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On the classification of certain ternary codes of length 12

Makoto ARAYA and Masaaki HARADA

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ABSTRACT. Shimada and Zhang studied the existence of polarizations on some supersingular $K3$ surfaces by reducing the existence of the polarizations to that of ternary $[12,5]$ codes satisfying certain conditions. In this note, we give a classification of ternary $[12,5]$ codes satisfying the conditions. To do this, ternary $[10,5]$ codes are classified for minimum weights 3 and 4.

1. Introduction

A ternary $[n,k]$ code C is a k-dimensional vector subspace of $\mathbf{F}_3^n$, where $\mathbf{F}_3$ denotes the finite field of order 3. The weight $\mathrm{wt}(x)$ of a vector x is the number of non-zero components of x. The minimum non-zero weight of all codewords in C is called the minimum weight of C. A ternary $[n,k,d]$ code is a ternary $[n,k]$ code with minimum weight d. Throughout this note, we denote the minimum weight of a code C by $d(C)$.

Shimada and Zhang [9] studied the existence of polarizations on the supersingular $K3$ surfaces in characteristic 3 with Artin invariant 1 (see [9, Theorem 1.5] for the details). This was done by reducing the problem of the existence of the polarizations to a problem of the existence of ternary $[12,5]$ codes C satisfying the following conditions:

$$\mathrm{wt}((x_1,x_2,\dots,x_{10})) \equiv y_1 y_2 \pmod 3, \tag{1}$$

$$\text{if } c \text{ is not the zero vector, then } \mathrm{wt}((x_1,x_2,\dots,x_{10})) \geq 3, \tag{2}$$

$$\text{if } \mathrm{wt}((x_1,x_2,\dots,x_{10})) = 3, \text{ then } (y_1,y_2) \neq (0,0), \tag{3}$$

for any codeword $c = (x_1,x_2,\dots,x_{10},y_1,y_2) \in C$ (see [9, Claim 5.2]). Seven ternary $[12,5]$ codes satisfying the conditions (1)–(3) were found by Shimada and Zhang [9]. This motivates us to classify all such ternary $[12,5]$ codes.

For ternary $[12,5]$ codes satisfying the conditions (1)–(3), the following equivalence is considered in [9]. We say that two ternary $[12,5]$ codes

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satisfying the conditions (1)–(3) are *SZ-equivalent* if one can be obtained from the other by using the following:

$$(x_1, \dots, x_{10}, y_1, y_2) \mapsto ((-1)^{\alpha_1} x_{\sigma(1)}, \dots, (-1)^{\alpha_{10}} x_{\sigma(10)}, (-1)^{\beta} y_{\tau(1)}, (-1)^{\beta} y_{\tau(2)}), \quad (4)$$

where $\alpha_1, \dots, \alpha_{10}, \beta \in \{0, 1\}$ and $\sigma \in S_{10}$, $\tau \in S_2$ (see [9, Remark 5.3]). Here, S_n denotes the symmetric group of degree n.

The main aim of this note is to give the following classification, which is based on a computer calculation.

THEOREM 1. *Any ternary* $[12, 5]$ *code satisfying the conditions* (1)–(3) *is SZ-equivalent to one of the seven codes given in* [9, Remark 5.3].

To complete the above classification, ternary $[10, 5, d]$ codes are classified for the cases $d = 3$ and 4.

2. Characterization of ternary $[12, 5]$ codes satisfying (1)–(3)

Let C be a ternary $[n, k]$ code. The code obtained from C by deleting some coordinates I in each codeword is called the *punctured code* of C on I. Throughout this note, we denote the punctured code of a ternary $[12, 5]$ code C on $\{11, 12\}$ by $Pun(C)$. Let $d_{max}(n, k)$ denote the largest minimum weight among ternary $[n, k]$ codes. It is known that $d_{max}(10, 5) = 5$ and $d_{max}(12, 5) = 6$ (see [2], [5]).

LEMMA 1. *If* C *is a ternary* $[12, 5]$ *code satisfying the condition* (2), *then* $Pun(C)$ *is a ternary* $[10, 5]$ *code and* $d(Pun(C)) \in \{3, 4, 5\}$.

PROOF. Suppose that $Pun(C)$ has dimension at most 4. Then we may assume without loss of generality that C has generator matrix whose first row is $(0, 0, \dots, 0, y_1, y_2)$, where $(y_1, y_2) \neq (0, 0)$. This contradicts with the condition (2). Hence, $Pun(C)$ is a ternary $[10, 5]$ code. Again, by the condition (2), $Pun(C)$ has minimum weight at least 3. Since $d_{max}(10, 5) = 5$, the result follows.

LEMMA 2. *Let* C *be a ternary* $[12, 5]$ *code satisfying the conditions* (1)–(3).
(i) $d(Pun(C)) \in \{4, 5\}$ *if and only if* $d(C) = 6$.
(ii) $d(Pun(C)) = 3$ *if and only if* $d(C) = 4$.

PROOF. By Lemma 1, $Pun(C)$ is a ternary $[10, 5]$ code and $d(Pun(C)) \in \{3, 4, 5\}$. It is trivial that $d(C) - d(Pun(C)) \in \{0, 1, 2\}$.

Suppose that $d(Pun(C)) \in \{4, 5\}$. Let $x = (x_1, \dots, x_{10})$ be a codeword of $Pun(C)$. If $\mathrm{wt}(x) = 4$ (resp. 5), then any corresponding codeword $(x_1, \dots, x_{10}, y_1, y_2)$ of C has weight 6 (resp. 7), by the condition (1). Since $d_{max}(12, 5) = 6$, we have that $d(C) = 6$. Conversely, if $d(C) = 6$, then it follows from $d_{max}(10, 5) = 5$ that $d(Pun(C)) \in \{4, 5\}$.

Suppose that $d(Pun(C)) = 3$. Let $x = (x_1, \dots, x_{10})$ be a codeword of $Pun(C)$. If $\mathrm{wt}(x) = 3$, then any corresponding codeword $(x_1, \dots, x_{10}, y_1, y_2)$ of C has weight 4, by the conditions (1) and (3). Hence, we have that $d(C) = 4$. Conversely, suppose that $d(C) = 4$. Then $d(Pun(C)) \in \{2, 3, 4\}$. By the condition (2), $d(Pun(C)) \in \{3, 4\}$. From the statement (i), $d(Pun(C)) = 3$.

Recall that two ternary codes are equivalent if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates. For ternary $[10, 5]$ codes, we consider this usual equivalence.

Lemma 3. *Let C and C' be ternary $[12, 5]$ codes satisfying the conditions (1)–(3). Suppose that C and C' are SZ-equivalent. Then $Pun(C)$ and $Pun(C')$ are equivalent.*

Proof. Suppose that C is obtained from C' by (4). Then $Pun(C)$ can be obtained from $Pun(C')$ by

$$(x_1, \dots, x_{10}) \mapsto ((-1)^{\alpha_1} x_{\sigma(1)}, \dots, (-1)^{\alpha_{10}} x_{\sigma(10)}).$$

By considering the inverse operation of puncturing, one can construct ternary $[12, 5]$ codes satisfying the conditions (1)–(3) as follows. Throughout this note, we denote the ternary code having generator matrix G by $C(G)$. Suppose that $C(G)$ is a ternary $[10, 5]$ code and $d(C(G)) \in \{3, 4, 5\}$. Let g_i denote the ith row of G. Consider the following generator matrix:

$$\left(\begin{array}{c|cc} & a_1 & b_1 \\ G & \vdots & \vdots \\ & a_5 & b_5 \end{array} \right), \tag{5}$$

where

$$(a_i, b_i) = \begin{cases} (0,0), (0,1), (0,2), (1,0), (2,0) & \text{if } \mathrm{wt}(g_i) \equiv 0 \pmod 3, \\ (1,1), (2,2) & \text{if } \mathrm{wt}(g_i) \equiv 1 \pmod 3, \\ (1,2), (2,1) & \text{if } \mathrm{wt}(g_i) \equiv 2 \pmod 3. \end{cases}$$

We denote this generator matrix by $G(a, b)$, where $a = (a_1, \dots, a_5)$ and $b = (b_1, \dots, b_5)$. The set of the codes $C(G(a, b))$ contains all ternary $[12, 5]$ codes C satisfying the conditions (1) and $Pun(C(G(a, b))) = C(G)$. Hence, in this way, every ternary $[12, 5]$ code satisfying the conditions (1)–(3) can be obtained from some ternary $[10, 5]$ code. Here, by Lemma 2, its minimum weight is 3, 4 or 5. In addition, if $C(G)$ and $C(G')$ are equivalent $[10, 5]$

codes, then the sets of all codes $C(G(a,b))$ satisfying the conditions (1)–(3) is obtained from the set of all codes $C(G'(a,b))$ satisfying the same conditions by considering (4) with $\beta = 0$ and τ is the identity permutation. Hence, it is sufficient to consider only inequivalent ternary $[10,5,d]$ codes with $d \in \{3,4,5\}$ for the classification of ternary $[12,5]$ codes satisfying the conditions (1)–(3). This is a reason why we consider the classification of ternary $[10,5,d]$ codes with $d \in \{3,4,5\}$ in the next section.

3. Ternary $[10,5,d]$ codes with $d \in \{3,4,5\}$

There is a unique ternary $[10,5,5]$ code, up to equivalence [6]. In this section, we give a classification of ternary $[10,5,d]$ codes with $d \in \{3,4\}$, which is based on a computer calculation.

We describe how ternary $[10,5,3]$ codes and $[10,5,4]$ codes were classified. Let C be a ternary $[10,5,3]$ code (resp. $[10,5,4]$ code). We may assume without loss of generality that C has generator matrix of the following form:

$$G = \begin{pmatrix} I_5 & A \end{pmatrix},$$

where A is a 5×5 matrix over $\mathbf{F}_3$ and I_5 denotes the identity matrix of order 5. Thus, we only need consider the set of A, rather than the set of generator matrices. The set of matrices A was constructed, row by row, as follows, by a computer calculation. Let r_i be the ith row of A. Then, we may assume without loss of generality that $r_1 = (0,0,0,1,1)$ (resp. $r_1 = (0,0,1,1,1)$), by permuting and (if necessary) changing the signs of the columns of A.

Let $e_1, \dots, e_5$ denote the vectors $(1,0,0,0,0), \dots, (0,0,0,0,1)$, respectively. We denote the ternary code generated by vectors $y_1, y_2, \dots, y_s$ by $\langle y_1, y_2, \dots, y_s \rangle$. For $x = (x_1, \dots, x_5) \in \mathbf{F}_3^5$, consider the following conditions:

- the first nonzero element of x is 1,
- $\mathrm{wt}(x) \geq 2$ (resp. $\mathrm{wt}(x) \geq 3$),
- the ternary code $\langle (e_1, r_1), (e_2, x) \rangle$ has minimum weight 3 (resp. 4),
- $x_1 \leq x_2 \leq x_3 \leq 1$ and $x_4 \leq x_5$ (resp. $x_1 \leq x_2 \leq 1$ and $x_3 \leq x_4 \leq x_5$), where we consider a natural order on the elements of $\mathbf{F}_3 = \{0,1,2\}$ by $0 < 1 < 2$.

The determination of the minimum weights was done by a computer calculation for all codes in this note. Let X_1 be the set of vectors $x \in \mathbf{F}_3^5$ satisfying the first three conditions. Let X_2 be the set of vectors $x \in X_1$ satisfying the fourth condition. Our computer calculation shows that $(\#X_1, \#X_2) = (115, 18)$ (resp. $(88, 14)$). Define a lexicographical order on X_1 induced by the above order of $\mathbf{F}_3$, that is, $(a_1, \dots, a_5) < (b_1, \dots, b_5)$ if $a_1 < b_1$, or $a_1 = b_1, \dots, a_k = b_k$ and $a_{k+1} < b_{k+1}$ for some $k \in \{1,2,3,4\}$. The matrices A were constructed, row by row, satisfying the following conditions:

- the ternary code $\langle (e_s, r_s) \mid s = 1,2,3 \rangle$ has minimum weight 3 (resp. 4), where $r_2 \in X_2,\ r_3 \in X_1$,
- the ternary code $\langle (e_s, r_s) \mid s = 1,2,3,4 \rangle$ has minimum weight 3 (resp. 4), where $r_2 \in X_2,\ r_3, r_4 \in X_1\ (r_3 < r_4)$,
- the ternary code $\langle (e_s, r_s) \mid s = 1,2,3,4,5 \rangle$ has minimum weight 3 (resp. 4), where $r_2 \in X_2,\ r_3, r_4, r_5 \in X_1\ (r_3 < r_4 < r_5)$.

It is obvious that the set of the matrices A which must be checked to achieve a complete classification, can be obtained in this way.

Then, by a computer calculation, we found 4328352 (resp. 650051) matrices A. Our computer calculation shows the 4328352 ternary $[10,5,3]$ codes (resp. 650051 ternary $[10,5,4]$ codes) are divided into 527 (resp. 64) classes by comparing their Hamming weight enumerators. For each Hamming weight enumerator, to test equivalence of codes, we use the algorithm given in [7, Section 7.3.3] as follows. For a ternary $[n,k]$ code C, define the digraph $\Gamma(C)$ with vertex set

$$(C - \{\mathbf{0}\}) \cup (\{1,2,\dots,n\} \times (\mathbf{F}_3 - \{0\}))$$

and arc set

$$\{(c,(j,c_j)) \mid c = (c_1,\dots,c_n) \in C - \{\mathbf{0}\}, c_j \neq 0, 1 \le j \le n\} \\ \cup \{((j,1),(j,2)),((j,2),(j,1)) \mid 1 \le j \le n\}.$$

Then, two ternary $[n,k]$ codes C and C' are equivalent if and only if $\Gamma(C)$ and $\Gamma(C')$ are isomorphic. We use the package GRAPE [10] of GAP [4] for digraph isomorphism testing. After checking whether codes are equivalent or not by a computer calculation for each Hamming weight enumerator, we have the following:

Proposition 1. *There are* 135 *ternary* $[10,5,4]$ *codes, up to equivalence. There are* 1303 *ternary* $[10,5,3]$ *codes, up to equivalence.*

We denote the 135 ternary $[10,5,4]$ codes by $C_{10,4,i}$ $(i = 1,2,\dots,135)$, and we denote the 1303 ternary $[10,5,3]$ codes by $C_{10,3,i}$ $(i = 1,2,\dots,1303)$. Generator matrices of all codes can be obtained electronically from [1].

The unique ternary $[10,5,5]$ code $C_{10,5}$ is formally self-dual, that is, the Hamming weight enumerators of the code and its dual code are identical. In addition, the supports of the codewords of minimum weight in $C_{10,5}$ form a 3-design [3]. We verified by a computer calculation that 38 ternary $[10,5,4]$ codes and 242 ternary $[10,5,3]$ codes are formally self-dual. In addition, we verified by a computer calculation that the supports of the codewords of minimum weight in only the code $C_{10,4,132}$ form a 2-design and the supports of the codewords of minimum weight in $C_{10,4,i}$ form a 1-design for only $i = 6, 86, 87, 89, 132$.

4. Ternary $[12, 5]$ codes satisfying (1)–(3)

In this section, we give a classification of ternary $[12, 5]$ codes satisfying the conditions (1)–(3), which is based on a computer calculation. This is obtained from the classification of ternary $[10, 5, d]$ codes with $d \in \{3, 4, 5\}$, by using the method given in Section 2.

4.1. From the $[10, 5, 5]$ code and the $[10, 5, 4]$ codes. As described in the previous section, there is a unique ternary $[10, 5, 5]$ code, up to equivalence [6]. It follows from [3] that this code $C_{10,5}$ has generator matrix $G_{10,5} = (\ I_5 \quad A \ )$, where A is the following circulant matrix:

$$A = \begin{pmatrix} 12210 \\ 01221 \\ 10122 \\ 21012 \\ 22101 \end{pmatrix}.$$

In order to construct all ternary $[12, 5]$ codes C satisfying the conditions (1) and $Pun(C) = C_{10,5}$, we consider generator matrices $G_{10,5}(a, b)$ of the form (5). Since the weight of each row of $G_{10,5}$ is 5, $(a_i, b_i) = (1, 2)$ or $(2, 1)$ for $i = 1, 2, 3, 4, 5$. By (4), we may assume that $(a_1, b_1) = (1, 2)$. Since the weight of the sum of the first row and the second row of $G_{10,5}$ is 5, (a_2, b_2) must be $(1, 2)$. Similarly, we have that $(a_i, b_i) = (1, 2)$ for $i = 3, 4, 5$, since A is circulant. In addition, we verified by a computer calculation that this code satisfies the condition (1). Note that the code automatically satisfies the conditions (2) and (3). We denote the code by $\mathscr{C}_{12,1}$.

Now, consider the ternary $[10, 5, 4]$ codes $C_{10,4,i}$ $(i = 1, 2, \ldots, 135)$. By considering generator matrices of the form (5), we found all ternary $[12, 5]$ codes C satisfying the conditions (1) and $Pun(C) = C_{10,4,i}$. This was done by a computer calculation. We denote by $G_{10,4,i}$ the generator matrix $(\ I_5 \quad A \ )$ of $C_{10,4,i}$ for each i. Since the weight of the first row of A is 3 (see Section 3), by (4), we may assume that $(a_1, b_1) = (1, 1)$ in (5). Under this situation, we verified by a computer calculation that only the codes $C_{10,4,60}$ and $C_{10,4,132}$ give ternary $[12, 5]$ codes satisfying the condition (1). Note that these codes automatically satisfy the conditions (2) and (3). In Table 1, we list the matrices A and (a^T, b^T) in $G_{10,4,i}(a, b)$ for $i = 60, 132$, where a^T denotes the transposed of a vector a. It can be seen by hand that the two codes $C(G_{10,4,60}(a, b))$ are SZ-equivalent. By Lemma 3, there are two ternary $[12, 5]$ codes C satisfying the conditions (1)–(3) and the condition that $Pun(C)$ is a ternary $[10, 5, 4]$ code. We denote the two codes by $\mathscr{C}_{12,2}$ and $\mathscr{C}_{12,3}$, respectively (note that take the first (a^T, b^T) for $i = 60$).

Table 1. Generator matrices $G_{10,4,i}(a,b)$ $(i = 60, 132)$

i	A	(a^T, b^T)
60	$\begin{pmatrix} 00111 \\ 01011 \\ 10101 \\ 11001 \\ 12210 \end{pmatrix}$	$\begin{pmatrix} 11 \\ 22 \\ 22 \\ 11 \\ 12 \end{pmatrix}, \quad \begin{pmatrix} 11 \\ 22 \\ 22 \\ 11 \\ 21 \end{pmatrix}$
132	$\begin{pmatrix} 00111 \\ 01011 \\ 10101 \\ 11001 \\ 11111 \end{pmatrix}$	$\begin{pmatrix} 11 \\ 22 \\ 22 \\ 11 \\ 00 \end{pmatrix}$

Lemma 2 shows that there are no other ternary $[12, 5, 6]$ codes satisfying the conditions (1)–(3). Hence, we have the following:

LEMMA 4. *Up to SZ-equivalence, there are three ternary* $[12, 5, 6]$ *codes satisfying the conditions* (1)–(3).

4.2. From the $[10, 5, 3]$ **codes.** By considering generator matrices of the form (5), we found all ternary $[12, 5]$ codes C satisfying the conditions (1) and $Pun(C) = C_{10,3,i}$ $(i = 1, 2, \ldots, 1303)$. This was done by a computer calculation. We denote by $G_{10,3,i}$ the generator matrix $(\ I_5 \quad A\ )$ of $C_{10,3,i}$ for each i. Since the weight of the first row of A is 2 (see Section 3), by (4), we may assume that $(a_1, b_1) = (0, 1)$ in (5). Under this situation, we verified by a computer calculation that only the codes $C_{10,3,i}$ give ternary $[12, 5]$ codes satisfying the condition (1) for

$$i = 302, 639, 662, 666, 667, 756, 878, 957, 958, 987,$$
$$1210, 1215, 1241, 1245, 1263, 1285, 1297, 1298, 1299.$$

In this case, there are codes satisfying the condition (1), but not (3). We verified by a computer calculation that only the codes $C_{10,3,i}$ give ternary $[12, 5]$ codes satisfying the conditions (1) and (3) for $i = 302, 666, 987, 1245$. Note that these four codes automatically satisfy the condition (2). In Table 2, we list the matrices A and (a^T, b^T) in $G_{10,4,i}(a,b)$ for $i = 302, 666, 987, 1245$. By Lemma 3, there are four ternary $[12, 5]$ codes satisfying the conditions (1)–(3) and the condition that $Pun(C)$ is a ternary $[10, 5, 3]$ code. We denote the four codes by $\mathscr{C}_{12,i}$ $(i = 4, 5, 6, 7)$, respectively.

Table 2. Generator matrices $G_{10,3,i}(a,b)$ $(i = 302, 666, 987, 1245)$

i	A	(a^T, b^T)	i	A	(a^T, b^T)
302	$\begin{pmatrix} 00011 \\ 01100 \\ 10101 \\ 11010 \\ 11221 \end{pmatrix}$	$\begin{pmatrix} 01 \\ 01 \\ 22 \\ 22 \\ 01 \end{pmatrix}$	987	$\begin{pmatrix} 00011 \\ 00101 \\ 01010 \\ 01100 \\ 11221 \end{pmatrix}$	$\begin{pmatrix} 01 \\ 20 \\ 20 \\ 01 \\ 00 \end{pmatrix}$
666	$\begin{pmatrix} 00011 \\ 01100 \\ 10101 \\ 11010 \\ 12222 \end{pmatrix}$	$\begin{pmatrix} 01 \\ 01 \\ 22 \\ 22 \\ 20 \end{pmatrix}$	1245	$\begin{pmatrix} 00011 \\ 00101 \\ 01010 \\ 01100 \\ 01111 \end{pmatrix}$	$\begin{pmatrix} 01 \\ 20 \\ 20 \\ 01 \\ 12 \end{pmatrix}$

Lemma 2 shows that there are no other ternary $[12,5,4]$ codes satisfying the conditions (1)–(3). Hence, we have the following:

Lemma 5. *Up to SZ-equivalence, there are four ternary* $[12,5,4]$ *codes satisfying the conditions* (1)–(3).

Up to SZ-equivalence, seven ternary $[12,5]$ codes satisfying the conditions (1)–(3) are known (see [9, Remark 5.3]). Lemmas 4 and 5 show that there are no other ternary $[12,5]$ codes satisfying the conditions (1)–(3). Therefore, we have Theorem 1.

4.3. Some properties. For the ternary $[12,5]$ codes C satisfying the conditions (1)–(3), instead of the Hamming weight enumerators, we consider the weight enumerators $\sum_{(x_1,\dots,x_{10},y_1,y_2)\in C} x^{\mathrm{wt}((x_1,\dots,x_{10}))} y^{n_1} z^{n_2}$, where n_1 and n_2 are the numbers of 1's and 2's in (y_1, y_2), respectively. We verified by a computer calculation that the codes $\mathscr{C}_{12,i}$ $(i = 1, 2, \dots, 7)$ have the following weight enumerators W_i:

$$W_1 = 1 + 72x^5yz + 60x^6 + 90x^8yz + 20x^9,$$

$$W_2 = 1 + 9x^4z^2 + 9x^4y^2 + 18x^5yz + 24x^6 + 36x^6z + 36x^6y + 18x^7z^2 + 18x^7y^2 + 36x^8yz + 2x^9 + 18x^9z + 18x^9y,$$

$$W_3 = 1 + 15x^4z^2 + 15x^4y^2 + 60x^6 + 60x^7z^2 + 60x^7y^2 + 20x^9 + 6x^{10}z^2 + 6x^{10}y^2,$$

$$W_4 = 1 + 2x^3z + 2x^3y + 4x^4z^2 + 4x^4y^2 + 24x^5yz + 18x^6 + 38x^6z + 38x^6y + 22x^7z^2 + 22x^7y^2 + 30x^8yz + 8x^9 + 14x^9z + 14x^9y + x^{10}z^2 + x^{10}y^2,$$

$$W_5 = 1 + 3x^3z + 3x^3y + 3x^4z^2 + 3x^4y^2 + 18x^5yz + 24x^6 + 39x^6z + 39x^6y + 21x^7z^2 + 21x^7y^2 + 36x^8yz + 2x^9 + 12x^9z + 12x^9y + 3x^{10}z^2 + 3x^{10}y^2,$$

$$W_6 = 1 + 4x^3z + 4x^3y + 5x^4z^2 + 5x^4y^2 + 24x^5yz + 18x^6 + 34x^6z + 34x^6y + 20x^7z^2 + 20x^7y^2 + 30x^8yz + 8x^9 + 16x^9z + 16x^9y + 2x^{10}z^2 + 2x^{10}y^2,$$

$$W_7 = 1 + 6x^3z + 6x^3y + 9x^4z^2 + 9x^4y^2 + 36x^5yz + 24x^6 + 42x^6z + 42x^6y + 18x^7z^2 + 18x^7y^2 + 18x^8yz + 2x^9 + 6x^9z + 6x^9y,$$

respectively. These weight enumerators guarantee that the codes $\mathscr{C}_{12,i}$ $(i = 1, 2, \dots, 7)$ satisfy the conditions (1)–(3). By putting $y = z = 1$, the above weight enumerators determine the Hamming weight enumerators of $Pun(\mathscr{C}_{12,i})$ $(i = 1, 2, \dots, 7)$. This implies that $\mathscr{C}_{12,1}$ is SZ-equivalent to C_7 in [9, Table 5.1]. In addition, by comparing generator matrices, it is easy to see that $\mathscr{C}_{12,i}$ $(i = 2, 3, \dots, 7)$ are equal to C_6, C_5, C_3, C_4, C_2 and C_1 in [9, Table 5.1], respectively.

Remark 1. *Shimada and Zhang* [9] *also considered the existence of ternary* $[12, 4, 6]$ *codes satisfying the condition that all codewords have weight divisible by three, in the proof of Theorem* 1.4 *(see* [9, Claim 6.2]*). We point out that a code satisfying the condition is self-orthogonal. There is a unique self-orthogonal ternary* $[12, 4, 6]$ *code, up to equivalence* [8, Table 1].

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Makoto Araya
Department of Computer Science
Faculty of Informatics
Shizuoka University
Hamamatsu 432-8011, *Japan*
E-mail: araya@inf.shizuoka.ac.jp

Masaaki Harada
Research Center for Pure and Applied Mathematics
Graduate School of Information Sciences
Tohoku University
Sendai 980-8579, *Japan*
E-mail: mharada@m.tohoku.ac.jp