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Free involutions on torus semi-bundles and the Borsuk-Ulam Theorem for maps into $\mathbf{R}^n$

Alexandre Paiva Barreto, Daciberg Lima Gonçalves and Daniel Vendrúscolo

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Abstract. In this article we classify the free involutions of every torus semi-bundle Sol 3-manifold. Moreover, we classify all the triples $(M, \tau; \mathbf{R}^n)$, where M is as above, τ is a free involution on M, and n is a positive integer, for which the Borsuk-Ulam Property holds.

1. Introduction

The classical Borsuk-Ulam Theorem states that, for any continuous map $f : S^n \to \mathbf{R}^n$, there exists a point $x \in S^n$ such that $f(x) = f(-x)$. This theorem has motivated the following quite natural and general question. Given a topological space M, a free involution τ on M, and a positive integer n, we say that the *Borsuk-Ulam Property* holds for the triple $(M, \tau; \mathbf{R}^n)$ (or the triple $(M, \tau; \mathbf{R}^n)$ satisfies the *Borsuk-Ulam Property*), if for any continuous map $f : M \to \mathbf{R}^n$, there exists a point $x \in X$ such that $f(x) = f(\tau(x))$. The question consists in classifying the triples $(M, \tau; \mathbf{R}^n)$ for which the Borsuk-Ulam Property holds.

The above question has been studied by several authors, see for example [1, 3, 4, 5, 7] among others. The classification of free involutions of a space is closely related to the above question and it is a problem in its own right. The classification of free involutions together with the study of the Borsuk-Ulam Property for many Seifert 3-manifolds can be found in [7], [1] and [2].

The family of torus semi-bundles (see [11]), also called sapphire manifolds in [9], contains a subfamily consisting of those manifolds which admit Sol geometry. We call the manifolds of this subfamily *sapphire Sol manifolds*.

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The purpose of this work is to classify all free involutions on each sapphire Sol manifold M, up to an equivalence relation (see section 3), as well as the triples $(M, \tau; \mathbf{R}^n)$ which satisfy the Borsuk-Ulam Property.

The main results are:

Theorem 1. *For any sapphire Sol manifold M and free involution τ on M, the triple triple $(M, \tau; \mathbf{R}^2)$ has the Borsuk-Ulam Property.*

Theorem 2. *Given a sapphire Sol manifold S_A with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have:*

I) *If c is odd, the manifold does not admit involutions.*

II) *If b and c are even, we have two cases:*

II-a) *If $|a| \neq |d|$, then the manifold admits a unique class of free involutions and the quotient is homeomorphic to the sapphire Sol manifold S_B with $B = \begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}$.*

II-b) *If $|a| = |d|$, then the manifold admits three distinct classes of free involutions and the quotients are the sapphire Sol manifolds S_B where B runs over $\begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}$, $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ and $\begin{pmatrix} s & r \\ u & t \end{pmatrix}$. Here (r, s, t, u) is one of the solutions given by Proposition 4.*

III) *If b is odd and c is even, then the manifold admits a unique class of free involutions and the quotient is homeomorphic to the sapphire Sol manifold S_B with $B = \begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}$.*

By applying a result of J. Hillman [8], we obtain the following corollary.

Corollary 1. *Given a sapphire Sol manifold S_A with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the Borsuk-Ulam Property holds for the triple $(S_A, \tau; \mathbf{R}^3)$ if and only if c is even and b is odd. Furthermore, in this case the manifold admits only one class of free involutions.*

This paper contains three sections besides this introduction.

In Section 2, we recall the definition and a classification of the torus semi-bundles (also called sapphires), in particular, those which are not torus bundles. The former are classified in terms of certain integral matrices in $GL(2, \mathbf{Z})$ by a result of K. Morimoto in [9].

In Section 3, we determine all double coverings of a given sapphire Sol manifold in terms of the classification given by Morimoto [9]. Then we obtain the equivalence classes of the double coverings.

In Section 4, we provide the classification of free involutions and we give the classification of all triples $(M, \tau; \mathbf{R}^2)$ and $(M, \tau; \mathbf{R}^3)$ for which the Borsuk-Ulam Property holds. The latter case uses a recent result by J. Hillman [8].

A great amount of information about involutions (not necessarily free) on torus bundles is given in the work of Sakuma [10]. His work might be useful to study similar questions for torus bundles. This work is in progress.

2. Sapphire Sol manifolds

The Sol geometry is one of the eight geometries mentioned in Thurston's geometrization conjecture. Using [11] we have that these manifolds can be divided into two disjoint subfamilies:

a) Torus bundles where the gluing map is an Anosov map;

b) The subfamily of the torus semi-bundle which admit Sol geometry (see below, or [11] in more detail, the definition of torus semi-bundles, which are also called sapphires in [9]).

A torus semi-bundle (sapphire) is obtained by gluing two orientable twisted bundles over the Klein bottle. We will describe them following Morimoto [9]. For $i = 1, 2$, let K_iI be two copies of the same orientable twisted I-bundles over the Klein bottle with $\pi_1(\partial K_iI) \cong \langle \alpha_i, \beta_i \mid \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1} \rangle$. The generators α_i, β_i of the fundamental group of the torus ∂K_iI can be seen as the fibers of two Seifert fibrations of K_iI. Namely, α_i is an oriented fiber of a Seifert fibration with orbit manifold a disk with two exceptional points and β_i is an oriented fiber of a Seifert fibration with orbit manifold a Möbius band without exceptional points.

Let $A = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ be an element of $GL_2(\mathbf{Z})$ and $\phi : \partial K_2I \to \partial K_1I$ be a homeomorphism that induces isomorphism $A = \phi_\# : \pi_1(\partial K_2I) \to \pi_1(\partial K_1I)$ such that $\phi_\#(\alpha_2) = r\alpha_1 + s\beta_1$ and $\phi_\#(\beta_2) = t\alpha_1 + u\beta_1$. By identifying $x \in \partial K_2I$ with $\phi(x) \in \partial K_1I$, we obtain an orientable 3-manifold $S_A = K_1I \cup_\phi K_2I$. By Proposition 1.5 in [11] (or Remark 1.8 in [9]) S_A is a Sol manifold if and only if all entries of A are nonzero. We call such a sapphire manifold S_A a *sapphire Sol manifold.*

A presentation of the fundamental group of S_A is given by [9]:

$$\pi_1(S_A) \cong \langle a, b, c \mid aba^{-1}b,\ c^2a^{-2r}b^{-s},\ ca^{2t}b^uc^{-1}a^{2t}b^u \rangle. \tag{1}$$

So, we can also compute the first homology group of S_A.

In [9] we also find when two gluing matrices produce the same manifold up to homeomorphisms. The topological types of sapphire manifolds are given by:

Theorem 3 (Morimoto). *Let A, A' be two elements of $GL(2,\mathbf{Z})$. Then S_A is homeomorphic to $S_{A'}$ if and only if A' is equal to one of the following matrices:* $\pm A^{\pm}$, $\pm BA^{\pm}$ $\pm A^{\pm}B$, $\pm BA^{\pm}B$, *where* $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let S_A be a sapphire manifold, using the classification given by the above theorem, we may suppose that $A = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ is such that all four numbers r, s, t and u are positive, and moreover $r \leq u$.

The first homology group with integral coefficients of a sapphire may be obtained from the above presentation, and it is explicitly given in [9, Proposition 1.6].

The family of closed sapphire Sol manifolds can be divided into two subfamilies. In the first subfamily, we consider those with first integral homology isomorphic to $\mathbf{Z}_{4t} + \mathbf{Z}_4$. This corresponds to the case where s is odd in the gluing map. In the second subfamily, we consider those with first integral homology isomorphic to $\mathbf{Z}_{4t} + \mathbf{Z}_2 + \mathbf{Z}_2$. This corresponds to the case where s is even in the gluing map.

3. Classification of the double coverings of sapphire Sol manifolds

In this section, we will study double coverings of a sapphire Sol manifold S_A, with $A = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$. We will describe double coverings of those manifolds using the kernels of epimorphisms from $\pi_1(S_A)$ to $\mathbf{Z}_2$. In terms of the presentation given in Section 1, we have to describe the images of a, b and c, so we will have at most seven cases.

We will see in Proposition 1 that for the purpose of classification of free involutions on sapphire Sol manifolds it will suffice to know the double coverings associated to three epimorphisms φ_1, φ_2 and φ_3 introduced below. We will present some of the details on the calculations of these three cases and some comments about one of the others.

Case I: $\varphi_1(a) = \overline{1}$, $\varphi_1(b) = \overline{0}$, $\varphi_1(c) = \overline{0}$. Making use of the Reidemeister-Schreier method, taking $\{1, a\}$ as a Schreier system for right cosets of the kernel (H) of φ_1, we obtain $\{b, c, a^2, aba^{-1}, aca^{-1}\}$ as a system of generators and the following relations:

- $aba^{-1}b$;
- $c^2a^{-2r}b^{-s}$;
- $ca^{2t}b^u$;
- $ba^{-1}ba$;

- $a^{-1}c^2a^{-2r}b^{-s}a$;
- $a^{-1}ca^{2t}b^uc^{-1}a^{2t}b^ua$.

Now we make $\alpha = b,\ \beta = c,\ \gamma = a^2,\ \delta = aba^{-1},\ \lambda = aca^{-1}$ and we obtain

$$H = \left\langle \begin{matrix} \alpha,\beta,\gamma,\delta,\lambda \mid \delta\alpha,\ \beta^2\gamma^{-r}\alpha^{-s},\ \beta\gamma^t\alpha^u\beta^{-1}\gamma^t\alpha^u,\ \alpha\gamma^{-1}\delta\gamma, \\ \lambda^2\gamma^{-r}\delta^{-s},\ \lambda\gamma^t\delta^u\lambda^{-1}\gamma^t\delta^u \end{matrix} \right\rangle.$$

Setting $a_1 = \beta,\ b_1 = \alpha^u\gamma^t$ and $c_1 = \lambda$ we have:

$$H = \langle a_1, b_1, c_1 \mid a_1b_1a_1^{-1}b_1,\ c_1^2a_1^{-2R}b_1^S,\ c_1a_1^{2T}b_1^Uc_1^{-1}a_1^{2T}b_1^U \rangle,$$

where

- $R = ru + st,\ S = -2rs,\ T = 2tu,\ U = -(ru + st)$ if $ru - st = 1$;
- $R = -(ru + st),\ S = 2rs,\ T = -2tu,\ U = ru + st$ if $ru - st = -1$.

So the double covering defined by φ_1 is the sapphire Sol manifold S_B where B is given as follows:

$$\begin{pmatrix} ru + st & -2rs \\ 2tu & -(ru + st) \end{pmatrix} \qquad \text{if } ru - st = 1,$$

$$\begin{pmatrix} -(ru + st) & 2rs \\ -2tu & ru + st \end{pmatrix} \qquad \text{if } ru - st = -1.$$

Case II: $\varphi_2(a) = \overline{0},\ \varphi_2(b) = \overline{0},\ \varphi_2(c) = \overline{1}$. Taking $\{1, c\}$ as a Schreier system for right cosets of the kernel (H) of φ_2, we obtain $\{a, b, cac^{-1}, cbc^{-1}, c^2\}$ as a system of generators, and the following relations:

- $aba^{-1}b$;
- $c^2a^{-2r}b^{-s}$;
- $ca^{2t}b^uc^{-1}a^{2t}b^u$;
- $c^{-1}aba^{-1}bc$;
- $ca^{-2r}b^{-s}c$;
- $a^{2t}b^uc^{-1}a^{2t}b^uc$.

Taking $\alpha = a,\ \beta = b,\ \gamma = cac^{-1},\ \delta = cbc^{-1},\ \lambda = c^2$, we obtain

$$H = \left\langle \begin{matrix} \alpha,\beta,\gamma,\delta,\lambda \mid \alpha\beta\alpha^{-1}\beta,\ \lambda\alpha^{-2r}\beta^{-s},\ \gamma^{2t}\delta^u\alpha^{2t}\beta^u,\ \gamma\delta\gamma^{-1}\delta \\ \gamma^{-2r}\delta^{-s}\lambda,\ \alpha^{2t}\beta^u\lambda^{-1}\gamma^{2t}\delta^u\lambda \end{matrix} \right\rangle.$$

Setting $a_1 = \alpha,\ b_1 = \beta$ and $c_1 = \gamma$, we have:

$$H = \langle a_1, b_1, c_1 \mid a_1b_1a_1^{-1}b_1,\ c_1^2a_1^{-2R}b_1^S,\ c_1a_1^{2T}b_1^Uc_1^{-1}a_1^{2T}b_1^U \rangle,$$

where

- $R = ru + st,\ S = 2su,\ T = -2rt,\ U = -(ru + st)$ if $ru - st = 1$;
- $R = -(ru + st),\ S = -2su,\ T = 2rt,\ U = ru + st$ if $ru - st = -1$.

Thus, the double covering defined by φ_2 is the sapphire Sol manifold S_B where B is given as follows:

$$\begin{pmatrix} ru+st & 2su \\ -2rt & -(ru+st) \end{pmatrix} \quad \text{if } ru-st=1,$$

$$\begin{pmatrix} -(ru+st) & -2su \\ 2rt & ru+st \end{pmatrix} \quad \text{if } ru-st=-1.$$

Case III: $\varphi_3(a)=\bar{0}$, $\varphi_3(b)=\bar{1}$, $\varphi_3(c)=\bar{0}$. First of all, we note that s must be even and u odd. This happens not only in this case but in all other cases with nontrivial image of b. Therefore, we denote $s=2k$ and $u=2l-1$. Taking $\{1,b\}$ as a Schreier system for right cosets of the kernel (H) of φ_3, we obtain $\{a,c,bab^{-1},b^2,bcb^{-1}\}$ as a system of generators and the following relations:

- $aba^{-1}b$;
- $c^2a^{-2r}b^{-s}$;
- $ca^{2t}b^uc^{-1}a^{2t}b^u$;
- $b^{-1}aba^{-1}b^2$;
- $b^{-1}c^2a^{-2r}b^{-s+1}$;
- $b^{-1}ca^{2t}b^uc^{-1}a^{2t}b^{u+1}$.

Taking $\alpha=a$, $\beta=c$, $\gamma=bab^{-1}$, $\delta=b^2$, $\lambda=bcb^{-1}$, we obtain

$$H=\left\langle \begin{matrix} \alpha,\beta,\gamma,\delta,\lambda \mid \alpha\gamma^{-1}\delta,\ \beta^2\alpha^{-2r}\delta^{-k},\ \beta\alpha^{2t}\delta^{l-1}\lambda^{-1}\gamma^{2t}\delta^l,\ \gamma\delta\alpha^{-1} \\ \lambda^2\gamma^{-2r}\delta^{-k},\ \lambda\gamma^{2t}\delta^l\beta^{-1}\alpha^{2t}\delta^{l-1} \end{matrix} \right\rangle.$$

Setting $a_1=\alpha$, $b_1=\delta$ and $c_1=\lambda$ we will have:

$$H=\langle a_1,b_1,c_1 \mid a_1b_1a_1^{-1}b_1,\ c_1^2a_1^{-2R}b_1^S,\ c_1a_1^{2T}b_1^Uc_1^{-1}a_1^{2T}b_1^U\rangle,$$

where $R=r$, $S=k=\dfrac{s}{2}$, $T=2t$ and $U=u$.

So the double covering defined by φ_3 is the sapphire Sol manifold S_B with $B=\begin{pmatrix} r & \frac{s}{2} \\ 2t & u \end{pmatrix}$.

Case IV: $\varphi_4(a)=\bar{1}$, $\varphi_4(b)=\bar{0}$, $\varphi_4(c)=\bar{1}$. This case was solved in [6]. The double covering is a torus bundle with Anosov gluing map given by the matrix:

$$\begin{pmatrix} ru+ts & -2rt \\ -2su & ru+ts \end{pmatrix},$$

if $\det A=1$. It is not hard to see, using similar arguments that in case $\det A=-1$ the gluing map is given by minus the matrix above. Recall from

[6] that the fundamental group of this double covering corresponds to the subgroup generated by $\{a^2, b, a^{-1}c\}$.

We will not present the calculation of the remaining three cases. Nevertheless we observe that the calculations are similar to the case III, and for the sake of completeness we provide the result for all cases in the Table 1 below.

If S_A is the sapphire Sol manifold with $A = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$, it follows from the calculations that one double covering of S_A is a torus bundle and the others are sapphire Sol manifolds. In both cases, the manifold is determined by a matrix (in general not unique) which will appear in the table. For simplicity, in the case that the double covering is a sapphire Sol manifold, we used the classification given by Morimoto [9, Theorem 1] to choose the matrices. We recall that we are assuming r, s, t and u are positive integers and $r \leq u$.

Table 1. Double coverings of a sapphire Sol manifold

Case	Hom.	Sapphire	Sol torus bundle
I	$\varphi_1 \begin{cases} a \mapsto \bar{1} \\ b \mapsto \bar{0} \\ c \mapsto \bar{0} \end{cases}$	$\begin{pmatrix} ru+st & 2rs \\ 2tu & ru+st \end{pmatrix}$	—
II	$\varphi_2 \begin{cases} a \mapsto \bar{0} \\ b \mapsto \bar{0} \\ c \mapsto \bar{1} \end{cases}$	$\begin{pmatrix} ru+st & 2su \\ 2rt & ru+st \end{pmatrix}$	—
III	$\varphi_3 \begin{cases} a \mapsto \bar{0} \\ b \mapsto \bar{1} \\ c \mapsto \bar{0} \end{cases}$ (s even)	$\begin{pmatrix} r & \frac{s}{2} \\ 2t & u \end{pmatrix}$	—
IV	$\varphi_4 \begin{cases} a \mapsto \bar{1} \\ b \mapsto \bar{0} \\ c \mapsto \bar{1} \end{cases}$	—	$\begin{pmatrix} ru+ts & -2rt \\ -2su & ru+ts \end{pmatrix}$
V	$\varphi_5 \begin{cases} a \mapsto \bar{1} \\ b \mapsto \bar{1} \\ c \mapsto \bar{0} \end{cases}$ (s even)	$\begin{pmatrix} r & \frac{s}{2} \\ 2t & u \end{pmatrix}$	—
VI	$\varphi_6 \begin{cases} a \mapsto \bar{0} \\ b \mapsto \bar{1} \\ c \mapsto \bar{1} \end{cases}$ (s even)	$\begin{pmatrix} r & \frac{s}{2} \\ 2t & u \end{pmatrix}$	—
VII	$\varphi_7 \begin{cases} a \mapsto \bar{1} \\ b \mapsto \bar{1} \\ c \mapsto \bar{1} \end{cases}$ (s even)	$\begin{pmatrix} r & \frac{s}{2} \\ 2t & u \end{pmatrix}$	—

Definition 1 [7, Corollary 2.3]. Two classes $x_1 \in H^1(W_1, \mathbf{Z}/2\mathbf{Z})$, $x_2 \in H^1(W_2, \mathbf{Z}/2\mathbf{Z})$ are equivalent if there is a homotopy equivalence $h : W_1 \to W_2$ such that the induced homomorphism by h on H^1 maps x_2 to x_1.

For the sapphire Sol manifold S_A, the mapping class group is isomorphic to the outer-automorphism group of the fundamental group. Therefore, two cohomology classes φ and φ' in $H^1(S_A, \mathbf{Z}/2\mathbf{Z})$ are equivalent if and only if there is an automorphism, θ, of $\pi_1(S_A)$ such that $\varphi' = \varphi \circ \theta$.

The above equivalence relation is closely related to an equivalence relation of involutions which is given at the beginning of the next section.

Now we classify the cohomololgy classes corresponding to the epimorphisms in the table by the equivalence relation.

By considering the isomorphism which sends $a \to ab$, $b \to b$ and $c \to c$, we obtain that case III is equivalent to case V, and case VI is equivalent to case VII. The following proposition shows that all of these cases are in fact equivalent.

Proposition 1. *The cases* III, V, VI *and* VII *are equivalent.*

Proof. We know by the above considerations that case III is equivalent to case V and case VI is equivalent to case VII. So, to prove the proposition, it suffices to show that cases III and VI are also equivalent. To do this, we need to construct an automorphism $\theta : \pi_1(S_A) \to \pi_1(S_A)$, where s is even, such that $\theta(\varphi_3) = \varphi_6$.

Recall the presentation

$$\pi_1(S_A) \cong \langle a, b, c \mid aba^{-1}b,\ c^2a^{-2r}b^{-s},\ ca^{2t}b^uc^{-1}a^{2t}b^u \rangle.$$

We first assume that $\det A = 1$. We will find an element $w \in \langle a^2, b \rangle \cong \mathbf{Z} \oplus \mathbf{Z}$ such that (i) the exponent of b in w is odd and (ii) there is an automorphism θ of $\pi_1(S_A)$ such that $(\theta(a), \theta(b), \theta(c)) = (a, b, wc)$. It is easy to see that $\theta(\varphi_3) = \varphi_6$ so this automorphism has the desired property.

Let us find w such that θ preserves the relations. The first relation $aba^{-1}b = 1$ is clearly preserved by θ. The third relation $ca^{2t}b^uc^{-1}a^{2t}b^u = 1$ is also preserved since the image of the relation is $wca^{2t}b^uc^{-1}w^{-1}a^{2t}b^u = w(a^{2t}b^u)^{-1}w^{-1}a^{2t}b^u$. But the subgroup $\langle a^2, b \rangle$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ and it follows that $w(a^{2t}b^u)^{-1}w^{-1}a^{2t}b^u = 1$. Thus, it suffices to find w with the above properties and the relation $c^2 = b^sa^{2r}$ preserved by θ. Note that this condition is equivalent to the condition $(wc)^2 = wcwc^{-1}c^2 = b^sa^{2r}$, which in turn is equivalent to the condition $cwc^{-1} = w^{-1}b^sa^{2r}c^{-2} = w^{-1}b^sa^{2r}a^{-2r}b^{-s} = w^{-1}$. Hence it suffices to find w an eigenvector for the eigenvalue -1 (for the action of c by conjugation on the subgroup $\langle a^2, b \rangle$) such that the exponent of b is odd. From [6], we know that the matrix of the automorphism of the subgroup $\langle a^2, b\rangle$ given by conjugation by $a^{-1}c$ is the matrix

$$\begin{pmatrix} ru+ts & -2rt \\ -2su & ru+ts \end{pmatrix}.$$

So the matrix of the automorphism given by conjugation by c is

$$\begin{pmatrix} ru+ts & -2rt \\ 2su & -ru-ts \end{pmatrix}.$$

Now we will show that this integral matrix has eigenvalue -1 and a corresponding eigenvector where the exponent of b is odd. The first part follows because the determinant of the matrix

$$\begin{pmatrix} ru+ts+1 & -2rt \\ 2su & -ru-ts+1 \end{pmatrix}$$

is zero, so we have an eigenvector for the eigenvalue -1. It remains to show that there is one such eigenvector with the exponent of b odd. Let (x, y) be an eigenvector. We have

$$(ru+ts+1)x - (2rt)y = 0.$$

But $ru+ts+1 = (ru-ts+1)+2ts = 2+2ts$, where the last equality follows from $\det A = 1$.

So we obtain $(1+st)x = rty$, where $1+st$ is odd since s is even. Therefore there is a solution with y odd and the result follows.

If $\det A = -1$, $A = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$, we can chose $A' = \begin{pmatrix} -r & s \\ -t & u \end{pmatrix}$ that determines the same sapphire manifold (by Morimoto's classification) with $\det A' = 1$. Using variables a, b and c to describe the presentation of $\pi_1(S_A)$ and x, y and z for $\pi_1(S_{A'})$, we see that the isomorphism γ that sends $a \to x^{-1}$, $b \to y$ and $c \to z$ is such that the composition $\gamma^{-1}\varphi\gamma$ sends φ_3 to φ_6 for the presentation of $\pi_1(S_A)$ and the result follows. □

Now we will see a situation where cases I and II are equivalent.

Proposition 2. *In the case that A is of the form $\begin{pmatrix} r & s \\ t & r \end{pmatrix}$, there is an automorphism θ of $\pi_1(S_A)$ such that $\theta(\varphi_1) = \varphi_2$, i.e. case* I *and case* II *are equivalent.*

Proof. Let us assume that $\det A = 1$. We write a new presentation for $\pi_1(S_A)$, namely

$$\pi_1(S_A) = \langle d, b, v, a \mid a^2d^{-1},\ aba^{-1}b,\ avavb^{-s}d^{-r},\ avd^tb^rv^{-1}a^{-1}b^rd^t\rangle.$$

The above presentation is obtained from the presentation given in Section 2 by introducing the variable $d = a^2$ and setting $v = a^{-1}c$. Now we try to find an automorphism θ of the fundamental group such that $\theta(a) = va$, $\theta(v) = d^x b^y v$, $\theta(d) = d^k b^m$, $\theta(d) = d^l b^n$. If such an automorphism θ exists, then we will have $\theta(\varphi_1) = \varphi_2$. We must find integers x, y, k, l, m, n such that the map above extends to an automorphism of the group.

By straightforward calculation, using the relations $a^2 = d$, $aba^{-1} = b^{-1}$, we conclude that $k = r$, $m = -s$, $l = -\varepsilon t$, $n = \varepsilon r$, where ε is either 1 or -1. Because θ restricted to the index two subgroup given by the kernel of φ_4 is also an automorphism, the solution must be $k = r$, $m = -s$, $l = -t$, $n = r$.

We need to verify two more relations. It is staightforward to verify that independently of x and y the relation $avd^t b^r v^{-1} a^{-1} = d^{-t} b^{-r}$ holds after applying θ using the values of k, l, m, n already found. Thus, it remains to find x, y so that $\theta(a)\theta(v)\theta(a)\theta(v) = \theta(d^r b^s)$ or $vad^x b^y vvad^x b^y v = d$. The latter equation can be rewritten as $(vava)a^{-1}(v^{-1}d^x b^y v)aa^{-1}(vava)aa^{-2}v^{-1}d^x b^y v = d$. Using the action of v on the subgroup $\langle d, b\rangle$ (see the proof of the Proposition 1) we obtain the following equation

$$(r^2 + st)x + (2rt)y = 1 - r.$$

Since $r^2 + st$, $2rt$ are relatively prime integers it follows that there is a linear combination with integral coefficients of these two numbers equal 1. Therefore the equation above has solution over the integers and the result follows.

The case when $\det A = -1$ is similar where the matrix which gives the action of v on $\langle d, b\rangle$ is given by minus the matrix used for the case above. We leave the details to the reader. This concludes the proof. $\square$

So we obtain the following result.

Proposition 3. *For the epimorphisms φ_i given by the Table 1, we have:*

(1) *The subset $\{\varphi_3, \varphi_5, \varphi_6, \varphi_7\}$ (when it exists) forms a single equivalence class.*

(2) *The subset $\{\varphi_4\}$ forms a single equivalence class.*

(3) *if $r = u$, then φ_1 and φ_2 are equivalent, namely $\{\varphi_1, \varphi_2\}$ forms a single equivalent class. If $r \neq u$, then φ_1 is not equivalent to φ_2, namely $\{\varphi_1\}$ and $\{\varphi_2\}$ are two distinct equivalence classes.*

Proof. Part 1) follows from Proposition 1.

Observe that if two pairs (S_A, α), (S_A, β) are equivalent, then the corresponding double coverings are homeomorphic. Part 2) follows from the fact that the double covering which corresponds to the case $\{(S_A, \varphi_4)\}$ is a torus bundle, so it cannot be equivalent to any other case.

Part 3) If $r \neq u$ the double coverings corresponding to the two cases are not homeomorphic. In the case $r = u$ we use Proposition 2. Thus, the result follows. $\square$

4. Classification of free involutions and the Borsuk-Ulam Theorem

In this section we classify the involutions of the sapphire Sol manifolds and, for every free involution τ of S_A, we compute the values of n for which the triple $(S_A, \tau; \mathbf{R}^n)$ satisfies the Borsuk-Ulam Property.

Given a free involution τ on a manifold M, the projection $M \to M/\tau$ is a double covering of the orbit space M/τ, which in turn defines an element of $\mathrm{Hom}(\pi_1(M/\tau), \mathbf{Z}/2\mathbf{Z}) \in H^1(M/\tau, Z/2\mathbf{Z})$ called *characteristic class of the involution* τ.

We say that free involutions τ_1 on M_1 and τ_2 on M_2 are *equivalent* if there is a homeomorphism $f : M_1 \to M_2$ satisfying $\tau_2 = f\tau_1 f^{-1}$. Note that this condition is satisfied if and only if the correspondent characteristic classes are equivalent (see Definition 1). For more about the relation between involutions and characteristic classes, see [7].

We first prove Theorem 1 by using a general fact about the Borsuk-Ulam Property which is independent of the free involution.

Proof of Theorem 1. By the comments at the end of Section 2, $H_1(S_A, \mathbf{Z})$ is finite. The result follows immediately from [1, Corollary 3.3]. $\square$

We also observe that for any 3-dimensional manifold M it follows from [7, Lemma 2.4] that for any free involution τ on M, the triple $(M, \tau; \mathbf{R}^n)$ with $n > 3$ does note have the Borsuk-Ulam Property. Thus, it remains to study the case $n = 3$.

Let S_A be a sapphire Sol manifold. Using Table 1, we may detect whether such a manifold is a double covering of another sapphire Sol manifold.

In case I, we must have $a = d$, a is odd, b and c are even. The following proposition shows that these conditions are sufficient to guarantee that S_A is the double covering of some sapphire Sol manifold which corresponds to the kernel of the epimorphism φ_1. Here we have $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

To simplify our exposition, let us fix the following notation. Given an integer x and a prime p, denote by $|x|_p$ the largest integer n such that p^n divides x. If p does not divide x then we define $|x|_p$ to be zero.

Proposition 4. *If $a = d$, a is odd, b and c are even, then S_A is the double covering of two sapphire Sol manifolds S_B, with $B = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ associated to φ_1,* *i.e. (see Table 1) the system*

$$a = ru + ts, \qquad b = 2rs, \qquad c = 2tu \qquad \text{and} \qquad d = ru + st \tag{2}$$

admits 2 solutions with r, s, t and u positive. More precisely, B is one of the following matrices:

$$\begin{pmatrix} \prod_{j=1}^{n} q_j^{|b/2|_{q_j}} & \prod_{i=1}^{m} p_i^{|b/2|_{p_i}} \\ \prod_{i=1}^{m} p_i^{|c/2|_{p_i}} & \prod_{j=1}^{n} q_j^{|c/2|_{q_j}} \end{pmatrix} \qquad \begin{pmatrix} \prod_{i=1}^{m} p_i^{|b/2|_{p_i}} & \prod_{j=1}^{n} q_j^{|b/2|_{q_j}} \\ \prod_{j=1}^{n} q_j^{|c/2|_{q_j}} & \prod_{i=1}^{m} p_i^{|c/2|_{p_i}} \end{pmatrix}$$

where p_i runs over the set of all divisors of $(a+1)/2$, and q_i runs over the set of all divisors of $(a-1)/2$.

Proof. We can write A as

$$A = \begin{pmatrix} \alpha & 2\beta \\ 2\gamma & \alpha \end{pmatrix} \in M_{2\times 2}(\mathbf{Z} - \{0\}), \tag{3}$$

where $\alpha, \beta, \gamma \in \mathbf{N} - \{0\}$, α are odd, and $\det A = \pm 1$.

We will show that there are exactly two different homeomorphism classes of sapphire Sol manifolds that admit S_A as a double covering.

Suppose that A satisfies (3). Observe that

$$1 = \det A = \alpha^2 - 4\beta\gamma \Rightarrow \beta\gamma = \frac{\alpha+1}{2} \cdot \frac{\alpha-1}{2}.$$

So $\det A = \alpha^2 - 4\beta\gamma = 1$ (the case $\det A = -1$ is not possible because $\alpha = 2k+1 \Rightarrow \alpha^2 - 4\beta\gamma = 4(n^2 + n - \beta\gamma) + 1$).

If there exists a matrix B satisfying (2), we must have

$$rstu = \beta\gamma = \frac{\alpha+1}{2} \cdot \frac{\alpha-1}{2}. \tag{4}$$

Now we define $R = \frac{\alpha+1}{2}$, $S = \frac{\alpha-1}{2} \in \mathbf{Z}$. So $R, S \neq 0$ since

$$RS = rstu \neq 0$$

and

$$R = \prod_{i=1}^{m} p_i^{|R|_{p_i}}, \qquad S = \prod_{j=1}^{n} q_j^{|S|_{q_j}},$$

where p_i and q_j are positive primes and $p_i \neq q_j$, for all i and j (R and S are consecutive integers).

Let $x = st \in \mathbf{Z} - \{0\}$. It follows from (2) and (4) that

$$(\alpha - x)x = rux = RS$$

and so

$$x^2 - \alpha x + RS = 0.$$

An easy calculation shows that $x = R$ or $x = S$.

Suppose that $ts = x = R$ ($\Rightarrow ru = S$). In such a situation, we may write

$$t = \prod_{i=1}^{m} p_i^{|t|_{p_i}}, \qquad s = \prod_{i=1}^{m} p_i^{|s|_{p_i}}, \qquad r = \prod_{j=1}^{n} q_j^{|r|_{q_j}} \qquad \text{and} \qquad u = \prod_{j=1}^{n} q_j^{|u|_{q_j}},$$

where

$$|t|_{p_i} + |s|_{p_i} = |R|_{p_i} \qquad \text{and} \qquad |r|_{q_j} + |u|_{q_j} = |S|_{q_j}.$$

Using (2), we obtain

$$|r|_{q_j} = |\beta|_{q_j}, \qquad |s|_{p_i} = |\beta|_{p_i}, \qquad |t|_{p_i} = |\gamma|_{p_i} \qquad \text{and} \qquad |u|_{q_j} = |\gamma|_{q_j}.$$

Then we have

$$B = \begin{pmatrix} \prod_{j=1}^{n} q_j^{|\beta|_{q_j}} & \prod_{i=1}^{m} p_i^{|\beta|_{p_i}} \\ \prod_{i=1}^{m} p_i^{|\gamma|_{p_i}} & \prod_{j=1}^{n} q_j^{|\gamma|_{q_j}} \end{pmatrix}$$

Now, if $ts = x = S$ ($\Rightarrow ru = R$) we will have

$$t = \prod_{j=1}^{n} q_j^{|t|_{q_j}}, \qquad s = \prod_{j=1}^{n} q_j^{|s|_{q_j}}, \qquad r = \prod_{i=1}^{m} p_i^{|r|_{p_i}} \qquad \text{and} \qquad u = \prod_{i=1}^{m} p_i^{|u|_{p_i}},$$

where

$$|t|_{q_j} + |s|_{q_j} = |S|_{q_j} \qquad \text{and} \qquad |r|_{p_i} + |u|_{p_i} = |R|_{p_i}.$$

Again, using (2), we obtain

$$|r|_{p_i} = |\beta|_{p_i}, \qquad |s|_{q_j} = |\beta|_{q_j}, \qquad |t|_{q_j} = |\gamma|_{q_j} \qquad \text{and} \qquad |u|_{p_i} = |\gamma|_{p_i}.$$

Then

$$B = \begin{pmatrix} \prod_{i=1}^{m} p_i^{|\beta|_{p_i}} & \prod_{j=1}^{n} q_j^{|\beta|_{q_j}} \\ \prod_{j=1}^{n} q_j^{|\gamma|_{q_j}} & \prod_{i=1}^{m} p_i^{|\gamma|_{p_i}} \end{pmatrix}$$

and we get one possible solution:

$$B = \begin{pmatrix} \prod_{i=1}^{m} p_i^{|\beta|_{p_i}} & \prod_{j=1}^{n} q_j^{|\beta|_{q_j}} \\ \prod_{j=1}^{n} q_j^{|\gamma|_{q_j}} & \prod_{i=1}^{m} p_i^{|\gamma|_{p_i}} \end{pmatrix}.$$

Now observe that the matrix in the second case is not equivalent to the matrix obtained in the first case. □

We also have a similar proposition for the solution of double coverings of case II of the table. The system of equations is that given above where we simply replace r by u and u by r. Therefore the solution is as above where the rôles of r and u are interchanged. More precisely:

Proposition 5. *If $a = d$, a is odd, b and c are even, then S_A is the double covering of two sapphire Sol manifolds S_B, with $B = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ associated to φ_2, i.e. (see table 1) the system*

$$a = ru + ts, \qquad b = 2su, \qquad c = 2rt \qquad \textit{and} \qquad d = ru + st \tag{5}$$

admits 2 solutions with r, s, t and u positive. More precisely, B is one of the following matrices:

$$\begin{pmatrix} \prod_{j=1}^{n} q_j^{|c/2|_{q_j}} & \prod_{i=1}^{m} p_i^{|b/2|_{p_i}} \\ \prod_{i=1}^{m} p_i^{|c/2|_{p_i}} & \prod_{j=1}^{n} q_j^{|b/2|_{q_j}} \end{pmatrix} \qquad \begin{pmatrix} \prod_{i=1}^{m} p_i^{|c/2|_{p_i}} & \prod_{j=1}^{n} q_j^{|b/2|_{q_j}} \\ \prod_{j=1}^{n} q_j^{|c/2|_{q_j}} & \prod_{i=1}^{m} p_i^{|b/2|_{p_i}} \end{pmatrix}$$

where p_i runs over the set of all divisors of $(a+1)/2$ and q_i runs over the set of all divisors of $(a-1)/2$.

We note that in Proposition 4 if $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ is one of the solutions then the other is $\begin{pmatrix} y & x \\ w & z \end{pmatrix}$ and the two solutions given in Proposition 5 are $\begin{pmatrix} w & y \\ z & x \end{pmatrix}$ and $\begin{pmatrix} z & x \\ w & y \end{pmatrix}$. Thus, when $x \neq w$ and $y \neq z$, only two of them appear in the table above, in two different positions. In the case that $x = w$ or $y = z$ two of them appear in the table but one of them appears twice, in two different positions, namely in first and second lines. Observe that at most one of the equalities $x = w$, $y = z$ can occur.

We are now in a position to prove our main results, Theorem 2 and Corollary 1.

Proof of Theorem 2. Table 1 describes matrices of all double coverings of S_A. In fact, Table 1 can be constructed without the condition that all entries of the matrix are positive. In this case, if we apply the (new) Table 1 to another matrix B such that S_B is homeomorphic to S_A, the matrices obtained in this way will be representatives of the same manifolds (this correspondence may change its position in the table).

Part I) and III) follows easily from the table.

Part II) We start applying the table. Item II-a follows directly from Proposition 1, and item II-b is a consequence of Propositions 1, 4 and 5. □

Proof of Corollary 1. When c is odd, there is no free involution. For the other parts, it will be important to describe, among all epimorphisms, those which factor through $\pi_1(S_A)/\sqrt{\pi}$ (here $\sqrt{\pi}$ is the unique maximal abelian normal subgroup of $\pi_1(S_A)$, see [8] for details). They are precisely those where the generator b is sent to zero. We can conclude that φ_3, φ_5, φ_6 and φ_7 do not factor through $\pi_1(S_A)/\sqrt{\pi}$ and so the cube of the associated cohomology class is nontrivial precisely when the $(1,2)$-entry of the matrix A is divisible by 2 but not by 4.

Applying the result item 11 in Section 5 of [8], we can see that the cube of the cohomology class associated to φ_i is nontrivial only in part III of Theorem 2. □

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Alexandre Paiva Barreto
Departamento de Matemática
Universidade Federal de São Carlos
São Carlos, 13565-905, *Brazil*
E-mail: alexandre@dm.ufscar.br

Daciberg Lima Gonçalves
Departamento de Matemática
IME-USP, Caixa Postal 66.281
CEP 05314-970
São Paulo-SP, Brazil
E-mail: dlgoncal@ime.usp.br

Daniel Vendrúscolo
Departamento de Matemática
Universidade Federal de São Carlos
São Carlos, 13565-905, *Brazil*
E-mail: daniel@dm.ufscar.br