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Products of parts in class regular partitions

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Abstract. A q-analogue of a partition identity is presented.

1. Introduction

Let $\lambda = (1^{m_1} 2^{m_2} \ldots)$ be a partition. Define

$$a_\lambda := \prod_{i \geq 1} i^{m_i}, \qquad b_\lambda := \prod_{i \geq 1} m_i!.$$

It is well known that the product of a_λ over all partitions λ of n is equal to that of b_λ. In 2003 Olsson [3] found a "regular version" of this remarkable fact. Let $r \geq 2$ be an integer. A partition $\lambda = (1^{m_1} 2^{m_2} \ldots)$ is said to be r-class regular if $m_{ri} = 0$ for all i. Denote by $P^r(n)$ the set of all r-class regular partitions of n. Define

$$a_{r,n} := \prod_{\lambda \in P^r(n)} a_\lambda, \qquad b_{r,n} := \prod_{\lambda \in P^r(n)} b_\lambda.$$

Then one has $b_{r,n} = r^{c_{r,n}} a_{r,n}$, where $c_{r,n}$ is defined by the following generating function:

$$\sum_{n \geq 0} c_{r,n} q^n = \Phi_r(q) \sum_{m \geq 1} \frac{q^{rm}}{1 - q^{rm}},$$

with

$$\Phi_r(q) = \prod_{k \geq 1} \frac{1 - q^{rk}}{1 - q^k} = \sum_{n \geq 0} |P^r(n)| q^n.$$

When r is prime, $a_{r,n}$ equals the determinant of the irreducible Brauer character table $\Psi_n^{(r)}$, and $r^{c_{r,n}}$ equals the r-part of $b_{r,n}$ and hence is equal to the determinant of the Cartan matrix for r-modular representations of the symmetric group $\mathfrak{S}_n$ ([3], see also [2]).

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In this short note we present a q-analogue of Olsson's formula in a natural combinatorial way.

2. Result

For an r-class regular partion $\lambda = (1^{m_1} 2^{m_2} \dots)$, a non-negative integer ℓ and a positive integer i which is not a multiple of r, put

$$D_\ell(i, \lambda) := \{(j,k) \in \mathbf{Z}^2 \mid j \geq \ell, 1 \leq k \leq m_i, r^j \mid k\}.$$

Here is an example. If $r = 2$ and λ be such that $m_i = 10$ for some odd i, then $D_0(i, \lambda)$ looks

$$\begin{array}{cccccccccc} \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\ & \bigcirc & & \bigcirc & & \bigcirc & & \bigcirc & & \bigcirc \\ & & & \bigcirc & & & & \bigcirc & & \\ & & & & & & & \bigcirc & & \end{array}.$$

The k-axis is horizontal from left to right, and the j-axis is vertical from top to bottom. Define also the set of "cells" for λ by

$$\mathscr{D}_\ell(\lambda) := \{c = (\lambda; i, j, k) \in \{\lambda\} \times \mathbf{Z}^3 \mid i \geq 1, r \nmid i, (j,k) \in D_\ell(i, \lambda)\}$$

and the disjoint union

$$\mathscr{D}_\ell(r, n) := \bigsqcup_{\lambda \in P^r(n)} \mathscr{D}_\ell(\lambda).$$

For each cell $c = (\lambda; i, j, k) \in \mathscr{D}_0(\lambda)$, attach the A-weight $A(c)$ and the B-weight $B(c)$, respectively, by $A(c) := ir^j$ and $B(c) := k/r^j$. In the example above with odd i, the A-weights and the B-weights are tabulated as follows.

$$\begin{array}{cccccccccc} i & i & i & i & i & i & i & i & i & i \\ & 2i & & 2i & & 2i & & 2i & & 2i \\ & & & 4i & & & & 4i & & \\ & & & & & & & 8i & & \end{array} \quad \text{and} \quad \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ & 1 & & 2 & & 3 & & 4 & & 5 \\ & & & 1 & & & & 2 & & \\ & & & & & & & 1 & & \end{array}.$$

Let Q_k $(k \geq 1)$ be a family of indeterminates. Define the A-monomial and B-monomial, respectively, for $\lambda \in P^r(n)$ and $\ell \geq 0$ by

$$w_A^\ell(\lambda) := \prod_{c \in \mathscr{D}_\ell(\lambda)} Q_{A(c)}, \qquad w_B^\ell(\lambda) := \prod_{c \in \mathscr{D}_\ell(\lambda)} Q_{B(c)}.$$

In the example, we see that

$$w_A^0(\lambda) = Q_i^{10} Q_{2i}^5 Q_{4i}^2 Q_{8i}, \qquad w_B^0(\lambda) = Q_1^4 Q_2^3 Q_3^2 Q_4^2 Q_5^2 Q_6 Q_7 Q_8 Q_9 Q_{10}.$$

and

$$w_A^1(\lambda) = Q_{2i}^5 Q_{4i}^2 Q_{8i}, \qquad w_B^1(\lambda) = Q_1^3 Q_2^2 Q_3 Q_4 Q_5.$$

THEOREM. *For a non-negative integer* ℓ,

$$\prod_{\lambda \in P^r(n)} w_A^\ell(\lambda) = \prod_{\lambda \in P^r(n)} w_B^\ell(\lambda)|_{Q_k \mapsto Q_{r^\ell k}}.$$

PROOF. Let $\ell \geq 0$ be fixed. One can construct an involution

$$\theta_\ell : \mathscr{D}_\ell(r,n) \to \mathscr{D}_\ell(r,n)$$

as follows. Take $c = (\lambda; i, j, k) \in \mathscr{D}_\ell(\lambda)$. Since $k \leq m_i$ and $r^j \mid k$, we can write $k = i^* r^{j+j^*}$ with some i^* with $r \nmid i^*$, and $j^* \geq 0$. Put $k^* = i r^{j+j^*}$ so that $ik = i^* k^*$. There exists an r-class regular partition $\mu \in P^r(n - ik)$ such that λ is the Young diagrammatic union of μ and (i^k). Let λ^* be the union of partitions μ and $((i^*)^{k^*})$, which is in $P^r(n)$. Let $\theta_\ell(c) := (\lambda^*; i^*, j^* + \ell, k^*) \in \mathscr{D}_\ell(\lambda^*)$. It is easy to verify that $(\theta_\ell)^2 = id$. We also have

$$A(\theta_\ell(c)) = i^* r^{j^*+\ell} = \frac{ik}{k^*} r^{j^*+\ell} = \frac{ikr^{j^*+\ell}}{ir^{j+j^*}} = r^\ell \frac{k}{r^j} = r^\ell B(c)$$

as desired.

Here is an example. Let $r = 2$, $\ell = 0$ and $\lambda = (1\,3^2) \in P^2(7)$. If $c = (\lambda; 3, 1, 2) \in \mathscr{D}_0(\lambda)$, then one sees that $i^* = 1$, $j^* = 0$, $k^* = 6$, and $\mu = (1)$. Hence one has $\lambda^* = (1^7)$ and $\theta_0(c) = (\lambda^*; 1, 0, 6)$. Therefore $A(\theta_0(c)) = B(c) = 1$.

Let us introduce another family of indeterminates R_k $(k \geq 1)$, subject to the relations $Q_{rk} = R_k Q_k$ for $k \geq 1$. Then the formula in Theorem in case $\ell = 1$ reads

$$\prod_{\lambda \in P^r(n)} w_A^1(\lambda)(Q) = \prod_{\lambda \in P^r(n)} w_B^1(\lambda)(R) \prod_{\lambda \in P^r(n)} w_B^1(\lambda)(Q).$$

Remark that, for $\lambda = (1^{m_1} 2^{m_2} \ldots) \in P^r(n)$,

$$\frac{w_A^0(\lambda)(Q)}{w_A^1(\lambda)(Q)} = \prod_{i \geq 1} Q_i^{m_i}, \qquad \frac{w_B^0(\lambda)(Q)}{w_B^1(\lambda)(Q)} = \prod_{i \geq 1} Q_{m_i} Q_{m_i - 1} \ldots Q_1.$$

These give a Q-analogue of a_λ and b_λ, respectively.

In order to relate our result with Olsson's formula, we specialize the indeterminates as

$$Q_k = \frac{1 - q^k}{1 - q}, \qquad R_k = \frac{1 - q^{rk}}{1 - q^k}$$

with another indeterminate q. We regard

$$a_{r,n}(q) := \prod_{\lambda \in P^r(n)} \frac{w_A^0(\lambda)(Q)}{w_A^1(\lambda)(Q)} \qquad \text{and} \qquad b_{r,n}(q) := \prod_{\lambda \in P^r(n)} \frac{w_B^0(\lambda)(Q)}{w_B^1(\lambda)(Q)}$$

as polynomials in q.

We also denote

$$c_{r,n}(q) := \prod_{\lambda \in P^r(n)} w_B^1(R)$$

with the specialization above. This is a q-analogue of $r^{c_{r,n}}$, and is known to equal the determinant of the "graded" Cartan matrix for the Iwahori Hecke algebra $H_n(\zeta)$ with ζ a primitive r-th root of unity ([1]).

Consequenty Olsson's formula is q-deformed as

$$b_{r,n}(q) = c_{r,n}(q)a_{r,n}(q).$$

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