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Construction of spines of two-bridge link complements and upper bounds of their Matveev complexities

Masaharu Ishikawa and Keisuke Nemoto

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Abstract. We give upper bounds of the Matveev complexities of two-bridge link complements by constructing their spines explicitly. In particular, we determine the complexities for an infinite sequence of two-bridge links corresponding to the continued fractions of the form $[2, 1, \dots, 1, 2]$. We also give upper bounds for the 3-manifolds obtained as meridian-cyclic branched coverings of the 3-sphere along two-bridge links.

1. Introduction

Let M be a compact connected 3-manifold possibly with boundary. If M has nonempty boundary then a polyhedron $P \subset M$ to which M collapses is called a *spine* of M. If M is closed then a spine of M means that of $M \setminus B^3$, where B^3 is a 3-ball in M. A spine P of M is said to be *almost-simple* if the link of any point can be embedded into the complete graph K_4 with four vertices. A point of almost-simple spine whose link is K_4 is called a *true vertex*. The minimal number $c(M)$ of true vertices among all almost-simple spines of M is called the *complexity* of M.

The notion of the complexity was introduced by S. Matveev in [8]. The complexity gives an efficient measure on the set of all compact 3-manifolds $\mathscr{M}$, because it has the following properties: the complexity is additive under connected sum, that is, $c(M_1 \# M_2) = c(M_1) + c(M_2)$, and it has a finiteness property, that is, for any $n \in \mathbf{Z}_{\geqslant 0}$, there exists finitely many closed irreducible manifolds $M \in \mathscr{M}$ with $c(M) = n$. Note that if M is closed, irreducible and other than S^3, $\mathbf{RP}^3$ and $L(3,1)$ then $c(M)$ coincides with the minimal number of ideal tetrahedra of all triangulations of M.

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Determining the complexity $c(M)$ of a given 3-manifold M is very difficult in general. For the complexity of the lens space $L(p,q)$, Matveev proved the upper inequality $c(L(p,q)) \leqslant S(p,q) - 3$, where $S(p,q)$ is the sum of all partial quotients in the expansion of p/q as a regular continued fraction with positive entries, and conjectured that the equality holds (see also [7]). In recent studies, Jaco, Rubinstein and Tillmann solved this conjecture positively for some infinite sequences of lens spaces [5]. Petronio and Vesnin studied the complexities of closed 3-manifolds which are obtained as meridian-cyclic branched coverings of S^3 along two-bridge links [10]. In the case of compact manifolds with nonempty boundary, Fominykh and Wiest obtained sharp upper bounds on the complexities of torus link complements [2]. A certain lower bound of the complexity of a two-bridge link complement is given in [10]. There are several related studies, see for instance [3, 4, 1, 6, 12, 13].

In this paper, we give upper bounds of the complexities of two-bridge link complements.

Let $K(p,q)$ be a two-bridge link in the 3-sphere S^3, where p, q are coprime integers with $p \geqslant 2$ and $q \neq 0$. We may represent it by using Conway's notation as $C(a_1,\dots,a_n)$, where the integers a_i are the partial quotients of a regular continued fraction of p/q. We represent the regular continued fraction of p/q as $p/q = [a_1,\dots,a_n]$. For each continued fraction, $K(p,q)$ has a diagram as shown in Figure 1. By taking the mirror image if necessary, we may assume that $p/q > 0$. In this paper we only consider the continued fraction of p/q such that each a_i is positive and a_1, $a_n > 1$. Let $N(K(p,q))$ be a compact tubular neighborhood of $K(p,q)$ in S^3 and $\operatorname{int} N(K(p,q))$ its interior.

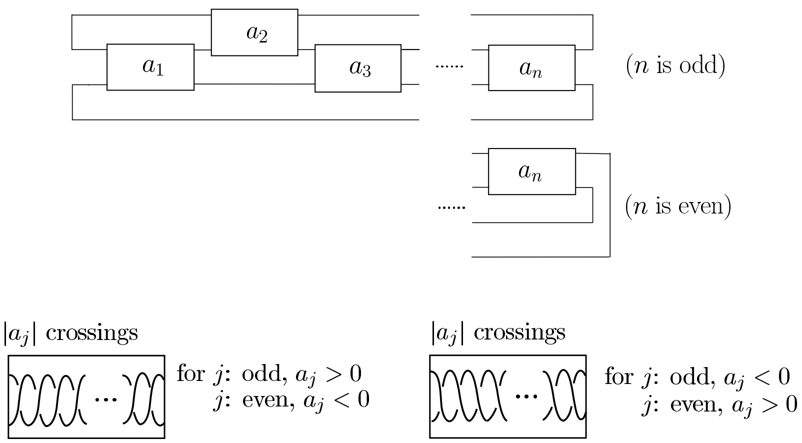

Fig. 1. Two-bridge link $C(a_1,\dots,a_n)$.

The main result of this paper is the following:

THEOREM 1.1. *Let $K(p,q)$ be a two-bridge link with $p/q = [a_1, \dots, a_n]$, $a_i > 0$ and $a_1, a_n > 1$. Then we have*

$$c(S^3 \backslash \operatorname{int} N(K(p,q))) \leqslant \sum_{i=1}^{n} a_i + 2(n-3) - \#\{a_i = 1\},$$

where $\#\{a_i = 1\}$ is the number of indices i such that $a_i = 1$.

As a corollary, we determine the complexities of an infinite sequence of two-bridge links.

COROLLARY 1.1. *Let $K(p,q)$ be a two-bridge link with $p/q = [2, \underbrace{1, \dots, 1}_{n-2}, 2]$, $n \geqslant 2$. Then we have*

$$c(S^3 \backslash \operatorname{int} N(K(p,q))) = 2n - 2.$$

As an application of Theorem 1.1, we obtain the following upper bounds on the complexity of meridian-cyclic d-fold branched coverings of S^3, which are sharper than the upper bounds given in [10].

COROLLARY 1.2. *Let $M_d(K(p,q))$ be the meridian-cyclic branched covering of S^3 along $K(p,q)$ of degree d. Then we have*

$$c(M_d(K(p,q))) \leqslant d \left(\sum_{i=1}^{n} a_i + 2(n-3) - \#\{a_i = 1\} \right) + rd,$$

where r is one if p is odd and three if p is even.

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2. Proof of Theorem 1.1

We will show the upper bound of the complexity in Theorem 1.1 by constructing an almost-simple spine of the two-bridge link complement explicitly.

Let $K(p,q)$ be a two-bridge link in the 3-sphere S^3, where p, q are coprime integers with $p \geqslant 2$ and $q > 0$. We may represent it as $C(a_1, \dots, a_n)$, where the integers a_i are the partial quotients of a regular continued fraction of p/q. For each $i = 1, \dots, n$, let T_i be the tangle containing the a_i twists in

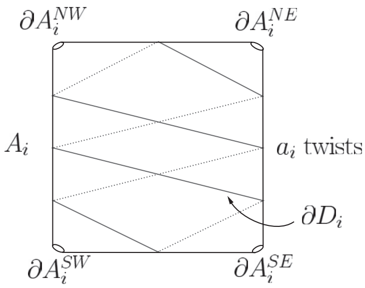

Fig. 2. Pillowcase A_i and its boundary components.

Figure 1, which is a 3-ball with two tunnels mutually twisted a_i-times. For each $i = 1, \dots, n$, let A_i be the union of a sphere with four holes and a disk which is placed inside the sphere twisted a_i-times as shown in Figure 2. Each A_i is a spine of T_i obtained by collapsing it from $\partial N(K(p,q))$ with keeping the holed sphere on the boundary of T_i. In this paper, we call A_i a *pillowcase*. We denote the disk lying in the middle of A_i by D_i. The pillowcase A_i has four boundary components. We denote these components by $\partial A_i^{\mathrm{NW}}$, $\partial A_i^{\mathrm{NE}}$, $\partial A_i^{\mathrm{SW}}$, $\partial A_i^{\mathrm{SE}}$, see Figure 2. We will make a spine of $S^3 \backslash \operatorname{Int} N(K(p,q))$ by gluing these pillowcases.

We first prove the following weaker version of Theorem 1.1.

Lemma 2.1. *Let $C(a_1, \dots, a_n)$ be a two-bridge link, where each a_i is positive and $a_1, a_n > 1$. Then we have*

$$c(S^3 \backslash \operatorname{int} N(C(a_1, \dots, a_n))) \leqslant \sum_{i=1}^{n} a_i + 2(n-3).$$

Proof. We first construct a spine P of $S^3 \backslash \operatorname{Int} N(C(a_1, \dots, a_n))$ by gluing the pillowcases $A_1, \dots, A_n$ together as follows:

- For each adjacent pair of pillowcases A_i and A_j, $1 \leqslant i, j \leqslant n-1$, we attach two tubes in order to pass the knot strands and then glue a disk for each region bounded by the tubes and pillowcases as shown in Figure 3.
- We attach three tubes and two disks to the pillowcase A_n as shown in Figure 4.

Next, we make a spine P_0 by collapsing P from ∂A_1 and ∂A_n, which decreases the number of true vertices. Let x_i be the number of true vertices on ∂D_i in the spine P_0. Since any true vertex of P lies on ∂D_i for some $1 \leqslant i \leqslant n$, there exists $\sum_{i=1}^{n} x_i$ true vertices on P_0.

We calculate the number of true vertices in the spine P_0.

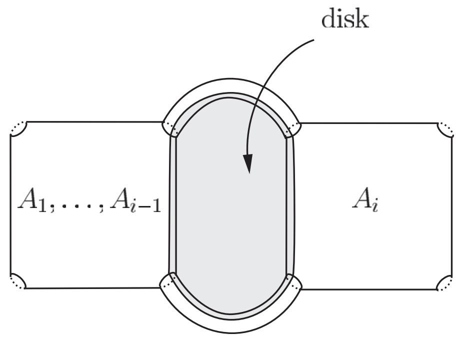

Fig. 3. Gluing a pair of adjacent pillowcases.

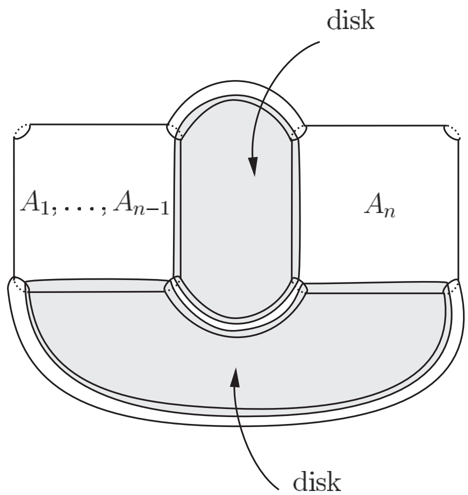

Fig. 4. Gluing A_{n-1} and A_n.

- For the pillowcase A_1:
  True vertices which lie on ∂D_1 are $y_1^{(1)}, \ldots, y_{a_1}^{(1)}, y_{a_1+1}^{(1)}$ shown in Figure 5. Since the true vertices $y_1^{(1)}$, $y_2^{(1)}$ are removed by the collapsing from $\partial A_1^{\mathrm{NW}}$, we get $x_1 = a_1 - 1$.
- For pillowcases A_i, where $i = 2, \ldots, n-1$:
  True vertices which lie on ∂D_i are $y_1^{(i)}, \ldots, y_{a_i}^{(i)}, y_{a_i+1}^{(i)}, y_{a_i+2}^{(i)}$ shown in Figure 6. Therefore, we get $x_i = a_i + 2$.
- For the pillowcase A_n:
  True vertices which lie on ∂D_n are $y_1^{(n)}, \ldots, y_{a_n}^{(n)}, y_{a_n+1}^{(n)}$ shown in Figure 7. Since the true vertices $y_{a_n-1}^{(n)}$, $y_{a_n}^{(n)}$ are removed by the collapsing from $\partial A_n^{\mathrm{SW}}$, we get $x_n = a_n - 1$.

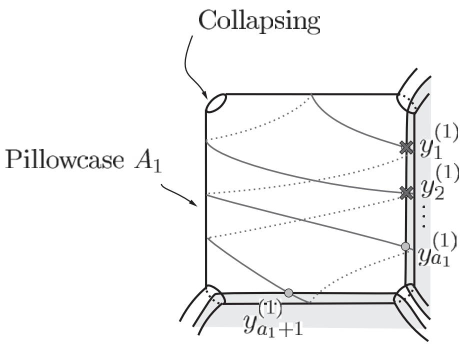

Fig. 5. True vertices in the pillowcase A_1.

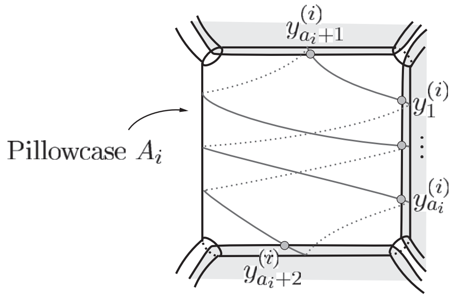

Fig. 6. True vertices in the pillowcase A_i.

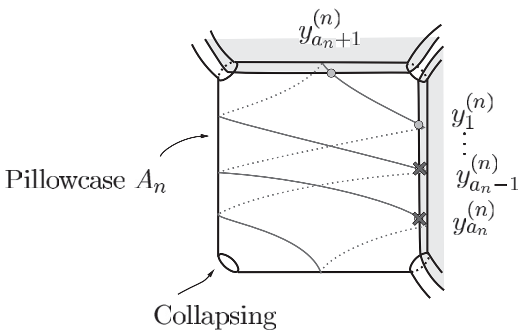

Fig. 7. True vertices in the pillowcase A_n.

In summary, the number of true vertices in each pillowcase is

$$\begin{cases} x_1 = a_1 - 1 \\ x_i = a_i + 2 \quad (i = 2, \dots, n-1) \\ x_n = a_n - 1. \end{cases}$$

Therefore, the number of true vertices in the spine P_0 is

$$\sum_{i=1}^{n} x_i = a_1 - 1 + \sum_{i=2}^{n-1} (a_i + 2) + a_n - 1$$

$$= \sum_{i=1}^{n} a_i + 2(n-3). \qquad \square$$

Proof of Theorem 1.1. We will make a new spine P' from P_0 constucted in Lemma 2.1 by collapsing it as follows (An example of P' is given in Example 2.1 below): Let $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n$ be the set of indices with $a_{i_j} = 1$, $j = 0, \dots, r-1$. For each j, let P_{j+1} be a spine obtained from P_j by applying the replacement shown in Figure 8. The left figure represents the replacement in the case of $a_{i_j - 1} > 1$ and the right one is in the case of $a_{i_j - 1} = 1$. Applying this replacement inductively for $j = 0, \dots, r-1$, we get a new spine P_r of $S^3 \backslash C(a_1, \dots, a_n)$.

Let P'_j be the spine obtained from P_j by collapsing it from the boundary. In the following we fix j, $0 \leqslant j \leqslant r$, and denote i_j by i for simplicity. Let $x_k^{(j)}$ be the number of true vertices on ∂D_k in the spine P'_j, where $k = 1, \dots, n$.

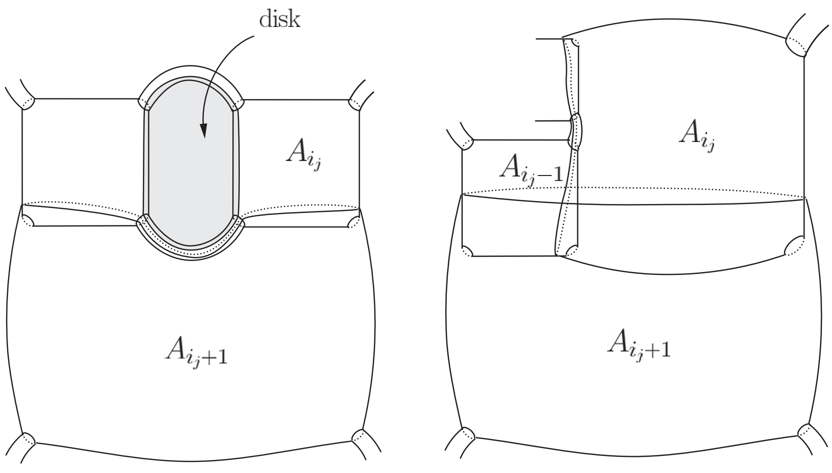

Fig. 8. Gluing pillowcases together in the case of $a_{i_j-1} > 1$ (shown in the left hand side) and $a_{i_j-1} = 1$ (shown in the right hand side).

All true vertices of P_j which do not lie on ∂D_k are removed in P_j' by the collapsing. Therefore, there exists $\sum_{k=1}^{n} x_k^{(j)}$ true vertices in P_j'.

We now calculate the difference of the numbers of true vertices between P_j' and P_{j+1}'.

(i) In the case of $a_{i-1} > 1$.

- For the pillowcase A_{i-1}:
  True vertices which lie on ∂D_{i-1} in P_j' are $y_1^{(i-1)}, \dots, y_{a_{i-1}}^{(i-1)}, y_{a_{i-1}+1}^{(i-1)}$, $y_{a_{i-1}+2}^{(i-1)}$ shown on the left hand side in Figure 9. Hence, we get $x_{i-1}^{(j)} = a_{i-1} + 2$. On the other hand, in P_{j+1}', true vertices which lie on ∂D_{i-1} are $z_1^{(i-1)}, \dots, z_{a_{i-1}}^{(i-1)}, z_{a_{i-1}+1}^{(i-1)}, z_{a_{i-1}+2}^{(i-1)}, z_{a_{i-1}+3}^{(i-1)}$ shown on the right hand side, and hence $x_{i-1}^{(j+1)} = a_{i-1} + 3$. Thus $x_{i-1}^{(j+1)} = x_{i-1}^{(j)} + 1$.
- For the pillowcase A_i:
  True vertices which lie on ∂D_i in P_j' are $y_{a_i}^{(i)}$, $y_{a_i+1}^{(i)}$, $y_{a_i+2}^{(i)}$ shown on the left hand side in Figure 9, and hence we get $x_i^{(j)} = a_i + 2 = 3$. On the other hand, true vertices which lie on ∂D_i in the P_{j+1} are $z_1^{(i)}, \dots, z_4^{(i)}$ shown on the right hand side. By collapsing from $\partial A_i^{\mathrm{SE}}$, the true vertices $z_1^{(i)}$, $z_2^{(i)}$ are removed in P_{j+1}'. Hence $x_i^{(j+1)} = 2$. Therefore we get $x_i^{(j+1)} = x_i^{(j)} - 1$.
- For the pillowcase A_{i+1}:
  True vertices which lie on ∂D_{i+1} in P_j' are $y_1^{(i+1)}, \dots, y_{a_{i+1}}^{(i+1)}, y_{a_{i+1}+1}^{(i+1)}$, $y_{a_{i+1}+2}^{(i+1)}$ shown on the left hand side in Figure 9, and hence we get

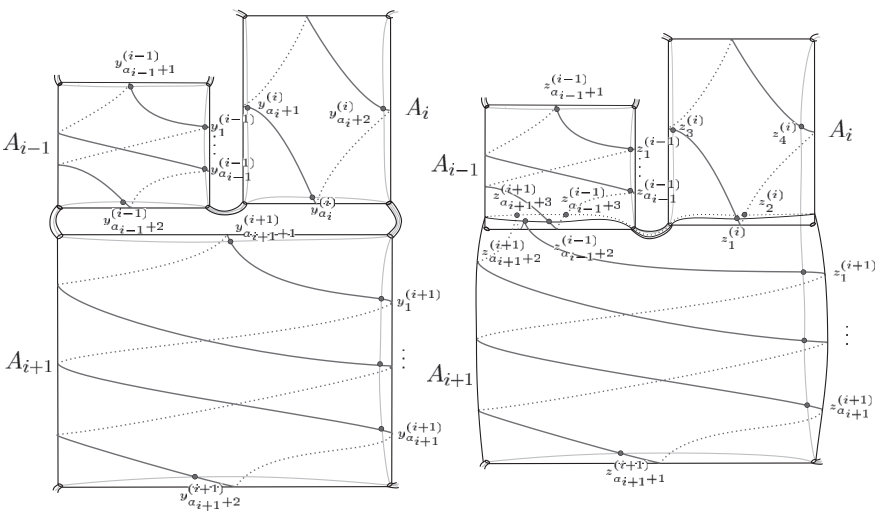

Fig. 9. True vertices in the spine P_j (shown in the left hand side) and in the spine P_{j+1} (shown in the right hand side) in the case of $a_{i-1} > 1$.

$x_{i+1}^{(j)} = a_{i+1} + 2$. On the other hand, true vertices which lie on ∂D_{i+1} in P_{j+1} are $z_1^{(i+1)}, \dots, z_{a_{i+1}}^{(i+1)}, z_{a_{i+1}+1}^{(i+1)}, z_{a_{i+1}+2}^{(i+1)}, z_{a_{i+1}+3}^{(i+1)}$ shown on the right hand side. By collapsing from $\partial A_{i-1}^{\mathrm{SW}}$, the true vertices $z_{a_{i+1}+2}^{(i+1)}$, $z_{a_{i+1}+3}^{(i+1)}$ are removed in P'_{j+1}. Hence $x_{i+1}^{(j+1)} = a_{i+1} + 1$. Thus we get $x_{i+1}^{(j+1)} = x_{i+1}^{(j)} - 1$.

- For the other pillowcases A_k, where $k \neq i-1, i, i+1$:
  Since the replacement $P_j \to P_{j+1}$ does not change the true vertices in A_k, we get $x_k^{(j+1)} = x_k^{(j)}$.

(ii) In the case of $a_{i-1} = 1$.

- For the pillowcase A_{i-1}:
  By the replacement $P_j \to P_{j+1}$, the true vertex $z^{(i-1)}$ shown in Figure 10 appears and this is not removed in P'_{j+1}. Therefore, we get $x_{i-1}^{(j+1)} = x_{i-1}^{(j)} + 1$.
- For the pillowcase A_i:
  True vertices which lie on ∂D_i in P_j are $y_{a_i}^{(i)}$, $y_{a_i+1}^{(i)}$, $y_{a_i+2}^{(i)}$, $y_{a_i+3}^{(i)}$ shown on the left hand side in Figure 10. By collapsing P_j from $\partial A_{i-2}^{\mathrm{NE}}$, the true vertices $y_{a_i+2}^{(i)}$, $y_{a_i+3}^{(i)}$ are removed in P'_{j+1}. Hence $x_i^{(j)} = a_i + 1 = 2$. On the other hand, true vertices which lie on ∂D_i in P_{j+1} are $z_{a_i}^{(i)}, \dots, z_{a_i+4}^{(i)}$ shown on the right hand side. By collapsing from $\partial A_{i-2}^{\mathrm{NE}}$ and $\partial A_i^{\mathrm{SE}}$, the true vertices $z_{a_i+2}^{(i)}$, $z_{a_i+3}^{(i)}$ and $z_{a_i}^{(i)}$, $z_{a_i+4}^{(i)}$ are removed in P'_{j+1}, respectively, and hence $x_i^{(j+1)} = 1$. Thus we get $x_i^{(j+1)} = x_i^{(j)} - 1$.

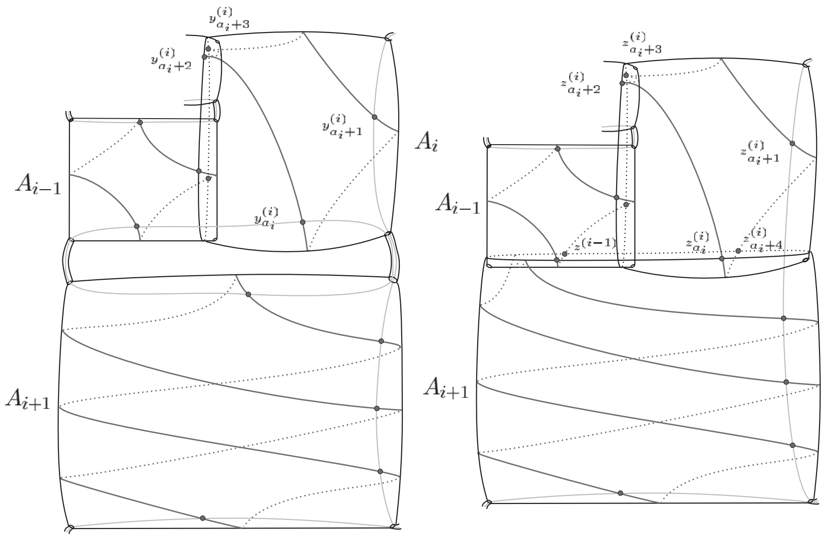

Fig. 10. True vertices in the spine P_j (shown in the left hand side) and in the spine P_{j+1} (shown in the right hand side) in the case of $a_{i-1} = 1$.

- For the pillowcase A_{i+1}:
  Applying the same argument as in case (i), we get $x_{i+1}^{(j+1)} = x_{i+1}^{(j)} - 1$.
- For the other pillowcases A_k, where $k \neq i-1, i, i+1$:
  Since the replacement $P_j \to P_{j+1}$ does not change the true vertices in A_k, we get $x_k^{(j+1)} = x_k^{(j)}$.

By the above arguments (i) and (ii), the number of true vertices in each pillowcase changes as

$$\begin{cases} x_{i-1}^{(j+1)} = x_{i-1}^{(j)} + 1 \\ x_{i}^{(j+1)} = x_{i}^{(j)} - 1 \\ x_{i+1}^{(j+1)} = x_{i+1}^{(j)} - 1 \\ x_{k}^{(j+1)} = x_{k}^{(j)} \quad (k \neq i-1, i, i+1). \end{cases}$$

Therefore, we get

$$\sum_{k=1}^{n} x_k^{(j+1)} = \sum_{k=1}^{n} x_k^{(j)} - 1.$$

Since P_j' has $\sum_{k=1}^{n} x_k^{(j)}$ true vertices, the replacement $P_j' \to P_{j+1}'$ decreases the number of true vertices in P_{j+1}' by one. Hence, by the inductive sequence $P_0' \to P_1' \to \cdots \to P_r' = P'$, the number of true vertices decreases by r. Now we apply Lemma 2.1. The number of true vertices in the spine P' is

$$\sum_{i=1}^{n} a_i + 2(n-3) - r = \sum_{i=1}^{n} a_i + 2(n-3) - \#\{a_i = 1\}.$$

This completes the proof. □

Example 2.1. The spine P_r of $S^3 \backslash \operatorname{int} N(C(3,2,1,3,3))$ is as shown in Figure 11. The spine P' is obtained by collapsing this from its boundary components.

Remark 2.1. Let $P(a_1, \ldots, a_n)$ be a pretzel link, where $|a_i| > 0$, $|a_1|$, $|a_n| > 1$. We can construct a spine P of the complement of $P(a_1, \ldots, a_n)$, by attaching $2(n-1)+1$ tubes and n disks to the n tangles with a_i-twists, $i = 1, \ldots, n$ by a way similar to what we did in the proof of Lemma 2.1. Then the number of true vertices of the spine obtained from P by collapsing becomes

$$|a_1| + 2\sum_{i=1}^{n} |a_i| + |a_n| + n - 4,$$

which is an upper bound of the complexity $c(S^3 \backslash P(a_1, \ldots, a_n))$. See [9] for precise discussion.

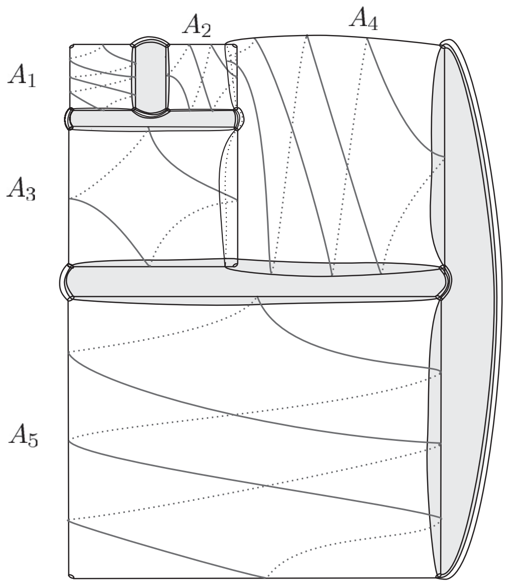

Fig. 11. The spine P_r of $S^3\backslash \mathrm{int}\, N(C(3,2,1,3,3))$.

3. Proof of Corollaries 1.1 and 1.2

To prove Corollary 1.1, we need to have a lower bound on the complexity of $S^3\backslash C(a_1,\dots,a_n)$. Let $\mathrm{vol}(M)$ denote the hyperbolic volume of a hyperbolic 3-manifold M, and v_3 denote the hyperbolic volume of the regular ideal tetrahedron in the hyperbolic 3-space $\mathbf{H}^3$, that is, $v_3 = 1.01494\dots$.

In order to prove Corollary 1.1, we will use the following theorems.

THEOREM 3.1 ([10]). *Let $K(p,q)$ be a hyperbolic two-bridge link with $p/q = [a_1,\dots,a_n]$, $a_i > 0$ and $a_1, a_n > 1$. Then,*

$$\mathrm{vol}(S^3\backslash \mathrm{int}\, N(K(p,q))) \geqslant v_3 \cdot \max\{2, 2n - 2.6667\dots\}. \tag{1}$$

A spine is said to be *simple* if the link of each point is either a circle, a theta-graph, or K_4, and it is said to be *special* if each 2-stratum of the simple spine is an open disk, and each 1-stratum is an open interval. Remark that if P is a special spine of a link complement M, then its dual is a topological ideal triangulation of M, and vice versa.

THEOREM 3.2 ([7]). *Let M be a compact, irreducible and boundary-irreducible 3-manifold which differs from a 3-ball, S^3, $\mathbf{RP}^3$, $L(3,1)$ and suppose that all proper annuli in M are inessential. Then, for any almost-simple spine of*

M, there exists a special spine of M which has the same or a fewer number of true vertices.

Now we give a few notations. Suppose that $K(p,q)$ is hyperbolic. Let $\mathscr{T}$ denote a topological ideal triangulation of $S^3 \setminus \operatorname{int} N(K(p,q))$, $n(\mathscr{T})$ denote the number of ideal tetrahedra of $\mathscr{T}$ and $\sigma_{\text{ideal}}(S^3 \setminus \operatorname{int} N(K(p,q)))$ denote the minimal number of $n(\mathscr{T})$. By inequality (1) we have

$$\begin{aligned} \sigma_{\text{ideal}}(S^3 \setminus \operatorname{int} N(K(p,q))) &\geqslant \frac{\operatorname{vol}(S^3 \setminus \operatorname{int} N(K(p,q)))}{v_3} \\ &\geqslant \max\{2, 2n - 2.6667\ldots\}. \end{aligned} \tag{2}$$

The next proposition suggests that we can replace the left hand side of inequality (2) by the complexity $c(S^3 \setminus \operatorname{int} N(K(p,q)))$.

PROPOSITION 3.1. *If L is a hyperbolic link then,*

$$\sigma_{\text{ideal}}(S^3 \setminus \operatorname{int} N(L)) = c(S^3 \setminus \operatorname{int} N(L)).$$

PROOF. Let us prove the inequality ($\leqslant$). Since $S^3 \setminus \operatorname{int} N(L)$ is hyperbolic, it is irreducible and boundary-irreducible, and contains no essential annuli. By Theorem 3.2, we can deform any almost-simple spine of $S^3 \setminus \operatorname{int} N(L)$ into a special one such that it has the same or a fewer number of true vertices. Since the dual of a special spine is a topological ideal triangulation of $S^3 \setminus \operatorname{int} N(L)$, we have $\sigma_{\text{ideal}}(S^3 \setminus \operatorname{int} N(L)) \leqslant c(S^3 \setminus \operatorname{int} N(L))$. The inverse inequality ($\geqslant$) is obvious, since $\{\text{Special spine}\} \subset \{\text{Almost-simple spine}\}$. □

From Theorem 1.1, Theorem 3.1 and Proposition 3.1, we can determine the exact values of the complexities for an infinite sequence of two-bridge links as mentioned in Corollary 1.1.

PROOF OF COROLLARY 1.1. Let $K(p,q)$ be hyperbolic. By Theorem 3.1 and Proposition 3.1, the following inequality holds:

$$\begin{aligned} 2n - 2.66 \leqslant \sigma_{\text{ideal}}(S^3 \setminus \operatorname{int} N(K(p,q))) &= c(S^3 \setminus \operatorname{int} N(K(p,q))) \\ &\leqslant \sum_{i=1}^{n} a_i + 2(n-3) - \#\{a_i = 1\}. \end{aligned}$$

In particular, if $K(p,q) = C(2,1,\ldots,1,2)$ then

$$\begin{aligned} 2n - 2.66 \leqslant \sigma_{\text{ideal}}(S^3 \setminus \operatorname{int} N(K(p,q))) &= c(S^3 \setminus \operatorname{int} N(K(p,q))) \\ &\leqslant 2n - 2. \end{aligned}$$ □

REMARK 3.1. Sakuma and Weeks constructed the canonical decompositions of hyperbolic two-bridge link complements explicitly in [11]. Calculating the number of ideal tetrahedra in their ideal triangulation, we get the upper bound

$$\sigma_{\mathrm{ideal}}(S^3\backslash \mathrm{int}\, N(K(p,q))) \leqslant 2\sum_{i=1}^{n} a_i - 6. \tag{3}$$

Let c be the number of true vertices of the spine constructed in Theorem 1.1. We can obtain a special spine with the same as or a fewer number of vertices than c by applying Theorem 3.2. Hence by considering its dual, we can obtain a topological ideal triangulation of $S^3\backslash K(p,q)$ consisting of at most c ideal tetrahedra. If $p/q = [2,1,\dots,1,2]$ then the upper bound c of the number of ideal tetrahedra constructed in Theorem 1.1 coincides with the upper bound in inequality (3). In general, the upper bounds obtained by our construction are better than those obtained in [11].

Finally we give a proof of Corollary 1.2.

Proof of Corollary 1.2. Since $a_1 > 1$, there exists a tube connecting A_1 and A_2 such that the union of the meridian-disk D of this tube and the spine P' constructed in the proof of Theorem 1.1 has only one true vertex on the boundary of D. Let P_d be a spine of the d-fold cyclic covering space $\tilde{M}$ of $S^3\backslash K(p,q)$ induced by P'.

Suppose that p is odd, that is, $K(p,q)$ is a knot. Recall that the complexity of a closed 3-manifold is by definition the complexity of that manifold minus an open ball. Therefore, the complexity of $M_d(K(p,q))$ is at most the number of true vertices of the spine obtained from P_d by attaching a meridian-disk along the preimage of the boundary of D. Thus we have

$$c(M_d(K(p,q))) \leqslant d\left(\sum_{i=1}^{n} a_i + 2(n-3) - \#\{a_i = 1\}\right) + d.$$

Suppose that p is even, that is, $K(p,q)$ is a link. We attach one more meridian-disk D' to the other boundary component of $\tilde{M}$ such that the union of P_d, D and D' has two true vertices on the boundary of D'. It is known in [7] that the complexity does not change even if we remove several open balls. Therefore, the complexity of $M_d(K(p,q))$ is bounded above by the number of true vertices of this union. Thus we have

$$c(M_d(K(p,q))) \leqslant d\left(\sum_{i=1}^{n} a_i + 2(n-3) - \#\{a_i = 1\}\right) + 3d.$$

This completes the proof. □

Remark 3.2. The complexities of 3-manifolds obtained as meridian-cyclic branched coverings along two-bridge links had been studied in [10]. We can easily check that the upper bound in Corollary 1.2 is better than theirs.

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Masaharu Ishikawa
Mathematical Institute
Tohoku University
Sendai, 980-8578, *Japan*
E-mail: ishikawa@math.tohoku.ac.jp

Keisuke Nemoto
Mathematical Institute
Tohoku University
Sendai, 980-8578, *Japan*
E-mail: keisuke.nemo0121@gmail.com