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# Higher level representation of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and its integrability

Rasha Mohamed Farghly

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**Abstract.** By using an elliptic analogue of the Drinfeld coproduct, we construct the level-$(k+1)$ representation of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ from the level-1 highest weight representation. The quantum $Z$-algebra of level-$(k+1)$ is realized. We also find the elliptic analogue of the condition of integrability for higher level modules constructed by the Drinfeld coproduct. This also enables us to express $\triangle^k(e(z))\triangle^k(e(zq^2))\dots\triangle^k(e(zq^{2(N-1)}))$ and $\triangle^k(f(z))\triangle^k(f(zq^{-2}))\dots\triangle^k(f(zq^{-2(N-1)}))$ as vertex operators of the level-$(k+1)$ bosons.

## 1. Introduction

Lepowsky and Primic [15] studied the condition of integrability of higher level representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. Ding and Feigin [1], Ding and Miwa [3] studied the quantum integrable condition and the $q$-parafermion of $U_q(\widehat{\mathfrak{sl}}_2)$ by using the Drinfeld coproduct [1] for the Drinfeld realization of $U_q(\widehat{\mathfrak{sl}}_2)$ [4]. The universal $R$ matrix $R_\infty$ associated with the Drinfeld coproduct is given in [2] for $U_q(\hat{\mathfrak{g}})$ for general untwisted affine Lie algebra $\hat{\mathfrak{g}}$. In [11, 8], Jimbo, Konno, Odake, Shiraishi gave an elliptic analogue $U_{q,p}(\hat{\mathfrak{g}})$ of the Drinfeld realization of $U_q(\hat{\mathfrak{g}})$. In particular in [8], the authors introduced the elliptic analogue of the Drinfeld coproduct for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Konno [13] defined the $H$-Hopf algebroid structure [5, 6, 10] of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ in term of the coproduct of the $L$-operator of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and defined the associated elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Farghly, Konno and Oshima [16] gave a new definition of $U_{q,p}(\hat{\mathfrak{g}})$ as a certain topological algebra over the ring of formal power series in $p$ and studied the dynamical quantum $Z$-algebra structure associated with the level-$k$ highest weight representation of $U_{q,p}(\hat{\mathfrak{g}})$. Also the authors constructed the induced $U_{q,p}(\hat{\mathfrak{g}})$-module from the dynamical quantum $Z$-module. The level-1 standard representations of $U_{q,p}(\hat{\mathfrak{g}})$ for $\hat{\mathfrak{g}} = A_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ and $B_l^{(1)}$ were also given. The purpose of this paper is to construct the higher level realization of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ from its standard level-1 realization

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[16] by using the elliptic Drinfeld coproduct [8, 14]. The higher level elliptic currents are expressed in term of the level-1 currents. In particular, we obtain the level-$(k+1)$ Heisenberg algebra, then we introduce the vertex operators $E^{\pm}_{(k)}(\alpha, z)$, $E^{\pm}_{(k)}(\alpha', z)$ and we define the level-$(k+1)$ quantum $Z$-operators from the level-$(k+1)$ elliptic currents. Also, we give the elliptic analogue of the quantum integrable condition for level-$(k+1)$ integrable module of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

This paper is organized as follows. In section 2, we define the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ in term of the elliptic Drinfeld generators. We use the Drinfeld coproduct to define the $H$-Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and formulate it as an elliptic quantum group. Also we recall the level-1 realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ following [16]. In section 3, we show a construction of the level-$(k+1)$ realization ($k \in \mathbf{Z}_{>0}$) of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ using the level-1 realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Also, we give a realization of the level-$(k+1)$ $Z$-algebra. In section 4, we present the elliptic analogue of quantum integrable condition for any level-$(k+1)$ integrable module of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

## 2. Elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

In this section we expose the definition of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and the level-1 realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ which we are going to use in the following sections.

**2.1. Definition of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ ([16]).** Let $\mathfrak{h} = \tilde{\mathfrak{h}} \oplus \mathbf{C}d$, $\tilde{\mathfrak{h}} = \mathbf{C}h \oplus \mathbf{C}c$ be the Cartan subalgebra of $\widehat{\mathfrak{sl}}_2$. Define $\delta, \Lambda_0, \alpha \in \mathfrak{h}^*$ by

$$\langle \alpha, h \rangle = 2, \qquad \langle \delta, d \rangle = 1 = \langle \Lambda_0, c \rangle, \tag{2.1}$$

the other pairings are 0. We also define $\bar{\Lambda}_1 \in \mathfrak{h}^*$ by

$$\langle \bar{\Lambda}_1, h \rangle = 1$$

We set $\tilde{\mathfrak{h}}^* = \mathbf{C}\Lambda_0 \oplus \mathbf{C}\bar{\Lambda}_1$, $\mathfrak{Q} = \mathbf{Z}\alpha$ and $\mathfrak{P} = \mathbf{Z}\bar{\Lambda}_1$.

We introduce another Heisenberg algebra generated by $P$ and $Q$ with the pairing $\langle P, Q \rangle = 1$. Now let us set $H = \tilde{\mathfrak{h}} \oplus \mathbf{C}P$ and denote its dual space by $H^* = \tilde{\mathfrak{h}}^* \oplus \mathbf{C}Q$. We define the paring by equation (2.1), and $\langle Q, h \rangle = \langle Q, c \rangle = \langle Q, d \rangle = 0 = \langle \alpha, P \rangle = \langle \delta, P \rangle = \langle \Lambda_0, P \rangle$. We define $\mathbf{F} = \mathfrak{M}_{H^*}$ to be the field of meromorphic functions on $H^*$. We regard a function of $P+h$, $P$ and $c$, $\hat{f} = f(P+h, P, c)$, as an element in $\mathbf{F}$ by $\hat{f}(\mu) = f(\langle \mu, P+h \rangle, \langle \mu, P \rangle, \langle \mu, c \rangle)$ for $\mu \in H^*$.

We use the following notations.

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad (x;q)_\infty = \prod_{n=0}^{\infty}(1 - xq^n),$$

$$(x;q,t)_\infty = \prod_{n,m=0}^{\infty}(1 - xq^n t^m), \qquad \Theta_p(z) = (z;p)_\infty (p/z;p)_\infty (p;p)_\infty.$$

Definition 2.1 ([16]). *The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is a topological algebra over $\mathbf{F}[[p]]$ generated by $\mathfrak{M}_{H^*}$, $e_m$, $f_m$, $\alpha_n$, $(m \in \mathbf{Z}, n \in \mathbf{Z}_{\neq 0})$, $K^\pm$, $d$ and the central element $c$. Let*

$$e(z) = \sum_{m \in \mathbf{Z}} e_m z^{-m}, \qquad f(z) = \sum_{m \in \mathbf{Z}} f_m z^{-m}$$

$$\psi^+(z) = K^+ \exp\left(-(q - q^{-1}) \sum_{n>0} \frac{\alpha_{-n}}{1 - p^n} (zq^{c/2})^n\right) \times \exp\left((q - q^{-1}) \sum_{n>0} \frac{p^n \alpha_n}{1 - p^n} (zq^{c/2})^{-n}\right),$$

$$\psi^-(z) = K^- \exp\left(-(q - q^{-1}) \sum_{n>0} \frac{p^n \alpha_{-n}}{1 - p^n} (zq^{-c/2})^n\right) \times \exp\left((q - q^{-1}) \sum_{n>0} \frac{\alpha_n}{1 - p^n} (zq^{-c/2})^{-n}\right).$$

*We call $e(z)$, $f(z)$, $\psi^\pm(z)$ the elliptic currents. They are formal Laurent series in $z$. The defining relations are*

$$g(P+h)e(z) = e(z)g(P+h), \qquad g(P)e(z) = e(z)g(P - \langle Q, P\rangle), \tag{2.2}$$

$$g(P+h)f(z) = f(z)g(P+h - \langle \alpha, P+h\rangle), \qquad g(P)f(z) = f(z)g(P), \tag{2.3}$$

$$[g(P), \alpha_m] = [g(P+h), \alpha_m] = 0, \tag{2.4}$$

$$g(P)K^\pm = K^\pm g(P - \langle Q, P\rangle), \tag{2.5}$$

$$g(P+h)K^\pm = K^\pm g(P+h - \langle Q, P\rangle), \tag{2.6}$$

$$[d, g(P+h, P)] = 0, \tag{2.7}$$

$$[d, \alpha_n] = n\alpha_n, \qquad [d, e(z)] = -z\frac{\partial}{\partial z}e(z), \qquad [d, f(z)] = -z\frac{\partial}{\partial z}f(z), \tag{2.8}$$

$$K^\pm e(z) = q^{\mp 2} e(z) K^\pm, \qquad K^\pm f(z) = q^{\pm 2} f(z) K^\pm, \tag{2.9}$$

$$[\alpha_m, \alpha_n] = \delta_{m+n,0} \frac{[2m][cm]}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm}, \tag{2.10}$$

$$[\alpha_m, e(z)] = \frac{[2m]}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm} z^m e(z), \tag{2.11}$$

$$[\alpha_m, f(z)] = -\frac{[2m]}{m} z^m f(z), \tag{2.12}$$

$$z_1 \frac{(q^2 z_2/z_1; p^*)_\infty}{(p^* q^{-2} z_2/z_1; p^*)_\infty} e(z_1)e(z_2) = -z_2 \frac{(q^2 z_1/z_2; p^*)_\infty}{(p^* q^{-2} z_1/z_2; p^*)_\infty} e(z_2)e(z_1), \tag{2.13}$$

$$z_1 \frac{(q^{-2} z_2/z_1; p)_\infty}{(pq^2 z_2/z_1; p)_\infty} f(z_1)f(z_2) = -z_2 \frac{(q^{-2} z_1/z_2; p)_\infty}{(pq^2 z_1/z_2; p)_\infty} f(z_2)f(z_1), \tag{2.14}$$

$$[e(z_1), f(z_2)] = \frac{1}{q-q^{-1}} (\delta(q^{-c} z_1/z_2)\psi^-(q^{c/2} z_2) - \delta(q^c z_1/z_2)\psi^+(q^{-c/2} z_2)), \tag{2.15}$$

*where* $p^* = pq^{-2c}$ *and* $\delta(z) = \sum_{n \in \mathbf{Z}} z^n$.

**2.2. Hopf algebroid structure of** $U_{q,p}(\widehat{\mathfrak{sl}}_2)$**.** Here we follow [13, 12, 14] to present the Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ using the Drinfeld coproduct of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [8].

**2.2.1.** *H-Hopf algebroid.* Let $\mathfrak{A}$ be a complex associative algebra, $\mathfrak{H}$ be a finite dimensional commutative subalgebra of $\mathfrak{A}$, and $\mathfrak{M}_{\mathfrak{H}^*}$ be the field of meromorphic functions on $\mathfrak{H}^*$ the dual space of $\mathfrak{H}$.

DEFINITION 2.2 ($\mathfrak{H}$-algebra). *An $\mathfrak{H}$-algebra is an associative algebra $\mathfrak{A}$ with* 1*, which is bigraded over $\mathfrak{H}^*$, $\mathfrak{A} = \bigoplus_{\alpha,\beta \in \mathfrak{H}^*} \mathfrak{A}_{\alpha\beta}$, and equipped with two algebra embeddings $\mu_l, \mu_r : \mathfrak{M}_{\mathfrak{H}^*} \to \mathfrak{A}_{00}$ (the left and right moment maps), such that*

$$\mu_l(\hat{f})a = a\mu_l(T_\alpha \hat{f}), \qquad \mu_r(\hat{f})a = a\mu_r(T_\beta \hat{f}), \qquad a \in \mathfrak{A}_{\alpha\beta},\ \hat{f} \in \mathfrak{M}_{\mathfrak{H}^*},$$

*where $T_\alpha$ denotes the automorphism $(T_\alpha \hat{f})(\lambda) = \hat{f}(\lambda + \alpha)$ of $\mathfrak{M}_{\mathfrak{H}^*}$.*

DEFINITION 2.3. *An $\mathfrak{H}$-algebra homomorphism is an algebra homomorphism $\pi : A \to B$ between two $\mathfrak{H}$-algebras $A$ and $B$ such that for $\alpha, \beta \in \mathfrak{H}^*$*

$$\pi(A_{\alpha\beta}) \subseteq B_{\alpha\beta}, \qquad \pi(\mu_l^A(\hat{f})) = \mu_l^B(\hat{f}), \qquad \pi(\mu_r^A(\hat{f})) = \mu_r^B(\hat{f}).$$

The tensor product $A \tilde{\otimes} B = \bigoplus_{\alpha,\beta \in \mathfrak{H}^*} (A \tilde{\otimes} B)_{\alpha\beta} = \bigoplus_{\alpha,\beta \in \mathfrak{H}^*} \left( \bigoplus_{\gamma \in \mathfrak{H}^*} (A_{\alpha\gamma} \otimes_{\mathfrak{M}_{\mathfrak{H}^*}} B_{\gamma\beta}) \right)$ is again an $\mathfrak{H}$-algebra with the multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$. The tensor product $\otimes_{\mathfrak{M}_{\mathfrak{H}^*}}$ refers to the usual tensor product modulo the following rule:

$$\mu_r^A(\hat{f})a \otimes b = a \otimes \mu_l^B(\hat{f})b, \qquad a \in A,\ b \in B,\ \hat{f} \in \mathfrak{M}_{\mathfrak{H}^*}. \tag{2.16}$$

The unit object $\mathfrak{D}$ in the category of $\mathfrak{H}$-algebras is an algebra of automorphisms $\mathfrak{M}_{\mathfrak{H}^*} \to \mathfrak{M}_{\mathfrak{H}^*}$

$$\mathfrak{D} = \left\{ \sum_i \hat{f}_i T_{\beta_i} \,|\, \hat{f}_i \in \mathfrak{M}_{\mathfrak{H}^*}, \beta_i \in \mathfrak{H}^* \right\} = \bigoplus_{\alpha \in \mathfrak{H}^*} \mathfrak{D}_{\alpha\alpha} \tag{2.17}$$

where $\mathfrak{D}_{\alpha\alpha} = \{\hat{f}T_{-\alpha} \,|\, \hat{f} \in \mathfrak{M}_{\mathfrak{H}^*}, \alpha \in \mathfrak{H}^*\}$ and the moment maps $\mu_l^{\mathfrak{D}}, \mu_r^{\mathfrak{D}} : \mathfrak{M}_{\mathfrak{H}^*} \to \mathfrak{D}_{00}$ are defined by $\mu_l^{\mathfrak{D}}(\hat{f}) = \mu_r^{\mathfrak{D}}(\hat{f}) = \hat{f}T_0$.

Definition 2.4. *An $\mathfrak{H}$-Hopf algebroid is an $\mathfrak{H}$-algebra $A$ equipped with two $\mathfrak{H}$-algebra homomorphisms: coproduct $\triangle : A \to A \tilde{\otimes} A$, counit $\varepsilon : A \to \mathfrak{D}$ and a $\mathbf{C}$-linear map: antipode $a : A \to A$. $\triangle$, $\varepsilon$, $a$ satisfy the following*

$$(\triangle \tilde{\otimes}\, id) \circ \triangle = (id \,\tilde{\otimes} \triangle) \circ \triangle \tag{2.18}$$

$$(\varepsilon \tilde{\otimes}\, id) \circ \triangle = (id \,\tilde{\otimes}\, \varepsilon) \circ \triangle \tag{2.19}$$

$$m \circ (id \,\tilde{\otimes}\, a) \circ \triangle(x) = \mu_l(\varepsilon(x)1), \qquad \forall x \in A \tag{2.20}$$

$$m \circ (a \tilde{\otimes}\, id) \circ \triangle(x) = \mu_r(T_\alpha(\varepsilon(x)1)), \qquad \forall x \in A_{\alpha\beta}. \tag{2.21}$$

*$m : A \tilde{\otimes} A \to A$ refers the multiplication and $\varepsilon(x)1$ $(x \in A)$ refers the action of the operator $\varepsilon(x)$ on the constant function $1 \in \mathfrak{M}_{\mathfrak{H}^*}$.*

**2.2.2.** *$H$-Hopf algebroid structure of $U = U_{q,p}(\widehat{\mathfrak{sl}}_2)$.*

Proposition 2.5. *$U = U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is an $H$-algebra by*

$$U = \bigoplus_{\alpha,\beta \in H^*} U_{\alpha\beta},$$

$$U_{\alpha\beta} = \{x \in U \,|\, q^{P+h} x q^{-(P+h)} = q^{\langle \alpha, P+h \rangle} x,\ q^P x q^{-P} = q^{\langle \beta, P \rangle} x$$
$$\forall P+h, P \in H\}$$

*and $\mu_l, \mu_r : \mathbf{F} \to U_{00}$ defined by*

$$\mu_l(\hat{f}) = f(P+h, p) \in \mathbf{F}[[p]], \qquad \mu_r(\hat{f}) = f(P, p^*) \in \mathbf{F}[[p]].$$

The tensor product $U \tilde{\otimes} U = \bigoplus_{\alpha,\beta \in H^*} (U \tilde{\otimes} U)_{\alpha\beta}$ is an $H^*$ bigraded algebra.

The $H$-algebra $\mathfrak{D}$ of the shift operators is

$$\mathfrak{D} = \left\{ \sum_i \hat{f}_i T_{\alpha_i} \,|\, \hat{f}_i \in \mathfrak{M}_{H^*}, \alpha_i \in H^* \right\}.$$

with the bigraded structure and the moments map as in Definition 2.2.

In [13], Konno defined the Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by the coproduct of $L$-operator. Here we define the Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by the Drinfeld coproduct [8, 14].

THEOREM 2.6 ([14]). *The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ has an elliptic analogue of the Drinfeld coproduct $\triangle : U_{q,p}(\widehat{\mathfrak{sl}}_2) \to U_{q,p}(\widehat{\mathfrak{sl}}_2) \tilde{\otimes} U_{q,p}(\widehat{\mathfrak{sl}}_2)$, the counit $\varepsilon : U_{q,p}(\widehat{\mathfrak{sl}}_2) \to \mathfrak{D}$ and the antipode $a : U_{q,p}(\widehat{\mathfrak{sl}}_2) \to U_{q,p}(\widehat{\mathfrak{sl}}_2)$*

$$\triangle(q^c) = q^c \tilde{\otimes} q^c, \qquad \triangle(q^h) = q^h \tilde{\otimes} q^h \tag{2.22}$$

$$\triangle(\psi^{\pm}(z)) = \psi^{\pm}(q^{\pm c^{(2)}/2} z) \tilde{\otimes} \psi^{\pm}(q^{\mp c^{(1)}/2} z) \tag{2.23}$$

$$\triangle(\mu_r(\hat{f})) = 1 \tilde{\otimes} \mu_r(\hat{f}), \qquad \triangle(\mu_l(\hat{f})) = \mu_l(\hat{f}) \tilde{\otimes} 1 \tag{2.24}$$

$$\triangle(e(z)) = e(q^{-c^{(2)}} z) \tilde{\otimes} \psi^{-}(q^{-c^{(2)}/2} z) + 1 \tilde{\otimes} e(z) \tag{2.25}$$

$$\triangle(f(z)) = f(z) \tilde{\otimes} 1 + \psi^{+}(q^{-c^{(1)}/2} z) \tilde{\otimes} f(zq^{-c^{(1)}}) \tag{2.26}$$

$$\varepsilon(q^c) = 1, \qquad \varepsilon(\psi^{+}(z)) = \varepsilon(\psi^{-}(z)) = 1 \tag{2.27}$$

$$\varepsilon(\mu_r(\hat{f})) = \varepsilon(\mu_l(\hat{f})) = \hat{f} T_0 \tag{2.28}$$

$$\varepsilon(e(z)) = \varepsilon(f(z)) = 0, \qquad \varepsilon(\alpha_n) = 0 \tag{2.29}$$

$$a(q^c) = q^{-c}, \qquad a(\psi^{\pm}(z)) = \psi^{\pm}(z)^{-1} \tag{2.30}$$

$$a(\mu_r(\hat{f})) = \mu_l(\hat{f}), \qquad a(\mu_l(\hat{f})) = \mu_r(\hat{f}) \tag{2.31}$$

$$a(e(z)) = -\psi^{-}(zq^{c/2})^{-1} e(q^c z) \tag{2.32}$$

$$a(f(z)) = -f(q^c z)\psi^{+}(zq^{c/2})^{-1}. \tag{2.33}$$

*Namely, the maps $\triangle$, $\varepsilon$ are algebra homomorphism and $a$ is an anti-algebra homomorphism satisfying the relations* (2.18)–(2.21) *in Definition* 2.4. *Therefore the $H$-algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with $\triangle$, $\varepsilon$, $a$ is an $H$-Hopf algebroid.*

PROOF. Let's check (2.19)

$$\begin{aligned} m \circ (\varepsilon \tilde{\otimes} id) \circ \triangle(e(z)) &= m(\varepsilon \tilde{\otimes} id) \circ (e(q^{-c^{(2)}} z) \tilde{\otimes} \psi^{-}(q^{-c^{(2)}/2} z) + 1 \tilde{\otimes} e(z)) \\ &= m \circ (\varepsilon(e(q^{-c^{(2)}} z)) \tilde{\otimes} \psi^{-}(q^{-c^{(2)}/2} z) + 1 \tilde{\otimes} e(z)) \\ &= m \circ (1 \tilde{\otimes} e(z)) = e(z), \\ m \circ (id \tilde{\otimes} \varepsilon) \circ \triangle(e(z)) &= m(id \tilde{\otimes} \varepsilon) \circ (e(q^{-c^{(2)}} z) \tilde{\otimes} \psi^{-}(q^{-c^{(2)}/2} z) + 1 \tilde{\otimes} e(z)) \\ &= m \circ (e(\varepsilon(q^{-c^{(2)}}) z) \tilde{\otimes} \varepsilon(\psi^{-}(\varepsilon(q^{-c^{(2)}/2}) z)) + 1 \tilde{\otimes} \varepsilon(e(z))) \\ &= m \circ (e(z) \tilde{\otimes} 1) = e(z). \end{aligned}$$

For (2.20)

$$\begin{aligned} m \circ (id \otimes a) \circ \triangle(e(z)) &= m \circ (id \otimes a) \circ (e(q^{-c^{(2)}} z) \tilde{\otimes} \psi^-(q^{-c^{(2)}/2} z) + 1 \tilde{\otimes} e(z)) \\ &= m \circ (e(a(q^{-c^{(2)}})z) \tilde{\otimes} a(\psi^-(a(q^{-c^{(2)}/2})z)) + 1 \tilde{\otimes} a(e(z))) \\ &= e(q^{c^{(2)}} z)\psi^-(q^{c^{(2)}/2} z)^{-1} - \psi^+(q^{-c^{(1)}/2} z)^{-1} e(q^{c^{(2)}} z) = 0 \\ &= \mu_l(\varepsilon(e(z))1). \end{aligned}$$

We call the $H$-Hopf algebroid $(U_{q,p}(\widehat{\mathfrak{sl}}_2), H, \mathfrak{M}_{H^*}, \mu_l, \mu_r, \triangle, \varepsilon, a)$ the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

From (2.24), a straight forward calculation shows the following relation

$$\triangle\left(\frac{\mu_l(\hat{f})}{\mu_r(\hat{f})}\right) = \frac{\mu_l(\hat{f})}{\mu_r(\hat{f})} \tilde{\otimes} \frac{\mu_l(\hat{f})}{\mu_r(\hat{f})}. \tag{2.34}$$

## 2.3. Level-1 highest weight representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

Definition 2.7 ([16]). *Let $\mathfrak{H}$, $\mathfrak{N}_+$, $\mathfrak{N}_-$ be the subalgebras of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ generated by $c$, $d$, $K^{\pm}$, by $\alpha_n$ $(n \in \mathbf{Z}_{>0})$, $e_n$ $(n \in \mathbf{Z}_{\geq 0})$, $f_n$ $(n \in \mathbf{Z}_{>0})$ and by $\alpha_{-n}$ $(n \in \mathbf{Z}_{>0})$, $e_{-n}$ $(n \in \mathbf{Z}_{>0})$, $f_{-n}$ $(n \in \mathbf{Z}_{\geq 0})$, respectively.*

The Heisenberg algebra $U_{q,p}(\mathfrak{H})$ is a subalgebra of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ generated by $\alpha_m$, $(m \neq 0)$ and $c$. From defining relations of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, we have

$$[\alpha_m, \alpha_n] = \frac{[2m][cm]}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm} \delta_{m+n,0}, \tag{2.35}$$

$$[\alpha'_m, \alpha'_n] = \frac{[2m][cm]}{m} \frac{1-p^{*m}}{1-p^m} q^{cm} \delta_{m+n,0}, \tag{2.36}$$

$$[\alpha_m, \alpha'_n] = \frac{[2m][cm]}{m} \delta_{m+n,0}, \tag{2.37}$$

where $\alpha'_m = \dfrac{1-p^{*m}}{1-p^m} q^{cm} \alpha_m$, $(m \neq 0)$.

Definition 2.8. *For $k \in \mathbf{C}$, a $U_{q,p}(\widehat{\mathfrak{sl}}_2)$-module $V(\lambda, \mu)$ is called the level-$k$ highest weight module with the highest weight $(\lambda, \mu)$, if there exists a highest weight vector $v \in V(\lambda, \mu)$ such that*

$$V(\lambda, \mu) = U_{q,p}(\widehat{\mathfrak{sl}}_2) \cdot v, \qquad \mathfrak{N}_+ \cdot v = 0,$$

$$c \cdot v = kv, \qquad f(P) \cdot v = f(\langle \mu, P \rangle)v, \qquad f(P+h) \cdot v = f(\langle \lambda, P+h \rangle)v.$$

Definition 2.9. *Define* $\Lambda_a(a = 0, 1) \in \mathfrak{h}^*$ *by*

$$\langle \Lambda_a, h \rangle = \delta_{a,1}, \qquad \langle \Lambda_a, c \rangle = \delta_{a,0},$$

*and the other pairings are* 0.

Theorem 2.10 ([16]). *For* $a = 0, 1$. *Define*

$$V(\Lambda_a + \mu, \mu) = \bigoplus_{\gamma, \kappa \in \mathfrak{Q}} (\mathbf{F} \otimes_{\mathbf{C}} (F_{\alpha,1} \otimes e^{\bar{\Lambda}_a + \gamma}) \otimes e^{Q_{\bar{\mu}+\kappa}}).$$

*Let* $\rho : U_{q,p}(\widehat{\mathfrak{sl}}_2) \to End(V(\Lambda_a + \mu, \mu))$ *by*

$$\rho(\psi^+(z)) = q^{-h} e^{-2Q} \exp\left( -(q - q^{-1}) \sum_{n>0} \frac{\rho(\alpha_{-n})}{1 - p^n} (zq^{1/2})^n \right)$$

$$\times \exp\left( (q - q^{-1}) \sum_{n>0} \frac{p^n \rho(\alpha_n)}{1 - p^n} (zq^{1/2})^{-n} \right) \tag{2.38}$$

$$\rho(\psi^-(z)) = q^h e^{-2Q} \exp\left( -(q - q^{-1}) \sum_{n>0} \frac{p^n \rho(\alpha_{-n})}{1 - p^n} (zq^{-1/2})^n \right)$$

$$\times \exp\left( (q - q^{-1}) \sum_{n>0} \frac{\rho(\alpha_n)}{1 - p^n} (zq^{-1/2})^{-n} \right) \tag{2.39}$$

$$\rho(e(z)) = :\exp\left( -\sum_{n \neq 0} \frac{\rho(\alpha_n)}{[n]} z^{-n} \right) : e^{\alpha} z^{h+1}$$

$$\rho(f(z)) = :\exp\left( \sum_{n \neq 0} \frac{\rho(\alpha'_n)}{[n]} z^{-n} \right) : e^{-\alpha} z^{-h+1}, \tag{2.40}$$

*where* $F_{\alpha,1}$ *is the polynomial ring* $\mathbf{C}[\alpha_{-m} \ (m > 0)]$. *For* $u \in \mathbf{C}[\alpha_{-m} \ (m > 0)]$

$$\rho(c) \cdot u = u, \qquad \rho(\alpha_{-n}) \cdot u = \alpha_{-n} u,$$

$$\rho(\alpha_n) \cdot u = \frac{[2n][n]}{n} \frac{1 - p^n}{1 - p^{*n}} q^{-n} \frac{\partial}{\partial \alpha_{-n}} u \qquad (n > 0).$$

*Then* $V(\Lambda_a + \mu, \mu)$ *is the level-*1 *irreducible highest weight module of* $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ *with the highest weight* $(\Lambda_a + \mu, \mu)$ *and the highest weight vector* $v_0 = 1 \otimes 1 \otimes e^{\Lambda_a} \otimes e^Q$.

For convention, we will drop $\rho$ to refer the elements in $End(V(\Lambda_a + \mu, \mu))$.

Proposition 2.11. *The level-*1 *elliptic operators satisfy the following relations*

$$e(z)e(w) = \frac{\left(q^{-2}p^* \frac{w}{z}; p^*\right)_\infty}{\left(q^2 p^* \frac{w}{z}; p^*\right)_\infty} \frac{\left(q^{-2} \frac{w}{z}; q^{2c}\right)_\infty}{\left(q^2 \frac{w}{z}; q^{2c}\right)_\infty} : e(z)e(w) : \tag{2.41}$$

$$\psi^-(z)e(w) = \frac{\left(q^{-2-c/2} \frac{w}{z}; pq^{-2c}\right)_\infty}{\left(q^{2-c/2} \frac{w}{z}; pq^{-2c}\right)_\infty} : \psi^-(z)e(w) : \tag{2.42}$$

$$f(z)f(w) = \frac{\left(q^{-2} \frac{w}{z}; q^{2c}\right)_\infty}{\left(q^2 \frac{w}{z}; q^{2c}\right)_\infty} \frac{\left(q^2 \frac{w}{z}; p\right)_\infty}{\left(q^{-2} \frac{w}{z}; p\right)_\infty} : f(z)f(w) : \tag{2.43}$$

$$f(z)\psi^+(w) = \frac{\left(q^{-2+c/2} \frac{w}{z}; p\right)_\infty}{\left(q^{2+c/2} \frac{w}{z}; p\right)_\infty} : f(z)\psi^+(w) : \tag{2.44}$$

$$\psi^\pm(z)\psi^\pm(w) = \frac{\left(q^{-2} \frac{w}{z}; pq^{-2c}\right)_\infty}{\left(q^2 \frac{w}{z}; pq^{-2c}\right)_\infty} \frac{\left(q^2 \frac{w}{z}; p\right)_\infty}{\left(q^{-2} \frac{w}{z}; p\right)_\infty} : \psi^\pm(z)\psi^\pm(w) :, \tag{2.45}$$

*where $c$ acts on $V(\Lambda_a + \mu, \mu)$ by* 1.

## 3. Higher level representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

In this section we show a construction of the higher level realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by using the Drinfeld coproduct of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Also, we will present the associated $Z$-operators.

**3.1. Higher level representation of** $U_{q,p}(\widehat{\mathfrak{sl}}_2)$**.** For $k > 0$, $\lambda_i \in \mathfrak{h}^*$, $\mu^{(i)} \in H^*$ ($i \in \{0, 1, \dots, k+1\}$). Let's consider a tensor product of $k+1$ copies of the level-1 highest weight modules $V(\Lambda_a + \mu, \mu)$ ($a = 0, 1$)

$$\begin{aligned} V_{k+1}(\lambda_i, \mu) = {} & V(\Lambda_{a^{(1)}} + \mu^{(1)}, \mu^{(1)}) \otimes \cdots \otimes V(\Lambda_{a^{(i)}} + \mu^{(i)}, \mu^{(i)}) \\ & \otimes V(\Lambda_{a^{(i+1)}} + \mu^{(i+1)}, \mu^{(i+1)}) \\ & \otimes \cdots \otimes V(\Lambda_{a^{(k+1)}} + \mu^{(k+1)}, \mu^{(k+1)}), \end{aligned} \tag{3.1}$$

such that $a^{(1)}, \dots, a^{(k+1)} \in \{0, 1\}$ and take $i$ of $a$'s as 0 and $k+1-i$ of $a$'s as 1.

THEOREM 3.1. *The space $V_{k+1}(\lambda_i, \mu)$ is the level-$(k+1)$ module of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with the highest weight*

$$(\lambda_i, \mu) = \left( i\Lambda_0 + (k+1-i)\Lambda_1 + \sum_{j=1}^{k+1} \mu^{(j)}, \sum_{j=1}^{k+1} \mu^{(j)} \right)$$

*by the action*

$$\begin{aligned}
\triangle^k(e(z)) &= \sum_{i=1}^{k+1} e^i(z),\\
e^i(z) &= 1\otimes\cdots 1\otimes e(zq^{-(c^{(i+1)}+\cdots+c^{(k+1)})})\\
&\quad \otimes \psi^-(zq^{-(c^{(i+1)}/2+c^{(i+2)}+\cdots+c^{(k+1)})})\\
&\quad \otimes \psi^-(zq^{-(c^{(i+2)}/2+c^{(i+3)}+\cdots+c^{(k+1)})})\otimes\cdots\otimes\psi^-(zq^{-c^{(k+1)}/2}),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\triangle^k(f(z)) &= \sum_{i=1}^{k+1} f^i(z),\\
f^i(z) &= \psi^+(zq^{-c^{(1)}/2})\otimes\psi^+(zq^{-(c^{(1)}+c^{(2)}/2)})\\
&\quad \otimes\cdots\otimes\psi^+(zq^{-(c^{(1)}+\cdots+c^{(i-2)}+c^{(i-1)}/2)})\otimes f(zq^{-(c^{(1)}+\cdots+c^{(i-2)}+c^{(i-1)})})\\
&\quad \otimes 1\cdots\otimes 1,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\triangle^k(\psi^\pm(z)) &= \psi^\pm(zq^{\pm(c^{(2)}+c^{(3)}+\cdots+c^{(k+1)})/2})\otimes\psi^\pm(zq^{\mp c^{(1)}/2\pm(c^{(3)}+c^{(4)}+\cdots+c^{(k+1)})/2})\\
&\quad \otimes\psi^\pm(zq^{\mp(c^{(1)}+c^{(2)})/2\pm(c^{(4)}+c^{(5)}+\cdots+c^{(k+1)})/2})\\
&\quad \otimes\cdots\otimes\psi^\pm(zq^{-(c^{(1)}+\cdots+c^{(k)})/2}),
\end{aligned} \tag{3.4}$$

*where* $c^{(i)} = 1\otimes\cdots\otimes c\otimes\cdots\otimes 1$ *and* $c^{(i)}$ *acts on* $V(\Lambda_{a^{(i)}}+\mu^{(i)},\mu^{(i)})$ *as* 1.

In order to show the proof of Theorem 3.1, we need the following OPE relations for $e^i(z)$ and $f^i(z)$ in the expansion of $\triangle^k(e(z))$ and $\triangle^k(f(z))$ respectively.

LEMMA 3.2. *Set* $i<j$. *For* $e^i(z)$

$$e^i(z)e^j(w) = \frac{\left(q^{-2}\frac{w}{z};pq^{-2\triangle^k(c)}\right)_\infty}{\left(q^{2}\frac{w}{z};pq^{-2\triangle^k(c)}\right)_\infty} : e^i(z)e^j(w):, \tag{3.5}$$

$$e^j(z)e^j(w) = \frac{\left(q^{-2+2c^{(j)}}\frac{w}{z};q^{2c^{(j)}}\right)_\infty}{\left(q^{2+2c^{(j)}}\frac{w}{z};q^{2c^{(j)}}\right)_\infty}\frac{\left(q^{-2}\frac{w}{z};pq^{-2\triangle^k(c)}\right)_\infty}{\left(q^{2}\frac{w}{z};pq^{-2\triangle^k(c)}\right)_\infty} : e^j(z)e^j(w):, \tag{3.6}$$

$$e^j(z)e^i(w) = \frac{\left(q^{-2-2\triangle^k(c)}p\frac{w}{z};pq^{-2\triangle^k(c)}\right)_\infty}{\left(q^{2-2\triangle^k(c)}p\frac{w}{z};pq^{-2\triangle^k(c)}\right)_\infty} : e^j(z)e^i(w):. \tag{3.7}$$

*For* $f^i(z)$

$$f^i(z)f^j(w) = \frac{\left(q^2\frac{w}{z};p\right)_\infty}{\left(q^{-2}\frac{w}{z};p\right)_\infty} : f^i(z)f^j(w) :, \tag{3.8}$$

$$f^j(z)f^j(w) = \frac{\left(q^2\frac{w}{z};p\right)_\infty}{\left(q^{-2}\frac{w}{z};p\right)_\infty}\frac{\left(q^{-2}\frac{w}{z};q^{2c^{(j)}}\right)_\infty}{\left(q^2\frac{w}{z};q^{2c^{(j)}}\right)_\infty} : f^j(z)f^j(w) :, \tag{3.9}$$

$$f^j(z)f^i(w) = \frac{\left(q^{-2}p\frac{w}{z};p\right)_\infty}{\left(q^2p\frac{w}{z};p\right)_\infty} : f^j(z)f^i(w) : . \tag{3.10}$$

Proof. This follows from Proposition 2.11.

Proof. Proof of Theorem 3.1. We can check directly that $\triangle^k(e(z))$, $\triangle^k(f(z))$ and $\triangle^k(\psi^\pm(z))$ satisfy the defining relations of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

Let's show that $\triangle^k(e(z))$ satisfies (2.13). By using the tensor product rules, relations (3.5)–(3.7) and (2.16), we have

$$\begin{aligned}
&z_1\frac{\left(q^2\frac{z_2}{z_1};pq^{-2\triangle^k(c)}\right)_\infty}{\left(q^{-2}pq^{-2\triangle^k(c)}\frac{z_2}{z_1};pq^{-2\triangle^k(c)}\right)_\infty}\triangle^k(e(z_1))\triangle^k(e(z_2))\\
&= z_1\frac{\left(q^2\frac{z_2}{z_1};pq^{-2\triangle^k(c)}\right)_\infty}{\left(q^{-2}pq^{-2\triangle^k(c)}\frac{z_2}{z_1};pq^{-2\triangle^k(c)}\right)_\infty}\sum_{i=1}^{k+1}e^i(z_1)\sum_{j=1}^{k+1}e^j(z_2)\\
&= z_1q^2\frac{\left(1-q^{-2}\frac{z_2}{z_1}\right)}{\left(1-q^2\frac{z_1}{z_2}\right)}\frac{\left(q^2\frac{z_1}{z_2};pq^{-2\triangle^k(c)}\right)_\infty}{\left(q^{-2}pq^{-2\triangle^k(c)}\frac{z_1}{z_2};pq^{-2\triangle^k(c)}\right)_\infty}\sum_{i<j}^{k+1}e^j(z_2)e^i(z_1)\\
&\quad + z_1\frac{\left(q^2\frac{z_1}{z_2};pq^{-2\triangle^k(c)}\right)_\infty}{\left(q^{-2}pq^{-2\triangle^k(c)}\frac{z_1}{z_2};pq^{-2\triangle^k(c)}\right)_\infty}\\
&\quad\times\sum_{i=j=1}^{k+1}\frac{\left(1-q^{-2}\frac{z_2}{z_1}\right)}{\left(1-q^{-2}\frac{z_1}{z_2}\right)}\frac{\left(q^{-2+2c^{(i)}}\frac{z_2}{z_1};q^{2c^{(i)}}\right)_\infty}{\left(q^{2+2c^{(i)}}\frac{z_2}{z_1};q^{2c^{(i)}}\right)_\infty}\frac{\left(q^{2+2c^{(i)}}\frac{z_1}{z_2};q^{2c^{(i)}}\right)_\infty}{\left(q^{-2+2c^{(i)}}\frac{z_1}{z_2};q^{2c^{(i)}}\right)_\infty}e^i(z_2)e^i(z_1)\\
&\quad + z_1q^{-2}\frac{\left(1-q^2\frac{z_2}{z_1}\right)}{\left(1-q^{-2}\frac{z_1}{z_2}\right)}\frac{\left(q^2\frac{z_1}{z_2};pq^{-2\triangle^k(c)}\right)_\infty}{\left(q^{-2}pq^{-2\triangle^k(c)}\frac{z_1}{z_2};pq^{-2\triangle^k(c)}\right)_\infty}\sum_{i>j}^{k+1}e^j(z_2)e^i(z_1)\\
&= -z_2\frac{\left(q^2\frac{z_1}{z_2};pq^{-2\triangle^k(c)}\right)_\infty}{\left(q^{-2}pq^{-2\triangle^k(c)}\frac{z_1}{z_2};pq^{-2\triangle^k(c)}\right)_\infty}\\
&\quad\times\left\{-q^2\left(\frac{z_1}{z_2}\right)\frac{\left(1-q^{-2}\frac{z_2}{z_1}\right)}{\left(1-q^2\frac{z_1}{z_2}\right)}\sum_{i<j}^{k+1}e^j(z_2)e^i(z_1)\right.
\end{aligned}$$

$$
-\left(\frac{z_1}{z_2}\right)\frac{\left(1-q^{-2}\frac{z_2}{z_1}\right)}{\left(1-q^{-2}\frac{z_1}{z_2}\right)}\sum_{i=j=1}^{k+1}\frac{\left(q^{-2+2c^{(i)}}\frac{z_2}{z_1};q^{2c^{(i)}}\right)_\infty}{\left(q^{2+2c^{(i)}}\frac{z_2}{z_1};q^{2c^{(i)}}\right)_\infty}
$$

$$
\times\frac{\left(q^{2+2c^{(i)}}\frac{z_1}{z_2};q^{2c^{(i)}}\right)_\infty}{\left(q^{-2+2c^{(i)}}\frac{z_1}{z_2};q^{2c^{(i)}}\right)_\infty}e^i(z_2)e^i(z_1)
$$

$$
-q^{-2}\left(\frac{z_1}{z_2}\right)\frac{\left(1-q^{2}\frac{z_2}{z_1}\right)}{\left(1-q^{-2}\frac{z_1}{z_2}\right)}\sum_{i>j}^{k+1}e^j(z_2)e^i(z_1)\Bigg\}.
$$

The factor

$$
-\left(\frac{z_1}{z_2}\right)\frac{\left(1-q^{-2}\frac{z_2}{z_1}\right)}{\left(1-q^{-2}\frac{z_1}{z_2}\right)}\sum_{i=j=1}^{k+1}\frac{\left(q^{-2+2c^{(i)}}\frac{z_2}{z_1};q^{2c^{(i)}}\right)_\infty}{\left(q^{2+2c^{(i)}}\frac{z_2}{z_1};q^{2c^{(i)}}\right)_\infty}\frac{\left(q^{2+2c^{(i)}}\frac{z_1}{z_2};q^{2c^{(i)}}\right)_\infty}{\left(q^{-2+2c^{(i)}}\frac{z_1}{z_2};q^{2c^{(i)}}\right)_\infty}
$$

$$
=-\left(\frac{z_1}{z_2}\right)\sum_{i=j=1}^{k+1}\frac{\left(q^{-2}\frac{z_2}{z_1};q^{2c^{(i)}}\right)_\infty}{\left(q^{2+2c^{(i)}}\frac{z_2}{z_1};q^{2c^{(i)}}\right)_\infty}\frac{\left(q^{2+2c^{(i)}}\frac{z_1}{z_2};q^{2c^{(i)}}\right)_\infty}{\left(q^{-2}\frac{z_1}{z_2};q^{2c^{(i)}}\right)_\infty}
$$

becomes 1 on account of the notation $\Theta_{q^{2c^{(i)}}}(z_1/z_2)=-(z_1/z_2)\Theta_{q^{2c^{(i)}}}(z_2/z_1)$. Similarly, we can show that $\triangle^k(f(z))$ realizes (2.14).

Also, we can prove that $\triangle^k(e(z))$ and $\triangle^k(f(z))$ satisfy (2.15)

$$
[\triangle^k(e(z_1)),\triangle^k(f(z_2))]
$$

$$
=\frac{1}{q-q^{-1}}\left(\delta\left(q^{-\triangle^k(c)}\frac{z_1}{z_2}\right)\psi^-(q^{c^{(1)}/2}z_2)-\delta\left(q^{2c^{(1)}-\triangle^k(c)}\frac{z_1}{z_2}\right)\psi^+(q^{-c^{(1)}/2}z_2)\right)
$$

$$
\otimes\psi^-(z_1q^{-(c^{(2)}/2+c^{(3)}+\cdots+c^{(k+1)})})\otimes\cdots\otimes\psi^-(z_1q^{-c^{(k+1)}/2})
$$

$$
+\psi^+(z_2q^{-c^{(1)}/2})\otimes\psi^+(z_2q^{-(c^{(1)}+c^{(2)}/2)})\otimes\cdots\otimes\psi^+(z_2q^{-(c^{(1)}+\cdots+c^{(i-2)}+c^{(i-1)}/2)})
$$

$$
\otimes\frac{1}{q-q^{-1}}\left(\delta\left(q^{-2c^{(k+1)}+\triangle^k(c)}\frac{z_1}{z_2}\right)\psi^-(q^{-c^{(k+1)}/2+\triangle^k(c)}z_2)-\delta\left(q^{\triangle^k(c)}\frac{z_1}{z_2}\right)\right.
$$

$$
\left.\times\psi^+(q^{-c^{(k+1)}/2-(c^{(1)}+\cdots+c^{(k)})}z_2)\right).
$$

Then use the property of the delta function and (3.4).

Denote by $v^{(k+1)}\in V_{k+1}(\lambda,\mu)$ the tensor product of the highest weight vectors in the tensor factors in relation (3.1). We calculate the highest weight by using the action of $\mathfrak{M}_{H^*}$ (2.34) on $v^{(k+1)}$ as follows

$$\triangle^k\left(\frac{f(P)}{f(P+h)}\right)\cdot v^{(k+1)}$$

$$=\frac{f(\langle \mu^{(1)}+\mu^{(2)}+\cdots+\mu^{(k+1)},P\rangle)}{f(\langle i\Lambda_0+(k+1-i)\Lambda_1+\mu^{(1)}+\mu^{(2)}+\cdots+\mu^{(k+1)},P+h\rangle)}v^{(k+1)}.$$

We also obtain the comultiplication formula $\triangle^k$ of boson operator $\alpha_n$ $(n\neq 0)$ from $\triangle^k(\psi^{\pm}(z))$.

Corollary 3.3. *For $k\geq 1$, $n\neq 0$. The boson operator is*

$$\begin{aligned}\triangle^k(\alpha_n)&=\alpha_n\otimes 1\dots 1\otimes 1+\frac{(1-p^n)q^{-c^{(1)}n}}{1-p^nq^{-2c^{(1)}n}}\otimes\alpha_n\otimes 1\cdots\otimes 1\\&\quad+\frac{(1-p^n)q^{-(c^{(1)}+c^{(2)})n}}{1-p^nq^{-2(c^{(1)}+c^{(2)})n}}\otimes 1\otimes\alpha_n\otimes 1\cdots\otimes 1\\&\quad+\cdots+\frac{(1-p^n)q^{-(c^{(1)}+c^{(2)}+\cdots+c^{(i-1)})n}}{1-p^nq^{-2(c^{(1)}+c^{(2)}+\cdots+c^{(i-1)})n}}\otimes 1\cdots\otimes\alpha_n\otimes 1\cdots\otimes 1\\&\quad+\cdots+\frac{(1-p^n)q^{-(c^{(1)}+c^{(2)}+\cdots+c^{(k)})n}}{1-p^nq^{-2(c^{(1)}+c^{(2)}+\cdots+c^{(k)})n}}\otimes 1\cdots\otimes\alpha_n,\end{aligned}\tag{3.11}$$

*where $c^{(i)}=1\otimes\cdots\otimes c\otimes\cdots\otimes 1$.*

Proof. Based on the relations (2.16), (2.38)–(2.39) and (3.4) in Theorem 3.1, we can write

$$\begin{aligned}\triangle^k(\psi^+(z))&=\triangle^k(q^{-h}e^{-2Q})\exp\left(-(q-q^{-1})\sum_{n>0}\frac{\triangle^k(\alpha_{-n})}{1-p^n}(zq^{\triangle^k(c)/2})^n\right)\\&\quad\times\exp\left((q-q^{-1})\sum_{n>0}\frac{p^n\triangle^k(\alpha_n)}{1-p^n}(zq^{\triangle^k(c)/2})^{-n}\right),\end{aligned}\tag{3.12}$$

$$\begin{aligned}\triangle^k(\psi^-(z))&=\triangle^k(q^{h}e^{-2Q})\exp\left(-(q-q^{-1})\sum_{n>0}\frac{p^n\triangle^k(\alpha_{-n})}{1-p^n}(zq^{-\triangle^k(c)/2})^n\right)\\&\quad\times\exp\left((q-q^{-1})\sum_{n>0}\frac{\triangle^k(\alpha_n)}{1-p^n}(zq^{-\triangle^k(c)/2})^{-n}\right),\end{aligned}\tag{3.13}$$

where $\triangle^k(q^{\pm h}e^{-2Q})=(q^{\pm h}e^{-2Q})\otimes\cdots\otimes(q^{\pm h}e^{-2Q})$. These imply Corollary 3.3.

The operators $\triangle^k(\alpha_n)$ $(n\neq 0)$ give a level-$(k+1)$ realization of the Heisenberg algebra $U_{q,p}(\mathfrak{H})$.

PROPOSITION 3.4. *The operators* $\triangle^k(\alpha_n)$ $(n \neq 0)$ *and* $\triangle^k(c)$ *satisfy*

$$[\triangle^k(\alpha_m), \triangle^k(\alpha_n)] = \frac{[2m][\triangle^k(c)m]}{m} \frac{1-p^m}{1-p^m q^{-2\triangle^k(c)m}} q^{-\triangle^k(c)m} \delta_{m+n,0}, \tag{3.14}$$

$$[\triangle^k(\alpha_m), \triangle^k(e(z))] = \frac{[2m]}{m} \frac{1-p^m}{1-p^m q^{-2\triangle^k(c)m}} q^{-\triangle^k(c)m} z^m \triangle^k(e(z)), \tag{3.15}$$

$$[\triangle^k(\alpha_m), \triangle^k(f(z))] = -\frac{[2m]}{m} \frac{1-p^m q^{-2\triangle^k(c)m}}{1-p^m} q^{\triangle^k(c)m} z^m \triangle^k(f(z)). \tag{3.16}$$

**3.2. $Z$-algebra.** Here we give a realization of the level-$(k+1)$ $Z$-algebra. The form of the vertex operators in [16] section 3 led us to introduce $E^{\pm}_{(k)}(\alpha, z)$ and $E^{\pm}_{(k)}(\alpha', z)$ in the following definition.

DEFINITION 3.1. *By using the level-$(k+1)$ elliptic bosons $\triangle^k(\alpha_n)$ $(n \neq 0)$, we define the vertex operators*

$$E^{\pm}_{(k)}(\alpha, z) = \exp\left(\pm \sum_{n>0} \frac{\triangle^k(\alpha_{\pm n})}{[\triangle^k(c)n]} z^{\mp n}\right),$$

$$E^{\pm}_{(k)}(\alpha', z) = \exp\left(\mp \sum_{n>0} \frac{\triangle^k(\alpha'_{\pm n}))}{[\triangle^k(c)n]} z^{\mp n}\right),$$

*which are formal Laurent series in $z$ with coefficient in End $V_{k+1}(\lambda_i, \mu)$.*

The following proposition is a consequence of the commutation relations (3.14)–(3.16) in Proposition 3.4 with $\triangle^k(c)$ acts as the scalar $k+1$.

PROPOSITION 3.5. *$E^{\pm}_{(k)}(\alpha, z)$ and $E^{\pm}_{(k)}(\alpha', z)$ satisfy the following relations:*

$$E^{+}_{(k)}(\alpha, z) E^{-}_{(k)}(\alpha, w) = \frac{(q^{-2+2(k+1)} w/z; q^{2(k+1)})_\infty (q^{-2} w/z; pq^{-2(k+1)})_\infty}{(q^{2+2(k+1)} w/z; q^{2(k+1)})_\infty (q^{2} w/z; pq^{-2(k+1)})_\infty} \times E^{-}_{(k)}(\alpha, w) E^{+}_{(k)}(\alpha, z), \tag{3.17}$$

$$E^{+}_{(k)}(\alpha', z) E^{-}_{(k)}(\alpha', w) = \frac{(q^{-2} w/z; q^{2(k+1)})_\infty (q^{2} w/z; p)_\infty}{(q^{2} w/z; q^{2(k+1)})_\infty (q^{-2} w/z; p)_\infty} \times E^{-}_{(k)}(\alpha', w) E^{+}_{(k)}(\alpha', z), \tag{3.18}$$

$$E^{+}_{(k)}(\alpha, z) E^{-}_{(k)}(\alpha', w) = \frac{(q^{2+(k+1)} w/z; q^{2(k+1)})_\infty}{(q^{-2+(k+1)} w/z; q^{2(k+1)})_\infty} E^{-}_{(k)}(\alpha', w) E^{+}_{(k)}(\alpha, z), \tag{3.19}$$

$$E^{+}_{(k)}(\alpha',z)E^{-}_{(k)}(\alpha,w)=\frac{(q^{2+(k+1)}w/z;q^{2(k+1)})_{\infty}}{(q^{-2+(k+1)}w/z;q^{2(k+1)})_{\infty}}E^{-}_{(k)}(\alpha,w)E^{+}_{(k)}(\alpha',z), \tag{3.20}$$

$$\begin{aligned}E^{\pm}_{(k)}(\alpha,z)\triangle^{k}(e(w))&=\frac{(q^{\pm 2+2(k+1)}(w/z)^{\pm 1};q^{2(k+1)})_{\infty}}{(q^{\mp 2+2(k+1)}(w/z)^{\pm 1};q^{2(k+1)})_{\infty}}\\&\quad\times\frac{(q^{\pm 2}(w/z)^{\pm 1};pq^{-2(k+1)})_{\infty}}{(q^{\mp 2}(w/z)^{\pm 1};pq^{-2(k+1)})_{\infty}}\triangle^{k}(e(w))E^{\pm}_{(k)}(\alpha,z),\end{aligned} \tag{3.21}$$

$$\begin{aligned}E^{\pm}_{(k)}(\alpha',z)\triangle^{k}(f(w))&=\frac{(q^{\pm 2}(w/z)^{\pm 1};q^{2(k+1)})_{\infty}(q^{\pm 2}(w/z)^{\pm 1};p)_{\infty}}{(q^{\mp 2}(w/z)^{\pm 1};q^{2(k+1)})_{\infty}(q^{\mp 2}(w/z)^{\pm 1};p)_{\infty}}\\&\quad\times\triangle^{k}(f(w))E^{\pm}_{(k)}(\alpha',z),\end{aligned} \tag{3.22}$$

$$E^{\pm}_{(k)}(\alpha',z)\triangle^{k}(e(w))=\frac{(q^{\mp 2+(k+1)}(w/z)^{\pm 1};q^{2(k+1)})_{\infty}}{(q^{\pm 2+(k+1)}(w/z)^{\pm 1};q^{2(k+1)})_{\infty}}\triangle^{k}(e(w))E^{\pm}_{(k)}(\alpha',z), \tag{3.23}$$

$$E^{\pm}_{(k)}(\alpha,z)\triangle^{k}(f(w))=\frac{(q^{\mp 2+(k+1)}(w/z)^{\pm 1};q^{2(k+1)})_{\infty}}{(q^{\pm 2+(k+1)}(w/z)^{\pm 1};q^{2(k+1)})_{\infty}}\triangle^{k}(f(w))E^{\pm}_{(k)}(\alpha,z). \tag{3.24}$$

Definition 3.2 ([16]). *For $k\in\mathbf{Z}_{>0}$. We define the level-$(k+1)$ quantum Z-operators by*

$$\triangle^{k}(e(z))=E(k,\alpha,z)Z^{+}(z)$$

$$\triangle^{k}(f(z))=E(k,\alpha',z)Z^{-}(z)$$

*where*

$$\begin{aligned}E(k,\alpha,z)&=E^{-}_{(k)}(-\alpha,z)E^{+}_{(k)}(\alpha,z)\\&=\exp\left(\sum_{n>0}\frac{\triangle^{k}(\alpha_{-n})}{[\triangle^{k}(c)n]}z^{n}\right)\exp\left(-\sum_{n>0}\frac{\triangle^{k}(\alpha_{n})}{[\triangle^{k}(c)n]}z^{-n}\right),\end{aligned} \tag{3.25}$$

$$\begin{aligned}E(k,\alpha',z)&=E^{-}_{(k)}(-\alpha',z)E^{+}_{(k)}(-\alpha',z)\\&=\exp\left(-\sum_{n>0}\frac{\triangle^{k}(\alpha'_{-n}))}{[\triangle^{k}(c)n]}z^{n}\right)\exp\left(\sum_{n>0}\frac{\triangle^{k}(\alpha'_{n}))}{[\triangle^{k}cn]}z^{-n}\right),\end{aligned} \tag{3.26}$$

$$Z^{\pm}(z)=\sum_{i=1}^{k+1}Z^{\pm}_{i}(z). \tag{3.27}$$

Since $\triangle^{k}(e(z))$ and $\triangle^{k}(f(z))$ satisfy the defining relations of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, we find that $Z^{\pm}(z)$ satisfy the following relations [16]:

Theorem 3.6 ([16]).

$$g(P+h)Z^+(z) = Z^+(z)g(P+h), \qquad g(P)Z^+(z) = Z^+(z)g(P - \langle Q, P\rangle), \quad (3.28)$$

$$g(P+h)Z^-(z) = Z^-(z)g(P+h-\langle \alpha, P+h\rangle),$$

$$g(P)Z^-(z) = Z^-(z)g(P), \quad (3.29)$$

$$[d, Z^{\pm}(z)] = -z\frac{\partial}{\partial z}Z^{\pm}(z), \quad (3.30)$$

$$[\triangle^k(\alpha_m), Z^{\pm}(w)] = 0, \quad (3.31)$$

$$\triangle^k(K^{\pm})Z^+(z) = q^{\mp 2(k+1)}Z^+(z)\triangle^k(K^{\pm}),$$

$$\triangle^k(K^{\pm})Z^-(z) = q^{\pm 2(k+1)}Z^-(z)\triangle^k(K^{\pm}), \quad (3.32)$$

$$z\frac{(q^{-2}w/z; q^{2(k+1)})_\infty}{(q^{2+2(k+1)}w/z; q^{2(k+1)})_\infty}Z^{\pm}(z)Z^{\pm}(w)$$

$$= -w\frac{(q^{-2}z/w; q^{2(k+1)})_\infty}{(q^{2+2(k+1)}z/w; q^{2(k+1)})_\infty}Z^{\pm}(w)Z^{\pm}(z), \quad (3.33)$$

$$\frac{(q^{2+(k+1)}w/z; q^{2(k+1)})_\infty}{(q^{-2+(k+1)}w/z; q^{2(k+1)})_\infty}Z^+(z)Z^-(w)$$

$$-\frac{(q^{2+(k+1)}z/w; q^{2(k+1)})_\infty}{(q^{-2+(k+1)}z/w; q^{2(k+1)})_\infty}Z^-(w)Z^+(z)$$

$$= \frac{1}{q-q^{-1}}(\triangle^k(K^-)\delta(q^{-(k+1)}z/w) - \triangle^k(K^+)\delta(q^{(k+1)}z/w)). \quad (3.34)$$

Proof. Let us show the relation (3.31). For $m > 0$, we have

$$[\triangle^k(\alpha_m), Z^+(w)] = [\triangle^k(\alpha_m), E^-_{(k)}(\alpha, w)]\triangle^k(e(w))E^+_{(k)}(\alpha, w)$$
$$+ E^-_{(k)}(\alpha, w)[\triangle^k(\alpha_m), \triangle^k(e(w))]E^+_{(k)}(\alpha, w).$$

This vanishes because of (3.15) and

$$[\triangle^k(\alpha_m), E^-_{(k)}(\alpha, w)] = -\frac{[2m]}{m}\frac{1-p^m}{1-p^mq^{-2(k+1)m}}q^{-(k+1)m}w^mE^-_{(k)}(\alpha, w).$$

By the same way, from relation (3.16) and

$$[\triangle^k(\alpha'_m), E^-_{(k)}(\alpha', w)] = \frac{[2m]}{m}\frac{1-p^mq^{-2(k+1)m}}{1-p^m}q^{(k+1)m}w^mE^-_{(k)}(\alpha', w),$$

we get $[\triangle^k(\alpha'_m), Z^-(w)] = 0$.

Similarly, the case $m < 0$ can be proved.

To prove the relation (3.33), we use equations (3.17) and (3.21) and obtain

$$\begin{aligned}
&Z^+(z)Z^+(w)\\
&\quad= E^-_{(k)}(\alpha,z)\triangle^k(e(z))E^+_{(k)}(\alpha,z)E^-_{(k)}(\alpha,w)\triangle^k(e(w))E^+_{(k)}(\alpha,w)\\
&\quad= \frac{(q^{-2+2(k+1)}w/z;q^{2(k+1)})_\infty(q^{-2}w/z;pq^{-2(k+1)})_\infty}{(q^{2+2(k+1)}w/z;q^{2(k+1)})_\infty(q^{2}w/z;pq^{-2(k+1)})_\infty}\\
&\qquad\times E^-_{(k)}(\alpha,z)\triangle^k(e(z))E^-_{(k)}(\alpha,w)E^+_{(k)}(\alpha,z)\triangle^k(e(w))E^+_{(k)}(\alpha,w)\\
&\quad= \frac{(q^{2+2(k+1)}w/z;q^{2(k+1)})_\infty(q^{2}w/z;pq^{-2(k+1)})_\infty}{(q^{-2+2(k+1)}w/z;q^{2(k+1)})_\infty(q^{-2}w/z;pq^{-2(k+1)})_\infty}\\
&\qquad\times E^-_{(k)}(\alpha,z)E^-_{(k)}(\alpha,w)\triangle^k(e(z))\triangle^k(e(w))E^+_{(k)}(\alpha,z)E^+_{(k)}(\alpha,w).
\end{aligned}$$

Since $\triangle^k(e(z))$ satisfy the defining relations of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and again use (3.21), we get the desired relation.

We also derive (3.34) as follows.

$$\begin{aligned}
&\frac{(q^{2+(k+1)}w/z;q^{2(k+1)})_\infty}{(q^{-2+(k+1)}w/z;q^{2(k+1)})_\infty}Z^+(z)Z^-(w)\\
&\quad= \frac{(q^{2+(k+1)}w/z;q^{2(k+1)})_\infty}{(q^{-2+(k+1)}w/z;q^{2(k+1)})_\infty}\\
&\qquad\times E^-_{(k)}(\alpha,z)\triangle^k(e(z))E^+_{(k)}(\alpha,z)E^-_{(k)}(\alpha',w)\triangle^k(f(w))E^+_{(k)}(\alpha',w)\\
&\quad= E^-_{(k)}(\alpha,z)E^-_{(k)}(\alpha',w)\triangle^k(e(z))\triangle^k(f(w))E^+_{(k)}(\alpha,z)E^+_{(k)}(\alpha',w)\\
&\quad= E^-_{(k)}(\alpha,z)E^-_{(k)}(\alpha',w)\\
&\qquad\times\left[\triangle^k(f(w))\triangle^k(e(z))+\frac{1}{q-q^{-1}}\delta\left(q^{-(k+1)}\frac{z}{w}\right)\triangle^k(\psi^-)(q^{(k+1)/2}w)\right.\\
&\qquad\quad\left.-\frac{1}{q-q^{-1}}\delta\left(q^{(k+1)}\frac{z}{w}\right)\triangle^k(\psi^+)(q^{-(k+1)/2}w)\right]E^+_{(k)}(\alpha,z)E^+_{(k)}(\alpha',w).
\end{aligned}$$

In the second equality, we used the relation (3.19). In the third equality we used the defining relation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ between $\triangle^k(e(z))$ and $\triangle^k(f(w))$.

By using

$$\begin{aligned}
\triangle^k(\psi^\pm)(q^{\mp(k+1)/2}w) &= \triangle^k(K^\pm)E^-_{(k)}(\alpha,q^{\mp(k+1)}w)^{-1}E^-_{(k)}(\alpha',q^{\mp 1/2}w)^{-1}\\
&\quad\times E^+(\alpha,q^{\mp(k+1)}w)^{-1}E^+_k(\alpha',q^{\mp 1/2}w)^{-1}
\end{aligned}$$

and the property of the delta function, we obtain relation (3.34).

From Definition 3.2, Theorem 3.1 and Theorem 2.10 with $c^{(i)} = 1 \otimes \cdots \otimes c \otimes \cdots \otimes 1$, we express the level-$(k+1)$ $Z$-operators as follows

$$
\begin{aligned}
Z^{+}(z) = \sum_{i=1}^{k+1} & E^{-}_{(k)}(\alpha, z)\mathfrak{e}^{-}_{i}(\alpha, z)\mathfrak{e}^{+}_{i}(\alpha, z)E^{+}_{(k)}(\alpha, z) \\
& \times (1 \otimes \cdots \otimes e^{\alpha} \otimes e^{-2Q} \otimes \cdots \otimes e^{-2Q}) \\
& \times (1 \otimes \cdots \otimes z^{h} q^{-(c^{(i+1)}+\cdots+c^{(k+1)})h} \otimes q^{h} \otimes \cdots \otimes q^{h}) \\
& \times z q^{-(c^{(i+1)}+\cdots+c^{(k+1)})} \\
Z^{-}(z) = \sum_{i=1}^{k+1} & E^{-}_{(k)}(\alpha', z)\mathfrak{f}^{-}_{i}(\alpha, z)\mathfrak{f}^{+}_{i}(\alpha, z)E^{+}_{(k)}(\alpha', z) \\
& \times (e^{-2Q} \otimes \cdots \otimes e^{-2Q} \otimes e^{-\alpha} \otimes 1 \cdots \otimes 1) \\
& \times (q^{-h} \otimes \cdots \otimes q^{-h} \otimes z^{-h} q^{(c^{(i+1)}+\cdots+c^{(k+1)})h} \otimes 1 \cdots \otimes 1) \\
& \times z q^{-(c^{(i+1)}+\cdots+c^{(k+1)})}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathfrak{e}^{-}_{i}(\alpha, z) = \exp\Bigg( (q^{-1} - q) \sum_{n>0} \Bigg\{ & 1 \otimes \cdots \otimes \frac{\alpha_{-n} q^{c^{(i)} n}}{1 - q^{2c^{(i)} n}} q^{-(c^{(i+1)}+\cdots+c^{(k+1)})n} \otimes 1 \cdots \otimes 1 \\
& + 1 \otimes \cdots \otimes \frac{p^{n} \alpha_{-n}}{1 - p^{n}} q^{-(c^{(i+2)}+\cdots+c^{(k+1)})n} \otimes 1 \cdots \otimes 1 \\
& + \cdots + 1 \otimes \cdots \otimes \frac{p^{n} \alpha_{-n}}{1 - p^{n}} \Bigg\} z^{n} \Bigg) \\
\mathfrak{e}^{+}_{i}(\alpha, z) = \exp\Bigg( (q - q^{-1}) \sum_{n>0} \Bigg\{ & 1 \otimes \cdots \otimes \frac{\alpha_{n}}{1 - q^{2c^{(i)} n}} q^{(c^{(i)}+\cdots+c^{(k+1)})n} \otimes 1 \cdots \otimes 1 \\
& + 1 \otimes \cdots \otimes \frac{\alpha_{n}}{1 - p^{n}} q^{-(c^{(i+2)}+\cdots+c^{(k+1)})n} \otimes 1 \cdots \otimes 1 \\
& + \cdots + 1 \otimes \cdots \otimes \frac{\alpha_{n}}{1 - p^{n}} \Bigg\} z^{-n} \Bigg) \\
\mathfrak{f}^{-}_{i}(\alpha, z) = \exp\Bigg( -(q - q^{-1}) \sum_{n>0} \Bigg\{ & \frac{\alpha'_{-n}}{1 - p^{n} q^{-2c^{(1)} n}} q^{-c^{(1)} n} \otimes 1 \cdots \otimes 1 \\
& + 1 \otimes \frac{\alpha'_{-n}}{1 - p^{n} q^{-2c^{(2)} n}} q^{-(c^{(1)}+c^{(2)})n} \otimes 1 \cdots \otimes 1 + \cdots
\end{aligned}
$$

$$+1 \otimes \cdots \otimes \frac{\alpha'_{-n}}{1 - p^n q^{-2c^{(i-1)}n}} q^{-(c^{(1)}+c^{(2)}+\cdots+c^{(i-1)})n} \otimes 1 \cdots \otimes 1$$

$$\left. \left. + 1 \otimes \cdots \otimes \frac{\alpha'_{-n}}{1 - q^{2c^{(i)}n}} q^{-(c^{(1)}+\cdots+c^{(i-1)}+c^{(i)})n} \otimes 1 \cdots \otimes 1 \right\} z^n \right)$$

$$\mathfrak{f}_i^+(\alpha, z) = \exp\left( (q - q^{-1}) \sum_{n>0} \left\{ \frac{p^n \alpha'_n}{1 - p^n q^{-2c^{(1)}n}} q^{-c^{(1)}n} \otimes 1 \cdots \otimes 1 \right. \right.$$

$$+ 1 \otimes \frac{p^n \alpha'_n}{1 - p^n q^{-2c^{(2)}n}} q^{(c^{(1)}-c^{(2)})n} \otimes 1 \cdots \otimes 1 + \cdots$$

$$+ 1 \otimes \cdots \otimes \frac{p^n \alpha'_n}{1 - p^n q^{-2c^{(i-1)}n}} q^{(c^{(1)}+\cdots+c^{(i-2)}-c^{(i-1)})n} \otimes 1 \cdots \otimes 1$$

$$\left. \left. + 1 \otimes \cdots \otimes \frac{\alpha'_n}{1 - q^{2c^{(i)}n}} q^{(c^{(1)}+c^{(2)}+\cdots+c^{(i-1)}+c^{(i)})n} \otimes 1 \cdots \otimes 1 \right\} z^{-n} \right)$$

## 4. Integrable condition of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ module

In this section we show that the products $\mathfrak{E}_N(z) = \triangle^k(e(z))\triangle^k(e(zq^2)) \dots \triangle^k(e(zq^{2(N-1)}))$ and $\mathfrak{F}_N(z) = \triangle^k(f(z))\triangle^k(f(zq^{-2})) \dots \triangle^k(f(zq^{-2(N-1)}))$ give the integrable condition for the level-$(k+1)$ $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ module at $N = k+2$, namely the nilpotent condition. Then at the same time we show that the products $\mathfrak{E}_N(z)$ and $\mathfrak{F}_N(z)$ at $N = k+1$ give certain vertex operators associated with the level-$(k+1)$ module of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

Theorem 4.1. *For $k \geq 1$. On the level-$(k+1)$ integrable module $V_{k+1}(\lambda_i, \mu)$ of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, we obtain a quantum analogue of the condition of integrability (an elliptic analogue of the Wheel condition) as*

$$\mathfrak{E}_{k+2}(z) = \triangle^k(e(z))\triangle^k(e(zq^2)) \dots \triangle^k(e(zq^{2(k+1)})) = 0 \tag{4.1}$$

$$\mathfrak{F}_{k+2}(z) = \triangle^k(f(z))\triangle^k(f(zq^{-2m})) \dots \triangle^k(f(zq^{-2(k+1)})) = 0. \tag{4.2}$$

*On the other hand, $\mathfrak{E}_{k+1}(z)$ and $\mathfrak{F}_{k+1}(z)$ give the following vertex operators*

$$\mathfrak{E}_{k+1}(z) = \mathfrak{S}(p,q)_e : \exp\left( \sum_{n \neq 0} -\frac{\triangle^k(\alpha_n)}{[n]} q^{-kn} z^{-n} \right) : (1 \otimes K^- \otimes K^- \otimes \cdots \otimes K^-)$$

$$\times (e^\alpha \otimes \cdots \otimes e^\alpha)(z^{h+1} \otimes \cdots \otimes z^{h+1})$$

$$\times (q^{kh} \otimes q^{(k-1)h} \otimes \cdots \otimes 1) q^{k(k+1)/2}, \tag{4.3}$$

$$\mathfrak{F}_{k+1}(z) = \mathfrak{S}(p,q)_f : \exp\left(\sum_{n\neq 0} \frac{\triangle^k(\alpha'_n)}{[n]} q^{kn} z^{-n}\right) : (K^+ \otimes K^+ \otimes \cdots \otimes K^+ \otimes 1)$$
$$\times (e^\alpha \otimes \cdots \otimes e^\alpha)(z^{-h+1} \otimes \cdots \otimes z^{-h+1})$$
$$\times (q^{(k+1)h} \otimes q^{(k)h} \otimes \cdots \otimes q^h) q^{-(k+1)(k+2)/2}, \tag{4.4}$$

*where*

$$\mathfrak{S}(p,q)_e = \frac{(q^{-2}pq^{-2(\triangle(c))}; pq^{-2(\triangle(c))})_\infty}{(q^2pq^{-2(\triangle(c))}; pq^{-2(\triangle(c))})_\infty}$$
$$\times \prod_{j=1}^{k}\prod_{i=1}^{j} \frac{(q^{-2}pq^{-2(\triangle^{(j)}(c)+(2i-1))}; pq^{-2(\triangle^{(j)}(c))})_\infty}{(q^2pq^{-2(\triangle^{(j)}(c)+(2i-1))}; pq^{-2(\triangle^{(j)}(c))})_\infty}$$
$$\times \prod_{j\leq l=1}^{k-1}\prod_{i=1}^{k-l} \frac{(q^{-2+2j}pq^{-2\triangle^{(i+l)}(c)}; pq^{-2\triangle^{(i+l)}(c)})_\infty}{(q^{2+2j}pq^{-2\triangle^{(i+l)}(c)}; pq^{-2\triangle^{(i+l)}(c)})_\infty}$$
$$\times \frac{(q^{2+2j}pq^{-2\triangle^{(i+l-1)}(c)}; pq^{-2\triangle^{(i+l-1)}(c)})_\infty}{(q^{-2+2j}pq^{-2\triangle^{(i+l-1)}(c)}; pq^{-2\triangle^{(i+l-1)}(c)})_\infty},$$

$$\mathfrak{S}(p,q)_f = \prod_{j=0}^{k-1}\prod_{i=1}^{k-j} \frac{(pq^{2-2i-2(c^{(1)}+\cdots+c^{(j)})}; pq^{-2(c^{(1)}+\cdots+c^{(j)})})_\infty}{(pq^{-2-2i-2(c^{(1)}+\cdots+c^{(j)})}; pq^{-2(c^{(1)}+\cdots+c^{(j)})})_\infty}$$
$$\times \prod_{j\leq l=1}^{k-1}\prod_{i=1}^{k-l} \frac{(q^{-2+2j}pq^{-2\triangle^{(i+l)}(c)}; pq^{-2\triangle^{(i+l)}(c)})_\infty}{(q^{2+2j}pq^{-2\triangle^{(i+l)}(c)}; pq^{-2\triangle^{(i+l)}(c)})_\infty}$$
$$\times \frac{(q^{2+2j}pq^{-2\triangle^{(i+l-1)}(c)}; pq^{-2\triangle^{(i+l-1)}(c)})_\infty}{(q^{-2+2j}pq^{-2\triangle^{(i+l-1)}(c)}; pq^{-2\triangle^{(i+l-1)}(c)})_\infty}.$$

Proof. Let us show the proof of (4.1). From the comultiplication (3.2) in Theorem 3.1, we have the following product on $V_{k+1}(\lambda_i, \mu)$ for some positive integer $N$ over all possible decompositions

$$\sum_{i_1,\dots,i_N \in \{1,\dots,k+1\}} e^{i_1}(z_{i_1}) e^{i_2}(z_{i_2}) \dots e^{i_N}(z_{i_N}), \tag{4.5}$$

where $c^{(i)} = 1$.

From the relations (3.5)–(3.7) in Lemma 3.2, one can show that for $z_{i_{j+1}}/z_{i_j} = q^2$ all terms in (4.5) are zero except for those with indices $i_1 > \cdots > i_{k+1}$. Suppose $N = k+2$ and $z_{i_{j+1}}/z_{i_j} = q^2$, then for $m \neq n$ there is $i_m = i_n$. Thus we get the first condition of integrability. Similarly one can prove the $\mathfrak{F}_{k+2}(z)$ case.

For the vertex operator $\mathfrak{E}_{k+1}(z)$, since the term with $i_1 > \cdots > i_{k+1}$ in (4.5) is not zero, we have

$$\begin{aligned}\mathfrak{E}_{k+1}(z) &= e(zq^k) \otimes e(zq^{k-1})\psi^-(zq^{k-1/2}) \otimes e(zq^{k-2})\psi^-(zq^{k-1/2})\psi^-(zq^{k+3/2}) \\ &\quad \otimes \cdots \otimes e(z)\psi^-(zq^{3/2})\ldots\psi^-(q^{2k-1/2}).\end{aligned}$$

We used relations (2.42) and (2.45) in Proposition 2.11 to write each factor of the tensor product in a normal order form. Then we get

$$\begin{aligned}\mathfrak{E}_{k+1}(z) = \mathfrak{S}(p,q)_e(e(zq^k) \otimes &: e(zq^{k-1})\psi^-(zq^{k-1/2}) : \\ &\otimes : e(zq^{k-2})\psi^-(zq^{k-1/2})\psi^-(zq^{k+3/2}) : \\ &\otimes \cdots \otimes : e(z)\psi^-(zq^{3/2})\ldots\psi^-(q^{2k-1/2}) :).\end{aligned} \tag{4.6}$$

Substitute (2.39) and (2.40) from Theorem 2.10 into the above relation and use (3.11), we get the desired relation (4.3). Relation (4.4) can be proved in a similar way.

Proposition 4.2. *On $V_{k+1}(\lambda_i,\mu)$, the vertex operators $\mathfrak{E}_{k+1}(z)$ and $\mathfrak{F}_{k+1}(z)$ satisfy the following difference equations*

$$\begin{aligned}\mathfrak{E}_{k+1}(zq^2) &= \triangle^k(q^{h+1})\exp\left((q-q^{-1})\sum_{n>0}\triangle^k(\alpha_{-n})(q^{k+1}z)^n\right) \\ &\quad \times \mathfrak{E}_{k+1}(z)\triangle^k(q^{h+1})\exp\left(-(q-q^{-1})\sum_{n>0}\triangle^k(\alpha_n)(q^{k+1}z)^{-n}\right),\end{aligned} \tag{4.7}$$

$$\begin{aligned}\mathfrak{F}_{k+1}(zq^2) &= \triangle^k(q^{-(h+1)})\exp\left((q-q^{-1})\sum_{n>0}\triangle^k(\alpha'_{-n})(q^{k+1}z)^n\right) \\ &\quad \times \mathfrak{F}_{k+1}(z)\triangle^k(q^{-(h+1)})\exp\left(-(q-q^{-1})\sum_{n>0}\triangle^k(\alpha'_n)(q^{k+1}z)^{-n}\right).\end{aligned} \tag{4.8}$$

By means of an elliptic analogue of the Drinfeld coproduct, we have found the higher level module of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

A highest weight $\mathfrak{sl}_2$-module is called integrable if the Chevally generators are locally nilpotent on this module [9]. Proposition VI.5 in Ref. [15] shows that on the level-$k$ standard $\widehat{\mathfrak{sl}}_2$-module, the currents $x_{\pm\alpha}(z)$ are nilpotent operators at $k+1$, $x_{\pm\alpha}(z)^{k+1} = 0$. The authors in [1, 3] found the nilpotent condition for $U_q(\widehat{\mathfrak{sl}}_2)$ integrable module. Here we obtained the elliptic analogue of the nilpotent condition for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ module. In quantum case, the vertex operators $x^{\pm k}(z)$ in [1] satisfy certain $q$-difference equations

$$x^{+k}(zq^2) = \triangle^k \phi^{-1}(zq^{(m+1)/2}) x^{+k}(z) \triangle^k \psi(zq^{3(m+1)/2}),$$
$$x^{-k}(zq^2) = \triangle^k \phi(zq^{-3(m+1)/2}) x^{-k}(z) \triangle^k \psi^{-1}(zq^{-(m+1)/2}),$$

where $\phi(z)$ and $\psi(z)$ are the generating functions of the bosons $a_{-n}, a_n$ $(n \in \mathbf{Z}_{>0})$ respectively. We found the elliptic analogue of these $q$-difference relations. It is clear that the operators $\triangle^k(\psi^{\pm}(z))$ do not appear on the both sides of $\mathfrak{E}_{k+1}(z)$ and $\mathfrak{F}_{k+1}(z)$ in (4.7) and (4.8) respectively unlikely in quantum case because the operators $\triangle^k(\psi^{\pm}(z))$ (3.12)–(3.13) are exponential functions of both annihilation operator $\triangle^k(\alpha_n)$ and creation operator $\triangle^k(\alpha_{-n})$ with $p$ factors.

The authors in [7] compute the correlation function of $U_q(\widehat{\mathfrak{sl}}_2)$ perfect vertex operators using the wheel condition. We expect that we can make a similar application in the elliptic case.

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*Rasha Mohamed Farghly*
*Department of Mathematics*
*Graduate School of Science*
*Hiroshima University*
*Higashi-Hiroshima* 739-8526*, Japan*
*E-mail: d102349@hiroshima-u.ac.jp*