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Uniform hyperbolicity for curve graphs of non-orientable surfaces

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Abstract. Hensel-Przytycki-Webb proved that the curve graphs of all orientable surfaces are 17-hyperbolic. In this paper, we show that the curve graphs of non-orientable surfaces are 17-hyperbolic by applying Hensel-Przytycki-Webb's argument. We also show that the arc graphs of non-orientable surfaces are 7-hyperbolic, and arc-curve graphs of (non-)orientable surfaces are 9-hyperbolic.

1. Introduction

For $g \geq 1$ and $n \geq 0$, let $N = N_{g,n}$ be a compact, connected, non-orientable surface of genus g with n boundary components. The *curve graph* $\mathscr{C}(N)$ of N is the graph whose vertex set is the set of homotopy classes of essential simple closed curves (or curves) and whose edges correspond to two disjoint curves. Curve graphs are often used to study mapping class groups of surfaces, geometric group theory, and hyperbolic geometry. In this paper, we consider a graph as a geodesic space. A triangle formed by geodesic edge-paths in the graph (we call such a triangle a *geodesic triangle*) has a *k-center* ($k \geq 0$) if there exists a vertex such that the distance from it to each side of T is not more than k. A connected graph is *k-hyperbolic* if every geodesic triangle in the graph has a k-center. We say that a graph is (*Gromov*) *hyperbolic* if it is k-hyperbolic for some $k \geq 0$, and we refer to such a constant k as a *hyperbolicity constant* for the graph. Bestvina-Fujiwara [2] first proved that $\mathscr{C}(N)$ is Gromov hyperbolic, and Masur-Schleimer [8] gave another proof. However, the uniform hyperbolicity for curve graphs of non-orientable surfaces was not known. The main result of this paper is to prove the following:

Theorem 1. *If $\mathscr{C}(N)$ is connected, then it is 17-hyperbolic.*

Let $S = S_{g,n}$ be an orientable surface of genus $g \geq 0$ with $n \geq 0$ boundary components. First, Masur-Minsky [7] proved that each curve graph $\mathscr{C}(S)$ of S is hyperbolic in 1999. After their original proof, various other proofs of

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hyperbolicity for curve graphs of orientable surfaces were given by several authors. Recently, Aougab [1], Bowditch [3], Clay-Rafi-Schleimer [4], and Hensel-Przytycki-Webb [5] independently proved that one can choose hyperbolicity constants which do not depend on the topological types of orientable surfaces. In particular, Hensel-Przytycki-Webb [5] showed that $\mathscr{C}(S)$ is 17-hyperbolic by a combinatorial argument, and their argument seems to give an optimum constant. We prove Theorem 1 by applying Hensel-Przytycki-Webb's argument to the case of non-orientable surfaces. They also showed that arc graphs of orientable surfaces are 7-hyperbolic. We prove a similar result for non-orientable surfaces:

Theorem 2. *The arc graph $\mathscr{A}(N)$ of N is 7-hyperbolic.*

We also consider arc-curve graphs. The hyperbolicity for arc-curve graphs of orientable surfaces was proved by Korkmaz-Papadopoulos [6, Corollary 1.4]. The uniform hyperbolicity, however, was not known. We also prove:

Theorem 3. *Set $F = S$ or N. If the arc-curve graph $\mathscr{AC}(F)$ of F is connected, then it is 9-hyperbolic.*

For the cases where a, b, d are arcs or where a, b, d are curves, Hensel-Przytycki-Webb proved a geodesic triangle $T = abd$ has a 7-center and a 9-center in $\mathscr{AC}(S)$ respectively. We show that a geodesic triangle $T = abd$ has an 8-center for the cases where a is a curve and b, d are arcs, or where a, b are curves and d is an arc to prove Theorem 3.

2. Preliminaries

A compact, connected, *non-orientable surface* of genus $g \geq 1$ with $n \geq 0$ boundary components is the connected sum of g projective planes which is removed n open disks. We denote it by $N = N_{g,n}$. Note that N is homeomorphic to the surface obtained from a sphere by removing $g + n$ open disks and attaching g Möbius bands along their boundaries, and we call each of the Möbius bands the crosscap. An arc a on N is *properly embedded* if $\partial a \subseteq \partial N$ and a is transverse to ∂N. A properly embedded arc a on N is called *essential* if it is not homotopic into ∂N. A curve on N is called *essential* if it does not bound a disk or a Möbius band, and it is not homotopic to a boundary component of N. We remark that a homotopy fixes each boundary component of N setwise. From now on, we consider arcs and curves which are properly embedded and essential. The *arc-curve graph* $\mathscr{AC}(N)$ of N is the graph whose vertex set $\mathscr{AC}^{(0)}(N)$ is the set of homotopy classes of arcs and curves on N. Two vertices form an edge if they can be represented by disjoint arcs or curves. The *arc graph* $\mathscr{A}(N)$ of N is the subgraph of the arc-curve graph induced by the vertex set $\mathscr{A}^{(0)}(N)$ which consists of homotopy classes of arcs on N. The *curve graph* $\mathscr{C}(N)$ of N is the subgraph of the arc-curve graph induced by the vertex set $\mathscr{C}^{(0)}(N)$ which consists of homotopy classes of curves on N. We deem that each edge of $\mathscr{AC}(N)$, $\mathscr{A}(N)$, and $\mathscr{C}(N)$ has unit length. We define the distances $d_{\mathscr{AC}}(\cdot,\cdot)$, $d_{\mathscr{A}}(\cdot,\cdot)$ and $d_{\mathscr{C}}(\cdot,\cdot)$ in $\mathscr{AC}(N)$, $\mathscr{A}(N)$, and $\mathscr{C}(N)$, respectively, by the minimal length of edge-paths connecting the two vertices. Now, we consider $\mathscr{AC}(N)$, $\mathscr{A}(N)$, and $\mathscr{C}(N)$, as geodesic spaces.

Two arcs a, b (or two curves a, b) on N are in *minimal position* if the number of intersections between a and b is minimal in the homotopy classes of a and b.

Proposition 1. *Two arcs a, b on N are in minimal position if and only if a and b intersect transversely and they do not form any bigons (i.e. an embedded disk on N bounded by a subarc of a and a subarc of b) or any half-bigons (i.e. an embedded disk on N bounded by a subarc of a, a subarc of b, and a part of a boundary component of N).*

We use the following proposition to prove Proposition 1.

Proposition 2 ([9, Proposition 2.1]). *Let N be a compact, non-orientable surface, and a and b essential curves on N. Then a and b are in minimal position if and only if a and b do not form a bigon.*

Proof (Proof of Proposition 1). If a and b bound bigons or half-bigons, then we can reduce intersection points by a homotopy through bigons or half-bigons.

Conversely, suppose that two arcs a and b on N are not in minimal position. We collect the boundary components which have endpoints of a and b in one side by a homeomorphism preserving intersections between a and b. We make a mirror reflective surface N' of N, and assume that a' and b' are arcs on N' corresponding to a and b on N respectively. Note that a' and b' are not in minimal position since a and b are not in minimal position. We attach each boundary component of N' which has the endpoints of a' and b' to the reflective part of N, and let M be the resulting surface. Then $a \cup a'$ and $b \cup b'$ are essential curves and not in minimal position on M. By Proposition 9, $a \cup a'$ and $b \cup b'$ form bigons. From the assumption that N and N' are mirror reflective surfaces each other, we have the following two cases. One is that a and b form bigons on N and a' and b' also form bigons on N' at the reflective parts. The other is that $a \cup a'$ and $b \cup b'$ form bigons on M which are mirror reflective for attached parts. The former implies that a and b form bigons on N, and the latter implies that a and b form half-bigons on N, as desired. $\square$

3. Unicorn paths

In this section, all lemmas come from Section 3 in [5] by changing the assumption of orientable surfaces to non-orientable surfaces, and so please see the proofs of [5] for the proofs of these lemmas.

Definition 1. Let a and b be two arcs on N which are in minimal position, and let α and β be one of the endpoints of a and b, respectively. Choose $\pi \in a \cap b$. Let a' be a subarc of a whose endpoints are α and π, and b' a subarc of b whose endpoints are β and π. If $a' \cup b'$ is an embedded arc on N, we say that $a' \cup b'$ is a *unicorn arc obtained from* a^α, b^β *and* π.

A unicorn arc is uniquely determined by π, although not all intersection points between a and b determine unicorn arcs since the resulting arcs may not be embedded on N. Note that a unicorn arc $a' \cup b'$ is an essential arc. Indeed, if $a' \cup b'$ is not essential, that is, if $a' \cup b'$ is homotopic into a boundary component of N, then a and b form a half-bigon, and this contradicts the assumption that a and b are in minimal position.

We define a total order among unicorn arcs obtained from a^α, b^β as follows.

Definition 2. Let $a' \cup b'$, $a'' \cup b''$ be two unicorn arcs obtained from a^α and b^β, where $a', a'' \subset a$ and $b', b'' \subset b$. We define $a' \cup b' \leq a'' \cup b''$ by $a'' \subset a'$ and $b' \subset b''$.

Definition 3. Let $(c_1, c_2, \dots, c_{n-1})$ be the ordered set of all unicorn arcs obtained from a^α and b^β. We call the sequence $\mathscr{P}(a^\alpha, b^\beta) = (a = c_0, c_1, \dots, c_{n-1}, c_n = b)$ the *unicorn path* between a^α and b^β.

Then, we have a natural question similar to that of the case of orientable surfaces whether a unicorn path $\mathscr{P}(a^\alpha, b^\beta)$ becomes a path in $\mathscr{A}(N)$. We gain the following:

Proposition 3. *Consecutive arcs in a unicorn path represent adjacent vertices of* $\mathscr{A}(N)$.

Proof. Let $c_i = a' \cup b'$ $(2 \leq i \leq n-1)$ and $\pi \in a' \cap b'$. Let π' be the point in $(a - a') \cap b$ which is nearest to α along a of the points determining a unicorn arc. Then, the intersection point π' determines the unicorn arc c_{i-1}.

The unicorn arc c_i does not pass any common points of a and b between π and π', otherwise the point becomes the next point determining the unicorn arc next to c_i and this contradicts the choice of π'. Thus, c_i and c_{i-1} do not intersect between π and π'. Furthermore, there exists an arc homotopic to c_i which is disjoint from c_{i-1}. Indeed, it is sufficient to choose a neighborhood of a' not intersecting c_{i-1} when c_i turns at π, and a neighborhood of b' not intersecting c_{i-1} at π'. For $i = 1, n$, the fact that c_{i-1} and c_i form an edge follows similarly. $\square$

We deduce that all arc graphs are connected by the existence of unicorn paths.

Corollary 1. *$\mathscr{A}(N)$ is connected.*

Lemma 1 (cf. [5, Lemma 3.3]). *Let a, b, and d be three arcs on N which are mutually in minimal position, and let α, β and δ be one of the endpoints of a, b and d, respectively. For each $c \in \mathscr{P}(a^\alpha, b^\beta)$, there exists $c^* \in \mathscr{P}(a^\alpha, d^\delta) \cup \mathscr{P}(b^\beta, d^\delta)$, such that c and c^* represent adjacent vertices of $\mathscr{A}(N)$.*

Note that c and d may not be in minimal position.

Lemma 2 (cf. [5, Lemma 3.4]). *Let a, b and d be three arcs on N which are mutually in minimal position, and let α, β and δ be one of the endpoints of a, b and d, respectively. Then there exist $c^1 \in \mathscr{P}(a^\alpha, b^\beta)$, $c^2 \in \mathscr{P}(b^\beta, d^\delta)$ and $c^3 \in \mathscr{P}(d^\delta, a^\alpha)$ such that c^i and c^j $(i \neq j$ and $i, j = 1, 2, 3)$ represent adjacent vertices of $\mathscr{A}(N)$.*

4. Arc graphs are uniformly hyperbolic

In this section, all the proofs are the same as those of [5, Section 4]. Slightly abusing the notation, we consider vertices a, b of $\mathscr{A}(N)$, $\mathscr{C}(N)$ and $\mathscr{AC}(N)$ as arcs or curves on N which are in minimal position from now on.

Definition 4. We define the following family $P(a,b)$ of unicorn paths to a pair of vertices a, b in $\mathscr{A}(N)$. Let (a,b) be an edge in $\mathscr{A}(N)$ connecting a and b. Let α_+ and α_- be the endpoints of a, and β_+ and β_- the endpoints of b. Then, we define

$$P(a,b) = \begin{cases} \{(a,b)\} & \text{if } a \cap b = \emptyset, \\ \{\mathscr{P}(a^{\alpha_+}, b^{\beta_+}), \mathscr{P}(a^{\alpha_+}, b^{\beta_-}), \mathscr{P}(a^{\alpha_-}, b^{\beta_+}), \mathscr{P}(a^{\alpha_-}, b^{\beta_-})\} & \text{if } a \cap b \neq \emptyset. \end{cases}$$

Proposition 4 (cf. [5, Proposition 4.2]). *Let $\mathscr{G}$ be a geodesic in $\mathscr{A}(N)$ between vertices a and b. Then any unicorn arc $c \in \mathscr{P} \in P(a,b)$ is at distance ≤ 6 from $\mathscr{G}$.*

Proof (Proof of Theorem 2). Let $T = abd$ be any geodesic triangle in $\mathscr{A}(N)$, where a, b and d are three vertices of $\mathscr{A}(N)$. By Lemma 2, for a, b and d, there exist $c_{ab} \in \mathscr{P}(a^{\alpha}, b^{\beta})$, $c_{bd} \in \mathscr{P}(b^{\beta}, d^{\delta})$ and $c_{da} \in \mathscr{P}(d^{\delta}, a^{\alpha})$ such that each pair represents adjacent vertices of $\mathscr{A}(N)$. Let ab, bd, and da be the sides of T connecting a and b, b and d, and d and a, respectively in $\mathscr{A}(N)$. By Proposition 4, c_{ab} is at distance ≤ 6 from ab, and ≤ 7 from both bd and da. Hence, c_{ab} is a 7-center of T. $\square$

5. Curve graphs are uniformly hyperbolic

By [10, Theorem 6.1], we obtain the following:

Proposition 5. *If $g = 1, 2$ and $g + n \geq 5$, or $g \geq 3$, then the curve graph $\mathscr{C}(N)$ of N is connected.*

We define a retraction $r : \mathscr{AC}^{(0)}(N) \to \mathscr{C}^{(0)}(N)$ as follows. If $a \in \mathscr{C}^{(0)}(N)$, then $r(a) = a$. If $a \in \mathscr{A}^{(0)}(N)$, then we assign a boundary component of a regular neighborhood of its union with ∂N to $r(a)$ (see Figure 1). If there are two boundary components of the regular neighborhood, we choose essential one (c.f. $r' : \mathscr{AC}^{(0)}(S) \to \mathscr{C}^{(0)}(S)$ in [5]). The difference from r' in [5] is as follows: if a is an arc on N which goes through crosscaps odd number of times, then $r(a)$ is "twisted" (see the left-hand side in Figure 2).

Lemma 3. *The retraction r is 2-Lipschitz, namely, for any $a, b \in \mathscr{AC}(N)$ $d_{\mathscr{C}}(r(a), r(b)) \leq 2 d_{\mathscr{AC}}(a, b)$.*

Proof. It is enough to prove that $d_{\mathscr{C}}(r(a), r(b)) \leq 2$ for $a, b \in \mathscr{AC}(N)$ with $d_{\mathscr{AC}}(a, b) = 1$. If $a, b \in \mathscr{C}^{(0)}(N)$, then $d_{\mathscr{C}}(r(a), r(b)) = d_{\mathscr{C}}(a, b) = d_{\mathscr{AC}}(a, b) = 1 < 2$. If $a \in \mathscr{C}^{(0)}(N)$ and $b \in \mathscr{A}^{(0)}(N)$, then we can take a regular neighborhood of the union of b and the boundary components which have endpoints of b without intersecting a. Note that $r(b)$ may coincide with a. Thus $d_{\mathscr{C}}(r(a), r(b)) = d_{\mathscr{C}}(a, r(b)) \leq 1 < 2$. From now, we assume $a, b \in \mathscr{A}^{(0)}(N)$. Then there are eight types of pairs of a, b which satisfy $d_{\mathscr{AC}}(a, b) = 1$

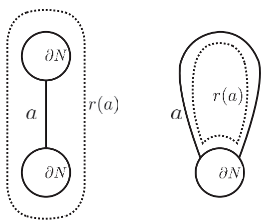

Fig. 1. Examples of the retraction r.

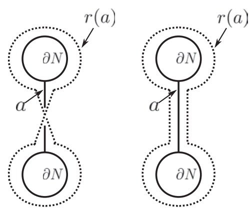

Fig. 2. Examples that $r(a)$ is twisted (left) and untwisted (right).

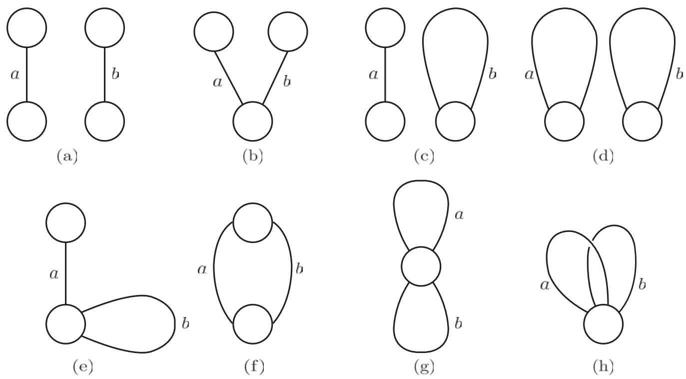

Fig. 3. Eight cases of $a, b \in \mathscr{A}^{(0)}(N)$ which satisfy $d_{\mathscr{A}\mathscr{C}}(a, b) = 1$.

(see Figure 3, where each circle represents a boundary component of N). Note that there are two cases where a (resp. b) passes through crosscaps odd number of times, and where it passes through crosscaps even number of times. In the former case, we say that $r(a)$ (resp. $r(b)$) is *twisted* (see the left-hand side in Figure 2), and in the latter case, we say that $r(a)$ (resp. $r(b)$) is *untwisted* (see the right-hand side in Figure 2). First we assume $(g, n) \neq (3, 1)$.

In the cases of (a), (c) and (d) in Figure 3, $r(a)$ and $r(b)$ are essential and disjoint curves. Note that $r(a)$ and $r(b)$ may coincide in (c) and (d). Hence, $d_{\mathscr{C}}(r(a), r(b)) \leq 1 < 2$.

In the case of (b) in Figure 3, there are three cases where both $r(a)$ and $r(b)$ are untwisted, $r(a)$ is untwisted and $r(b)$ is twisted, and both $r(a)$ and $r(b)$ are twisted. In all the three cases, we take a boundary component α of a regular neighborhood of the union of a and b with ∂N large enough to intersect neither $r(a)$ nor $r(b)$. Then it is sufficient to prove that α is essential. It is clear that α bounds three-punctured disk on one side. We show that α does not bound a disk, an annulus, or a Möbius band on the other side. By the calculation of the Euler characteristics, we see that α separates N into $S_{0,4}$ and $N_{g,n-2}$. If $g \geq 2$, then $N_{g,n-2}$ is not a disk, an annulus, or a Möbius band. If $g = 1$, then $N_{g,n-2}$ is also not a disk, an annulus, or a Möbius band, since $g + n \geq 5$. Therefore, α is essential and $d_{\mathscr{C}}(r(a), r(b)) \leq d_{\mathscr{C}}(r(a), \alpha) + d_{\mathscr{C}}(\alpha, r(b)) \leq 2$.

In the case of (e) in Figure 3, there are four cases where both $r(a)$ and $r(b)$ are untwisted, $r(a)$ is untwisted and $r(b)$ is twisted, $r(a)$ is twisted and $r(b)$ is untwisted, and both $r(a)$ and $r(b)$ are twisted. Let γ_1 and γ_2 be the boundary components of N which have endpoints of a and b. In the first case, i.e. both $r(a)$ and $r(b)$ are untwisted, there are two boundary components of a regular neighborhood of $a \cup \gamma_1 \cup \gamma_2 \cup b$. We denote by α the outer part of the regular neighborhood, and by α' the other (see Figure 4). Note that α and α' intersect neither $r(a)$ nor $r(b)$. It is sufficient to show that at least one of α and α' is essential. If α bounds a disk, an annulus, or a Möbius band, then we take α'. The curve α' separates N into $S_{0,3}$ and $N_{g,n-1}$, $S_{0,4}$ and $N_{g,n-2}$, $N_{1,3}$ and $N_{g-1,n-1}$, or $N_{1,3}$ and $S_{(g-1)/2,n-1}$. We can show that α' is essential by a similar argument in (b). If α does not bound a disk, an annulus, or a Möbius band, then we take α, and so α is essential. In the second case, i.e. $r(a)$ is untwisted and $r(b)$ is twisted, there is one boundary component of a regular neighborhood of $a \cup \gamma_1 \cup \gamma_2 \cup b$, and we denote it by α. It is sufficient to show that α is essential. The curve α separates N into $N_{1,3}$ and $N_{g-1,n-1}$, or $N_{1,3}$ and $S_{(g-1)/2,n-1}$, and we can see α is essential since $g + n \geq 5$. In the third case, i.e. $r(a)$ is twisted and $r(b)$ is untwisted, it is enough to follow a similar argument in the first case of (e). In the last case, i.e. both $r(a)$ and $r(b)$ are twisted, we can show it by a similar argument to that of the third case in (e).

In the case of (f) in Figure 3, there are three cases where both $r(a)$ and $r(b)$ are untwisted, $r(a)$ is untwisted and $r(b)$ is twisted, and both $r(a)$ and $r(b)$ are twisted. Let γ_1 and γ_2 be the boundary components of N which have endpoints of a and b. In the first case, i.e. both $r(a)$ and $r(b)$ are untwisted, there are two boundary components of a regular neighborhood of $a \cup \gamma_1 \cup \gamma_2 \cup b$. We denote by α the outer part of the regular neighborhood, and by α' the other (see Figure 4). If α bounds a disk, an annulus, or a Möbius band, we take α'. The curve α' separates N into $S_{0,3}$ and $N_{g,n-1}$, $S_{0,4}$ and $N_{g,n-2}$, $N_{1,3}$ and $N_{g-1,n-1}$, or $N_{1,3}$ and $S_{(g-1)/2,n-1}$, and so α' is essential. If α does not bound a disk, an annulus, or a Möbius band, then we take α, which is essential. In the second case, i.e. $r(a)$ is untwisted and $r(b)$ is twisted, there is one boundary component of a regular neighborhood of $a \cup \gamma_1 \cup \gamma_2 \cup b$, and we denote it by α. The curve α separates N into $N_{1,3}$ and $N_{g-1,n-1}$, or $N_{1,3}$ and $S_{(g-1)/2,n-1}$, and so α is essential. In the last case, i.e. both $r(a)$ and $r(b)$ are twisted, there are two boundary components of a regular neighborhood

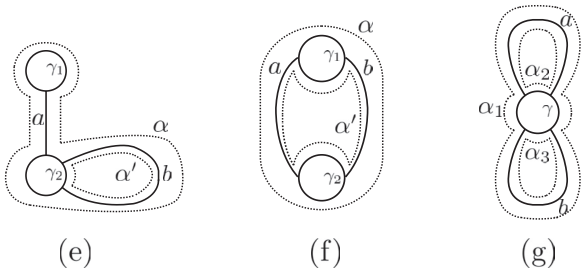

Fig. 4. The case where both $r(a)$ and $r(b)$ are untwisted in (e), (f), and (g) respectively.

of $a \cup \gamma_1 \cup \gamma_2 \cup b$. We take one of them and denote it by α. Then α is a non-separating curve on N. Therefore, α is essential.

In the case of (g) in Figure 3, there are three cases where both $r(a)$ and $r(b)$ are untwisted, $r(a)$ is untwisted and $r(b)$ is twisted, and both $r(a)$ and $r(b)$ are twisted. Let γ be a boundary component of N which has endpoints of a and b. In the first case, i.e. both $r(a)$ and $r(b)$ are untwisted, there are three boundary components of a regular neighborhood of $a \cup \gamma \cup b$. We denote by α_1 the component which encloses a, γ, and b, and by α_2 (resp. α_3) the component which lies in the inner part of a (resp. b) in Figure 4. Suppose that α_1 bounds a disk. It is sufficient to show that α_3 is essential if α_2 is not essential. (If α_2 is essential, then we take α_2.) When we assume that α_2 is not essential, α_2 bounds either an annulus or a Möbius band. Then, the curve α_3 separates N into $S_{0,3}$ and $N_{g,n-1}$, $N_{1,2}$ and $N_{g-1,n}$, or $N_{1,2}$ and $S_{(g-1)/2,n}$. Hence α_3 is essential. When α_1 bounds an annulus and α_2 is not essential, we can also take an essential curve α_3 which is disjoint from both $r(a)$ and $r(b)$. Suppose that α_1 bounds a Möbius band and α_2 is not essential. Then, α_2 bounds either an annulus or a Möbius band, and so the curve α_3 separates N into $N_{1,3}$ and $N_{g-1,n-1}$, $N_{1,3}$ and $S_{(g-1)/2,n-1}$, $N_{2,2}$ and $N_{g-2,n}$, or $N_{2,2}$ and $S_{(g-2)/2,n}$. By a similar argument to that of the third case in (e), $N_{g-1,n-1}$ is not a disk, an annulus, or a Möbius band. We consider $N_{g-2,n}$. If $g-2 \geq 2$, then $N_{g-2,n}$ is not a disk, an annulus, or a Möbius band. If $g-2=1$, then $N_{g-2,n}$ is not a disk, an annulus, or a Möbius band because we assume $(g,n) \neq (3,1)$. If $g-2=0$, then $N_{g-2,n}$ is not a disk, an annulus, or a Möbius band, since $g+n \geq 5$. When α_1 does not bound a disk, an annulus, or a Möbius band, we take α_1. In the second case, i.e. $r(a)$ is untwisted and $r(b)$ is twisted, there are two boundary components of a regular neighborhood of $a \cup \gamma \cup b$, and the regular neighborhood of $a \cup \gamma \cup b$ is a non-orientable surface of genus 1 with 3 boundary components. We denote by α_1 and α_2 the boundaries of this surface which are not γ. It is sufficient to show that, if α_1 is not essential, then α_2 is essential. If α_1 bounds a disk, an annulus, or a Möbius band, then α_2 separates N into $N_{1,2}$ and $N_{g-1,n}$, $N_{1,2}$ and $S_{(g-1)/2,n}$, $N_{1,3}$ and $N_{g-1,n-1}$, $N_{1,3}$ and $S_{(g-1)/2,n-1}$, $N_{2,2}$ and $N_{g-2,n}$, or $N_{2,2}$ and $S_{(g-2)/2,n}$. Hence α_2 is essential. In the third case, i.e. both $r(a)$ and $r(b)$ are twisted, there is one boundary component of a regular neighborhood of $a \cup \gamma \cup b$ (we denote it by α), and the regular neighborhood of $a \cup \gamma \cup b$ is a non-orientable surface of genus 2 with 2 boundary components. Then α separates N into $N_{2,2}$ and $N_{g-2,n}$, or $N_{2,2}$ and $S_{(g-2)/2,n}$, and so α is essential.

In the case of (h) in Figure 3, there are three cases where both $r(a)$ and $r(b)$ are untwisted, $r(a)$ is untwisted and $r(b)$ is twisted, and both $r(a)$ and $r(b)$ are twisted. Let γ be a boundary component of N which has endpoints of a and b. In the first case, i.e. both $r(a)$ and $r(b)$ are untwisted, a regular neighborhood of $a \cup \gamma \cup b$ is twice holed torus, and $r(a)$ and $r(b)$ intersect once. Hence, the complement of $r(a)$ and $r(b)$ is a twice holed disk, and then we can take an essential curve which goes around the twice holed disk. In the second case, i.e. $r(a)$ is untwisted and $r(b)$ is twisted, it is enough to give the same argument as we gave in the third case of (g). In the third case, i.e. both $r(a)$ and $r(b)$ are twisted, it is enough to give the same argument as we gave in the second case of (g). In the cases of (e), (f), (g), and (h), there is an essential curve α which intersects neither $r(a)$ nor $r(b)$. Therefore, $d_{\mathscr{C}}(r(a), r(b)) \leq d_{\mathscr{C}}(r(a), \alpha) + d_{\mathscr{C}}(\alpha, r(b)) \leq 2$.

Next we assume $(g, n) = (3, 1)$. By the argument mentioned above, it is enough to discuss only the case of (g). In all cases where $r(a)$ and $r(b)$ are untwisted or twisted, we can take a curve which passes through a Möbius band which intersects neither $r(a)$ nor $r(b)$. Therefore, $d_{\mathscr{C}}(r(a), r(b)) \leq d_{\mathscr{C}}(r(a), \alpha) + d_{\mathscr{C}}(\alpha, r(b)) \leq 2$, and we complete the proof of Lemma 3. □

Before proving Theorem 1, we need to show the following proposition.

Proposition 6. *If $g = 1, 2$ and $g + n \geq 5$, or $g \geq 3$, then $\mathscr{AC}(N)$ is connected.*

Proof. If $a, b \in \mathscr{A}^{(0)}(N)$ or $a, b \in \mathscr{C}^{(0)}(N)$, then there exists an edge-path connecting a and b in $\mathscr{A}(N)$ or $\mathscr{C}(N)$, respectively, by Corollaries 1 and 5. We consider it as an edge-path in $\mathscr{AC}(N)$. Therefore, we may assume $a \in \mathscr{C}^{(0)}(N)$ and $b \in \mathscr{A}^{(0)}(N)$. Fix any $a \in \mathscr{C}^{(0)}(N)$. We take an appropriate boundary component a' of a regular neighborhood of a, and we connect a' and a boundary component of N by an arc η which does not intersect a. Then the products $\eta * a' * \eta^{-1}$ is a properly embedded arc which is disjoint from a. Hence, we can connect the vertices a and $\eta * a' * \eta^{-1}$ by an edge in $\mathscr{AC}(N)$. On the other hand, for any $b \in \mathscr{A}^{(0)}(N)$, we connect it to $\eta * a' * \eta^{-1}$ in $\mathscr{A}(N)$ by a unicorn path in $P(\eta * a' * \eta^{-1}, b)$. Therefore, we can connect an arbitrary $a \in \mathscr{C}^{(0)}(N)$ and an arbitrary $b \in \mathscr{A}^{(0)}(N)$ by an edge-path in $\mathscr{AC}(N)$. □

We also need the following proposition to prove Theorem 1. The proof is the same as the second paragraph of the proof of [5, Theorem 1].

Proposition 7. *Let a, b be vertices of $\mathscr{C}(N)$, and $\bar{a}$, $\bar{b}$ vertices of $\mathscr{A}(N)$ which are adjacent to a, b, respectively. Let $\mathscr{G} = ab$ be a geodesic connecting a and b in $\mathscr{C}(N)$. Then any unicorn arc $\bar{c} \in \mathscr{P} \in P(\bar{a}, \bar{b})$ is at distance ≤ 8 from $\mathscr{G}$.*

Now, we give a proof of Theorem 1.

Proof (Proof of Theorem 1). First we assume that $\partial N \neq \varnothing$. We take any geodesic triangle $T = abd$ in $\mathscr{C}(N)$, where $a, b, d \in \mathscr{C}^{(0)}(N)$. Let $\bar{a}$, $\bar{b}$, and $\bar{d}$ be three vertices of $\mathscr{A}(N)$ which are adjacent to a, b and d in $\mathscr{AC}(N)$ respectively. We choose one of the endpoints α, β and δ of $\bar{a}$, $\bar{b}$ and $\bar{d}$, respectively. Let ab be the side of T connecting a and b in $\mathscr{C}(N)$. From Lemma 2, there exist $c_{\bar{a}\bar{b}} \in \mathscr{P}(\bar{a}^{\alpha}, \bar{b}^{\beta})$, $c_{\bar{b}\bar{d}} \in \mathscr{P}(\bar{b}^{\beta}, \bar{d}^{\delta})$, and $c_{\bar{d}\bar{a}} \in \mathscr{P}(\bar{d}^{\delta}, \bar{a}^{\alpha})$ such that each pair represents adjacent vertices of $\mathscr{A}(N)$. By Proposition 7, the vertex $c_{\bar{a}\bar{b}}$ of $\mathscr{AC}(N)$ is a 9-center of T. In particular, $c_{\bar{a}\bar{b}}$ is at distance ≤ 8 from a vertex of $\mathscr{G} = ab$, which is a curve. We connect this vertex with $c_{\bar{a}\bar{b}}$ by a geodesic in $\mathscr{AC}(N)$. By Lemma 3, the vertex $r(c_{\bar{a}\bar{b}}) \in \mathscr{C}^{(0)}(N)$ is a 17-center of the triangle T in $\mathscr{C}(N)$.

Secondly, we assume that $\partial N = \varnothing$. Note that N has a negative Euler characteristic, since the genus of N is at least 3. Let $\overline{N}$ be a surface obtained from N by removing an open disk. In this proof, we denote by $d_{\mathscr{C}(N)}(\cdot\,,\cdot)$ and $d_{\mathscr{C}(\overline{N})}(\cdot\,,\cdot)$ the distances in $\mathscr{C}(N)$ and $\mathscr{C}(\overline{N})$, respectively. We define a retraction $\mathrm{Ret} : \mathscr{C}(\overline{N}) \to \mathscr{C}(N)$ as follows: for any $\alpha \in \mathscr{C}(\overline{N})$, $\mathrm{Ret}(\alpha)$ is a homotopy class of α in $\mathscr{C}(N)$. Then Ret is 1-Lipschitz. We also define a section $\mathrm{Sec} : \mathscr{C}(N) \to \mathscr{C}(\overline{N})$ by choosing a hyperbolic metric on N, realizing curves as geodesics, and adding a puncture outside the union of the curves. Note that the composition $\mathrm{Ret} \circ \mathrm{Sec}$ is identity on $\mathscr{C}(N)$. Let $T = abd$ be any geodesic triangle in $\mathscr{C}(N)$, where a, b, and d are vertices of $\mathscr{C}(N)$. Since Sec is an embedding, $\mathrm{Sec}(T) = T$ has a 17-center $q \in \mathscr{C}^{(0)}(\overline{N})$ in $\mathscr{C}(\overline{N})$. Let ab, bd and da be the sides of T connecting a and b, b and d, and d and a in $\mathscr{C}(N)$. Then, for ab, we obtain

$$\begin{aligned} d_{\mathscr{C}(N)}(\mathrm{Ret}(q), ab) &= d_{\mathscr{C}(N)}(\mathrm{Ret}(q), (\mathrm{Ret} \circ \mathrm{Sec}(a))(\mathrm{Ret} \circ \mathrm{Sec}(b))) \\ &\leq d_{\mathscr{C}(\overline{N})}(q, \mathrm{Sec}(a)\ \mathrm{Sec}(b)) \\ &\leq 17. \end{aligned}$$

Here $(\mathrm{Ret} \circ \mathrm{Sec}(a))(\mathrm{Ret} \circ \mathrm{Sec}(b))$ is a geodesic in $\mathscr{C}(N)$ connecting $\mathrm{Ret} \circ \mathrm{Sec}(a)$ and $\mathrm{Ret} \circ \mathrm{Sec}(b)$, and $\mathrm{Sec}(a)\ \mathrm{Sec}(b)$ is a geodesic in $\mathscr{C}(\overline{N})$ connecting $\mathrm{Sec}(a)$ and $\mathrm{Sec}(b)$. For bd and da, we can show the same results that we showed for ab. Hence, $\mathrm{Ret}(q)$ is a 17-center of T in $\mathscr{C}(N)$. □

6. Arc-curve graphs are uniformly hyperbolic

Similarly to Propositions 4 and 7, we can prove the following.

Proposition 8. *Let a be a vertex of $\mathscr{C}(N)$, b a vertex of $\mathscr{A}(N)$, and $\bar{a}$ a vertex of $\mathscr{A}(N)$ which is adjacent to a in $\mathscr{AC}(N)$. Let $\mathscr{G} = ab$ be a geodesic connecting a and b in $\mathscr{AC}(N)$. Then any unicorn arc $\bar{c} \in \mathscr{P} \in P(\bar{a}, b)$ is at distance ≤ 7 from $\mathscr{G}$.*

Proof (Proof of Theorem 3). First we assume $F = N$. Fix any geodesic triangle $T = abd$ in $\mathscr{AC}(N)$, where a, b and d are vertices of $\mathscr{AC}(N)$. If $a, b, d \in \mathscr{A}^{(0)}(N)$, then T has a 7-center in $\mathscr{A}(N)$ by Theorem 1. Hence T has a 7-center in $\mathscr{AC}(N)$. If $a, b, d \in \mathscr{C}^{(0)}(N)$, then T has a 9-center in $\mathscr{AC}(N)$ by the proof of Theorem 1. If $a \in \mathscr{C}^{(0)}(N)$ and $b, d \in \mathscr{A}^{(0)}(N)$, then we take $\bar{a} \in \mathscr{A}^{(0)}(N)$ which is adjacent to a in $\mathscr{AC}(N)$. By Lemma 2, for $\bar{a}, b, d \in \mathscr{A}^{(0)}(N)$, there exist $c_{\bar{a}b} \in \mathscr{P}(\bar{a}^{\alpha}, b^{\beta})$, $c_{bd} \in \mathscr{P}(b^{\beta}, d^{\delta})$, and $c_{d\bar{a}} \in \mathscr{P}(d^{\delta}, \bar{a}^{\alpha})$ such that each pair represents adjacent vertices of $\mathscr{A}(N)$. Then $c_{\bar{a}b}$ is an 8-center of T in $\mathscr{AC}(N)$ by Proposition 6. If $a, b \in \mathscr{C}^{(0)}(N)$ and $d \in \mathscr{A}^{(0)}(N)$, then T also has an 8-center in $\mathscr{AC}(N)$. From the above four cases, $\mathscr{AC}(N)$ is 9-hyperbolic. Second we assume that $F = S$. Fix any geodesic triangle $T = abd$ in $\mathscr{AC}(S)$, where a, b and d are vertices of $\mathscr{AC}(S)$. If $a, b, d \in \mathscr{A}^{(0)}(S)$ or $a, b, d \in \mathscr{C}^{(0)}(S)$, then T has a 7-center or a 9-center in $\mathscr{AC}(S)$, respectively, by the proofs of [5, Theorems 1.1 and 1.2]. If $a \in \mathscr{C}^{(0)}(S)$ and $b, d \in \mathscr{A}^{(0)}(S)$, or $a, b \in \mathscr{C}^{(0)}(S)$ and $d \in \mathscr{A}^{(0)}(S)$, then we can show that T has an 8-center in $\mathscr{AC}(S)$ by the same argument that we gave in the proof of Theorem 3 (see the same cases in the proof of Theorem 3). Therefore $\mathscr{AC}(S)$ is 9-hyperbolic. $\square$

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