin{bmatrix} \lambda-y-2 \\ n+\lambda-1 \end{bmatrix}$$

  which can be determined by resolving the following linear system:

$$\begin{cases} P(0) = \begin{bmatrix} -2 \\ n+\lambda-1 \end{bmatrix} \begin{bmatrix} \lambda-2 \\ n+\lambda-1 \end{bmatrix} \{a + b(1+q^{\lambda})\}, \\ P(1) = \begin{bmatrix} -1 \\ n+\lambda-1 \end{bmatrix} \begin{bmatrix} \lambda-3 \\ n+\lambda-1 \end{bmatrix} \{a + b(q+q^{\lambda-1})\}. \end{cases}$$

Noticing further that there are two surviving terms with $k = 2n$ and $k = 2n+1$ in $P(0)$ and one surviving term corresponding to $k = 2n+1$ in $P(1)$, we get

$$P(0) = -q^{\binom{\lambda+n-1}{2}} \begin{bmatrix} 2n+1 \\ 2n+\lambda \end{bmatrix} \frac{(1+q^n)(1-q^{\lambda+n+1})}{1-q},$$

$$P(1) = -q^{\binom{\lambda+n-1}{2}} \begin{bmatrix} 2n+2 \\ 2n+\lambda \end{bmatrix}.$$

Substituting them into the last system of equations and then resolving it, we find, after simplifications, the following solution:

$$a = -\frac{q^{((\lambda+n)/2)(1+\lambda+3n)}}{q(1-q^{n+\lambda})} \frac{\begin{bmatrix} 2n+1 \\ n \end{bmatrix}}{\begin{bmatrix} 2n+\lambda \\ n+1 \end{bmatrix}} (q^{-n} - q^{\lambda} - q^{\lambda-1} - q^{\lambda+n+1}),$$

$$b = -\frac{q^{((\lambda+n)/2)(1+\lambda+3n)}}{q(1-q^{n+\lambda})} \frac{\begin{bmatrix} 2n+1 \\ n \end{bmatrix}}{\begin{bmatrix} 2n+\lambda \\ n+1 \end{bmatrix}}.$$

Therefore we have determined the extra factor

$$a + b(q^y + q^{\lambda-y}) = q^{((\lambda+n)/2)(1+\lambda+3n)} \frac{\begin{bmatrix} 2n+1 \\ n \end{bmatrix}}{\begin{bmatrix} 2n+\lambda \\ n+1 \end{bmatrix}} \frac{q^{\lambda} + q^{\lambda-1} + q^{\lambda+1+n} - q^{-n} - q^y - q^{\lambda-y}}{q(1-q^{n+\lambda})}$$

and consequently proved the theorem for $\lambda \le 3$.

When $\lambda > 3$, Theorem 5 can be confirmed analogously by examining the quotient $Q(y) = \frac{P(y)}{\begin{bmatrix} y-2 \\ \lambda-3 \end{bmatrix}\begin{bmatrix} \lambda-y-2 \\ \lambda-3 \end{bmatrix}}$, which is a Laurent polynomial of "degree" $|\ell| \le n+3$ with identified zeros $\{i, \lambda - i : \lambda - 1 \le i \le n + \lambda\}$, and then determining the extra factor $a + b(q^y + q^{\lambda-y})$ by letting $y = 0$ and $y = 1$. □

§ 6. Formula for $\Omega_n(\lambda+1, 1|\lambda, y)$.

Theorem 6 ($n \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$).

$$\sum_{k=0}^{2n+1} (-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix} \begin{bmatrix} k+y \\ 2n+\lambda+1 \end{bmatrix} \begin{bmatrix} k-y+\lambda \\ 2n+\lambda+1 \end{bmatrix} q^{\binom{\lambda+3n-k+1}{2}}$$

$$= \frac{\begin{bmatrix} 2n+1 \\ n \end{bmatrix}}{\begin{bmatrix} 2n+\lambda+1 \\ n+1 \end{bmatrix}} \begin{bmatrix} y-1 \\ n+\lambda \end{bmatrix} \begin{bmatrix} \lambda-y-1 \\ n+\lambda \end{bmatrix} \frac{q^{\lambda} + q^{1+n+\lambda} - q^y - q^{\lambda-y}}{1-q^{1+n+\lambda}} q^{((\lambda+n)/2)(1+\lambda+3n)}.$$

Proof. Define the Laurent polynomial in q^y by the q-binomial sum

$$P(y) = \sum_{k=0}^{2n+1} (-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix} \begin{bmatrix} k+y \\ 2n+\lambda+1 \end{bmatrix} \begin{bmatrix} k-y+\lambda \\ 2n+\lambda+1 \end{bmatrix} q^{\binom{\lambda+3n-k+1}{2}}$$

whose "degree" is limited to $|\ell| \le n+\lambda+1$.

When $\lambda \le 1$, we can verify the following assertions:

- For the q-binomial product $\begin{bmatrix} y-1 \\ n+\lambda \end{bmatrix}\begin{bmatrix} \lambda-y-1 \\ n+\lambda \end{bmatrix}$, its zeros $\{i, \lambda - i : 1 \le i \le n+\lambda\}$ are also $2n+2\lambda$ zeros of $P(y)$.
- There is an extra factor $a + b(q^y + q^{\lambda-y})$ such that

$$P(y) = \{a + b(q^y + q^{\lambda-y})\}\begin{bmatrix} y-1 \\ n+\lambda \end{bmatrix}\begin{bmatrix} \lambda-y-1 \\ n+\lambda \end{bmatrix}$$

which can be determined by resolving the following linear system:

$$\begin{cases} P(0) = \{a + b(1+q^\lambda)\}\begin{bmatrix} -1 \\ n+\lambda \end{bmatrix}\begin{bmatrix} \lambda-1 \\ n+\lambda \end{bmatrix}, \\ P(n+\lambda+1) = \{a + b(q^{n+\lambda+1} + q^{-n-1})\}\begin{bmatrix} -n-2 \\ n+\lambda \end{bmatrix}. \end{cases}$$

Evaluating both $P(0)$ and $P(n+\lambda+1)$ by

$$P(0) = -\begin{bmatrix} 2n+1 \\ 2n+\lambda+1 \end{bmatrix} q^{\binom{\lambda+n}{2}},$$

$$P(n+\lambda+1) = (-1)^{n+\lambda+1}\begin{bmatrix} 2n+1 \\ n \end{bmatrix} q^{-(2n+\lambda+1)};$$

and then resolving the last linear system, we get the following solution

$$a = \frac{\begin{bmatrix} 2n+1 \\ n \end{bmatrix}}{\begin{bmatrix} 2n+\lambda+1 \\ n+1 \end{bmatrix}} \frac{1+q^{n+1}}{1-q^{n+\lambda+1}} q^{((\lambda+n)/2)(1+\lambda+3n)+\lambda},$$

$$b = \frac{\begin{bmatrix} 2n+1 \\ n \end{bmatrix}}{\begin{bmatrix} 2n+\lambda+1 \\ n+1 \end{bmatrix}} \frac{q^{((\lambda+n)/2)(1+\lambda+3n)}}{q^{n+\lambda+1}-1}.$$

This leads explicitly to the following extra factor

$$a + b(q^y + q^{\lambda-y}) = q^{((\lambda+n)/2)(1+\lambda+3n)} \frac{\begin{bmatrix} 2n+1 \\ n \end{bmatrix}}{\begin{bmatrix} 2n+\lambda+1 \\ n+1 \end{bmatrix}} \frac{q^\lambda + q^{\lambda+1+n} - q^y - q^{\lambda-y}}{1-q^{n+\lambda+1}}$$

and proved accordingly the theorem for $\lambda \le 1$.

When $\lambda > 1$, Theorem 6 can be confirmed similarly by considering the quotient $Q(y) = \frac{P(y)}{\begin{bmatrix} y-1 \\ \lambda-1 \end{bmatrix}\begin{bmatrix} \lambda-y-1 \\ \lambda-1 \end{bmatrix}}$, which is a Laurent polynomial of "degree" $|\ell| \le n+2$ with identified zeros $\{i, \lambda - i : \lambda \le i \le n+\lambda\}$, and then determining the extra factor $a + b(q^y + q^{\lambda-y})$ by letting $y = 0$ and $y = n+\lambda+1$. $\square$

§7. Formula for $\Omega_n(\lambda+3, 1|\lambda, y)$.

THEOREM 7 ($n \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$).

$$\sum_{k=0}^{2n+1}(-1)^k\begin{bmatrix}2n+1\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+\lambda+3\end{bmatrix}\begin{bmatrix}k-y+\lambda\\2n+\lambda+3\end{bmatrix}q^{\binom{\lambda+3n-k+3}{2}}$$
$$=\frac{\begin{bmatrix}2n+1\\n\end{bmatrix}}{\begin{bmatrix}2n+\lambda+3\\n+1\end{bmatrix}}\begin{bmatrix}y\\n+\lambda+2\end{bmatrix}\begin{bmatrix}\lambda-y\\n+\lambda+2\end{bmatrix}q^{((\lambda+n)/2)(1+\lambda+3n)}$$
$$\times\frac{q^{\lambda+n}+q^{\lambda-1}-q^{\lambda+n+y+1}-q^{2\lambda+n-y+1}}{1-q^{3+n+\lambda}}.$$

Sketch of proof. Define the Laurent polynomial by the q-binomial sum

$$P(y)=\sum_{k=0}^{2n+1}(-1)^k\begin{bmatrix}2n+1\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+\lambda+3\end{bmatrix}\begin{bmatrix}k-y+\lambda\\2n+\lambda+3\end{bmatrix}q^{\binom{\lambda+3n-k+3}{2}}$$

which consists of the terms $q^{y\ell}$ with $|\ell|\leq n+\lambda+3$.

When $\lambda<0$, the theorem can be proved by checking the following statements:

- For the q-binomial product $\begin{bmatrix}y\\n+\lambda+2\end{bmatrix}\begin{bmatrix}\lambda-y\\n+\lambda+2\end{bmatrix}$, its zeros $\{i,\lambda-i:0\leq i\leq n+\lambda+1\}$ are also $2n+2\lambda+4$ zeros of $P(y)$.
- The extra factor $a+b(q^y+q^{\lambda-y})$ such that

$$P(y)=\{a+b(q^y+q^{\lambda-y})\}\begin{bmatrix}y\\n+\lambda+2\end{bmatrix}\begin{bmatrix}\lambda-y\\n+\lambda+2\end{bmatrix}$$

  is determined, by letting $y=n+\lambda+2$ and $y=n+\lambda+3$, as follows:

$$a+b(q^y+q^{\lambda-y})=\frac{q^{((\lambda+n)/2)(1+\lambda+3n)}}{1-q^{3+n+\lambda}}\frac{\begin{bmatrix}2n+1\\n\end{bmatrix}}{\begin{bmatrix}2n+\lambda+3\\n+1\end{bmatrix}}$$
$$\times\{q^{\lambda+n}+q^{\lambda-1}-q^{\lambda+n+y+1}-q^{2\lambda+n-y+1}\}.$$

When $\lambda\geq 0$, Theorem 7 can be confirmed analogously by examining the quotient $Q(y)=\frac{P(y)}{\begin{bmatrix}y\\\lambda+2\end{bmatrix}\begin{bmatrix}\lambda-y\\\lambda+2\end{bmatrix}}$, which is a Laurent polynomial of "degree" $|\ell|\leq n+1$ with identified zeros $\{i,\lambda-i:\lambda+2\leq i\leq n+\lambda+1\}$, and then determining the extra factor $a+b(q^y+q^{\lambda-y})$ by letting $y=n+\lambda+2$ and $y=n+\lambda+3$. □

§ 8. Formula for $\Omega_n(\lambda+4,1|\lambda,y)$.

Theorem 8 ($n\in\mathbb{N}_0$ and $\lambda\in\mathbb{Z}$).

$$\sum_{k=0}^{2n+1}(-1)^k\begin{bmatrix}2n+1\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+\lambda+4\end{bmatrix}\begin{bmatrix}k-y+\lambda\\2n+\lambda+4\end{bmatrix}q^{\binom{\lambda+3n-k+4}{2}}$$
$$=\frac{\begin{bmatrix}2n+1\\n\end{bmatrix}}{\begin{bmatrix}2n+\lambda+4\\n+1\end{bmatrix}}\begin{bmatrix}y\\n+\lambda+3\end{bmatrix}\begin{bmatrix}\lambda-y\\n+\lambda+3\end{bmatrix}q^{((\lambda+n)/2)(5+\lambda+3n)}$$
$$\times\frac{q+q^2+q^{-n}-q^{3+y}-q^{3+\lambda-y}-q^{5+\lambda+n}}{1-q^{4+n+\lambda}}.$$

SKETCH OF PROOF. Define the Laurent polynomial in q^y by the q-binomial sum

$$P(y)=\sum_{k=0}^{2n+1}(-1)^k\begin{bmatrix}2n+1\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+\lambda+4\end{bmatrix}\begin{bmatrix}k-y+\lambda\\2n+\lambda+4\end{bmatrix}q^{\binom{\lambda+3n-k+4}{2}}$$

whose "degree" is restricted to $|\ell|\le n+\lambda+4$.

When $\lambda<0$, the theorem is proved by verifying the following assertions:

- For the q-binomial product $\begin{bmatrix}y\\n+\lambda+3\end{bmatrix}\begin{bmatrix}\lambda-y\\n+\lambda+3\end{bmatrix}$, its zeros $\{i,\lambda-i:0\le i\le n+\lambda+2\}$ are also $2n+2\lambda+6$ zeros of $P(y)$.
- The extra factor $a+b(q^y+q^{\lambda-y})$ such that

$$P(y)=\{a+b(q^y+q^{\lambda-y})\}\begin{bmatrix}y\\n+\lambda+3\end{bmatrix}\begin{bmatrix}\lambda-y\\n+\lambda+3\end{bmatrix}$$

is determined, by letting $y=n+\lambda+3$ and $y=n+\lambda+4$, as follows:

$$a+b(q^y+q^{\lambda-y})=\frac{\begin{bmatrix}2n+1\\n\end{bmatrix}}{\begin{bmatrix}2n+\lambda+4\\n+1\end{bmatrix}}\frac{q^{((\lambda+n)/2)(5+\lambda+3n)}}{1-q^{4+n+\lambda}}$$
$$\times\{q+q^2+q^{-n}-q^{3+y}-q^{3+\lambda-y}-q^{5+n+\lambda}\}.$$

When $\lambda\ge 0$, Theorem 8 can be confirmed similarly by considering the quotient $Q(y)=\frac{P(y)}{\begin{bmatrix}y\\\lambda+3\end{bmatrix}\begin{bmatrix}\lambda-y\\\lambda+3\end{bmatrix}}$, which is a Laurent polynomial of "degree" $|\ell|\le n+1$ with identified zeros $\{i,\lambda-i:\lambda+3\le i\le n+\lambda+2\}$, and then determining the extra factor $a+b(q^y+q^{\lambda-y})$ by letting $y=n+\lambda+3$ and $y=n+\lambda+4$. $\square$

Finally, we are going to present three q-binomial formulae without λ-parameter.

§ 9. Formula for $\Omega_n(1,0|3,y)$.

THEOREM 9 $(n\in\mathbb{N}_0)$.

$$\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+1\end{bmatrix}\begin{bmatrix}k-y+3\\2n+1\end{bmatrix}q^{\binom{3n-k+1}{2}}$$
$$=(1+q^{1+n})\begin{bmatrix}y-3\\n\end{bmatrix}\begin{bmatrix}-y\\n\end{bmatrix}\left\{1+q^{1+3n}\frac{(1-q^{y-1})(1-q^{2-y})}{(1-q^{1+2n})(1-q^{2+2n})}\right\}q^{(n/2)(5+3n)}.$$

Proof. Define the Laurent polynomial by the q-binomial sum

$$\Omega(y)=\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+1\end{bmatrix}\begin{bmatrix}k-y+3\\2n+1\end{bmatrix}q^{\binom{3n-k+1}{2}}$$

which consists of the terms $q^{y\ell}$ with $|\ell|\le n+1$.

Then the theorem can be shown by justifying the following statements:

- For the q-binomial product $\left[\begin{smallmatrix}y-3\\n\end{smallmatrix}\right]\left[\begin{smallmatrix}-y\\n\end{smallmatrix}\right]$, its zeros $\{i,3-i:3\le i\le n+2\}$ are also $2n$ zeros of $\Omega(y)$.
- The extra factor $a+b(q^y+q^{3-y})$ such that

$$\Omega(y)=\{a+b(q^y+q^{3-y})\}\begin{bmatrix}y-3\\n\end{bmatrix}\begin{bmatrix}-y\\n\end{bmatrix}$$

can be determined by resolving the following linear system:

$$\begin{cases}\Omega(2)=\{a+b(q^2+q)\}\begin{bmatrix}-1\\n\end{bmatrix}\begin{bmatrix}-2\\n\end{bmatrix},\\ \Omega(n+3)=\{a+b(q^{n+3}+q^{-n})\}\begin{bmatrix}-n-3\\n\end{bmatrix}.\end{cases}$$

By evaluating further

$$\Omega(2)=q^{\binom{n+1}{2}}\begin{bmatrix}2n+2\\2n+1\end{bmatrix},$$
$$\Omega(n+3)=(-1)^n\begin{bmatrix}2n\\n-1\end{bmatrix}\frac{(1-q^{2n+2})(1+q^n+q^{n+1})}{1-q^{n+2}};$$

we get the solution

$$a=\frac{1-q^{2n+1}-q^{2n+2}+q^{1+3n}+q^{2+3n}+q^{3+4n}}{(1-q^{n+1})(1-q^{2n+1})}q^{(n/2)(5+3n)},$$
$$b=\frac{-q^{(n/2)(5+3n)+3n}}{(1-q^{n+1})(1-q^{2n+1})};$$

which gives rise to the following explicit expression

$$a+b(q^y+q^{3-y})=(1+q^{1+n})\left\{1+q^{1+3n}\frac{(1-q^{y-1})(1-q^{2-y})}{(1-q^{1+2n})(1-q^{2+2n})}\right\}q^{(n/2)(5+3n)}.\quad\square$$

§ 10. Formula for $\Omega_n(1,0|-3,y)$.

Theorem 10 ($n \in \mathbb{N}_0$).

$$\sum_{k=0}^{2n}(-1)^k \begin{bmatrix}2n\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+1\end{bmatrix}\begin{bmatrix}k-y-3\\2n+1\end{bmatrix} q^{\binom{3n-k+1}{2}}$$
$$= (1+q^{1+n})\begin{bmatrix}y\\n\end{bmatrix}\begin{bmatrix}-3-y\\n\end{bmatrix}\left\{1+q^{1+n}\frac{(1-q^{-y-1})(1-q^{2+y})}{(1-q^{1+2n})(1-q^{2+2n})}\right\}q^{(3/2)(n^2-3n-2)}.$$

Sketch of proof. Define the Laurent polynomial in q^y by the q-binomial sum

$$\Omega(y) = \sum_{k=0}^{2n}(-1)^k \begin{bmatrix}2n\\k\end{bmatrix}\begin{bmatrix}k+y\\2n+1\end{bmatrix}\begin{bmatrix}k-y+3\\2n+1\end{bmatrix} q^{\binom{3n-k+1}{2}}$$

whose "degree" is limited to $|\ell| \le n+1$.

Then the theorem can be proved by checking the following assertions:

- For the q-binomial of product $\begin{bmatrix}y\\n\end{bmatrix}\begin{bmatrix}-3-y\\n\end{bmatrix}$, its zeros $\{i, -3-i : 0 \le i \le n-1\}$ are also $2n$ zeros of $\Omega(y)$.
- The extra factor $a+b(q^y+q^{-3-y})$ such that

  $$\Omega(y) = \{a+b(q^y+q^{-3-y})\}\begin{bmatrix}y\\n\end{bmatrix}\begin{bmatrix}-3-y\\n\end{bmatrix}$$

  can be determined, by letting $y=-1$ and $y=n$, as follows:

  $$a+b(q^y+q^{-3-y})$$
  $$= (1+q^{1+n})\left\{1+q^{1+n}\frac{(1-q^{-y-1})(1-q^{2+y})}{(1-q^{1+2n})(1-q^{2+2n})}\right\}q^{(3/2)(n^2-3n-2)}. \qquad \square$$

§ 11. Formula for $\Omega_n(-1,0|1,y)$.

Theorem 11 ($n \in \mathbb{N}$).

$$\sum_{k=0}^{2n}(-1)^k \begin{bmatrix}2n\\k\end{bmatrix}\begin{bmatrix}k+y\\2n-1\end{bmatrix}\begin{bmatrix}k-y+1\\2n-1\end{bmatrix} q^{\binom{3n-k-1}{2}}$$
$$= \frac{1+q^n}{q}\begin{bmatrix}y-2\\n-1\end{bmatrix}\begin{bmatrix}-1-y\\n-1\end{bmatrix}\left\{1-\frac{(1+q^{n+1})(1-q^{n-1})(1-q^{2n+1})}{q^{1+3n-y}(1-q^{y+1})(1-q^{y-2})}\right\}q^{3\binom{n+1}{2}}.$$

Sketch of proof. Define the Laurent polynomial by the q-binomial sum

$$\Omega(y) = \sum_{k=0}^{2n}(-1)^k \begin{bmatrix}2n\\k\end{bmatrix}\begin{bmatrix}k+y\\2n-1\end{bmatrix}\begin{bmatrix}k-y+1\\2n-1\end{bmatrix} q^{\binom{3n-k-1}{2}}$$

which consists of the terms $q^{y\ell}$ with $|\ell| \le n-1$.

Then the theorem can be demonstrated by verifying the following claims:

- For the q-binomial product $\frac{\left[{y-2 \atop n-1}\right]\left[{-1-y \atop n-1}\right]}{(1-q^{y-2})(1-q^{-1-y})}$, its zeros $\{i, 1-i : 3 \le i \le n\}$ are also $2n-4$ zeros of $\Omega(y)$.
- The extra factor $a + b(q^y + q^{1-y})$ such that

$$\Omega(y) = \frac{a + b(q^y + q^{1-y})}{(1-q^{y-2})(1-q^{-1-y})} \begin{bmatrix} y-2 \\ n-1 \end{bmatrix} \begin{bmatrix} -1-y \\ n-1 \end{bmatrix}$$

is determined, by letting $y = 1$ and $y = 2$, as follows:

$$\frac{a + b(q^y + q^{1-y})}{(1-q^{y-2})(1-q^{-1-y})} = q^{3\binom{n+1}{2}-1}(1+q^n) \\ \times \left\{1 - \frac{(1+q^{n+1})(1-q^{n-1})(1-q^{2n+1})}{q^{1+3n-y}(1-q^{y+1})(1-q^{y-2})}\right\}. \qquad \square$$

Before concluding the paper, we would like to point out that these 11 identities are not exhaustive. For instance, by making use of the same method, we are able to show also the two summation formulae displayed in the following theorems.

Theorem 12 (Formula for $\Omega_n(3+\lambda, 0|\lambda, y)$: $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$).

$$\sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} \begin{bmatrix} k+y \\ 2n+\lambda+3 \end{bmatrix} \begin{bmatrix} k-y+\lambda \\ 2n+\lambda+3 \end{bmatrix} q^{\binom{3+\lambda+3n-k}{2}} \\ = \frac{\left[{2n \atop n}\right]}{\left[{2n+\lambda+3 \atop n}\right]} \begin{bmatrix} y \\ n+\lambda+3 \end{bmatrix} \begin{bmatrix} \lambda-y \\ n+\lambda+3 \end{bmatrix} q^{3+((\lambda+n)/2)(5+\lambda+3n)} \\ \times \left\{1 + \frac{q^{n+1}(1-q^n)(1-q^{3+n+\lambda})}{(1-q^{2+n+y})(1-q^{2+n+\lambda-y})}\right\}.$$

Theorem 13 (Formula for $\Omega_n(4+\lambda, 0|\lambda, y)$: $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$).

$$\sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} \begin{bmatrix} k+y \\ 2n+\lambda+4 \end{bmatrix} \begin{bmatrix} k-y+\lambda \\ 2n+\lambda+4 \end{bmatrix} q^{\binom{4+\lambda+3n-k}{2}} \\ = \frac{\left[{2n \atop n}\right]}{\left[{2n+\lambda+4 \atop n}\right]} \begin{bmatrix} y \\ n+\lambda+4 \end{bmatrix} \begin{bmatrix} \lambda-y \\ n+\lambda+4 \end{bmatrix} q^{6+((\lambda+n)/2)(7+\lambda+3n)} \\ \times \left\{1 + \frac{q^{n+1}(1+q-q^n-q^{2n+1})(1-q^{4+n+\lambda})}{(1-q^{3+n+y})(1-q^{3+n+\lambda-y})}\right\}.$$

However, it will be more difficult to determine the extra polynomial factors appearing in closed formulae when the discrepancy between two integer parameters ε and λ becomes large.

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