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Anisohedral spherical triangles and classification of spherical tilings by congruent kites, darts and rhombi

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Abstract. We classify all spherical monohedral (kite/dart/rhombus)-faced tilings, as follows: The set of spherical monohedral rhombus-faced tilings consists of (1) the central projection of the rhombic dodecahedron, (2) the central projection of the rhombic triacontahedron, (3) a series of non-isohedral tilings, and (4) a series of tilings which are topologically trapezohedra (here a trapezohedron is the dual of an antiprism.). The set of spherical tilings by congruent kites consists of (1) the central projection T of the tetragonal icosikaitetrahedron, (2) the central projection of the tetragonal hexacontahedron, (3) a non-isohedral tiling obtained from T by gliding a hemisphere of T with $\pi/4$ radian, and (4) a continuously deformable series of tilings which are topologically trapezohedra. The set of spherical tilings by congruent darts is a continuously deformable series of tilings which are topologically trapezohedra. In the above explanation, unless otherwise stated, the tilings we have enumerated are isohedral and admit no continuous deformation. We prove that if a spherical (kite/dart/rhombus) admits an edge-to-edge spherical monohedral tiling, then it also does a spherical isohedral tiling. We also prove that the set of anisohedral, spherical triangles (i.e., spherical triangles admitting spherical monohedral triangular tilings but not any spherical isohedral triangular tilings) consists of a certain, infinite series of isosceles triangles I, and an infinite series of right scalene triangles which are the bisections of I.

1. Introduction

Ueno-Agaoka [16] classified all spherical tilings by congruent triangles. Here, by a tiling, we mean an edge-to-edge tiling without two-valent vertices. This paper presents two applications of their classification.

As the first application, we prove that the set of *anisohedral*, spherical triangles consists of an infinite series of spherical isosceles triangles AI_n $(5 \neq n \geq 4)$ and an infinite series of spherical right scalene triangles ARS_m $(m \geq 4)$, each of which is the bisection of AI_m. See Figure 7 and Theorem 1. Here a tile T of a monohedral tiling is said to be *anisohedral*, provided that

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for *every* monohedral tiling $\mathscr{T}$ with the tile being congruent to T, $\mathscr{T}$ is not isohedral. A tiling on a constant curvature space is said to be *monohedral*, if the tiling consists of congruent tiles, while a tiling is said to be *isohedral* (or *tile-transitive*), if the symmetry group of the tiling acts transitively on the set of tiles. Monohedral tilings of the Euclidean space $\mathbf{R}^d$ have long been studied (see [10] for the survey). In his famous eighteenth problem, Hilbert posed three problems. The second problem of the three is

(†) "are there any anisohedral tile of the Euclidean space $\mathbf{R}^3$?" [12, 3].

As for the problem with the Euclidean space replaced by the two-dimensional sphere, the answer is yes. That is, there are anisohedral, spherical triangles, and they are the following: Recall that any spherical triangle is determined uniquely modulo congruence by the three inner angles [2, p. 62].

Theorem 1. *The set of anisohedral, spherical triangles consists of*

(1) *an infinite series of isosceles triangles AI_n such that the list of three inner angles is*

$$\left(\pi\left(\frac{1}{2}-\frac{1}{2n}\right), \pi\left(\frac{1}{2}-\frac{1}{2n}\right), \frac{2\pi}{n}\right) \qquad (5 \neq n \geq 4); \qquad \textit{and}$$

(2) *an infinite series of right scalene triangles ARS_m, which is the bisection of AI_m. The list of three inner angles of ARS_m is*

$$\left(\frac{\pi}{2},\ \pi\left(\frac{1}{2}-\frac{1}{2m}\right), \frac{\pi}{m}\right) \qquad (m \geq 4).$$

The second application of Ueno-Agaoka's complete classification [16] of spherical monohedral triangular tilings is a complete classification (Theorem 2) of all spherical tilings by congruent kites, all spherical tilings by congruent darts, and all spherical tilings by congruent rhombi. Here, by a *kite* (*dart*, resp.), we mean a convex (non-convex, resp.) quadrangle such that the cyclic list of edge-lengths is $aabb$ ($a \neq b$). Figure 1 is the list of all spherical monohedral (kite/dart/rhombus)-faced tilings. Each tiling in a square frame is an instance of a continuously deformable tiling, while the other tilings admit no continuous deformation. The tilings left to the double strokes are sporadic, and the others are instances of series. All the tilings in Figure 1 are isohedral except a spherical monohedral kite-faced tiling gK_{24} (see the image in the topmost leftmost oval frame in Figure 1) and a spherical monohedral rhombus-faced tiling gTR_{4n+2} ($n \geq 2$; see the image in the bottom rightmost oval frame in Figure 1). However, the tile of the spherical monohedral, non-isohedral kite-faced tiling gK_{24} is congruent to the tile of a spherical isohedral kite-faced tiling K_{24} (see the image next to the image of gK_{24} in Figure 1), and the tile of

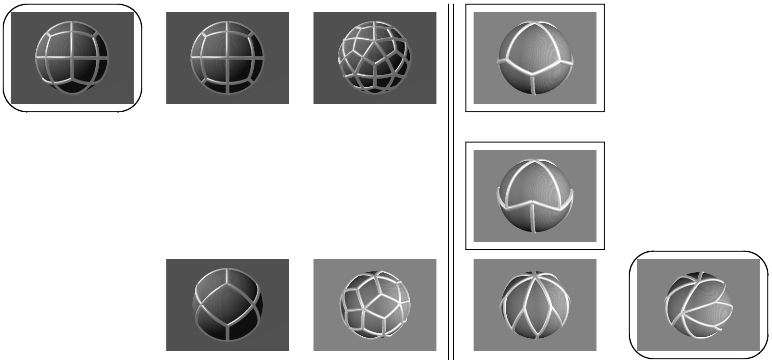

Fig. 1. The first line consists of spherical monohedral kite-faced tilings gK_{24}, K_{24}, K_{60} and $TR_8^{0.55}$. The second line is a spherical monohedral dart-faced tiling $TR_8^{0.45}$. The third line consists of spherical monohedral rhombus-faced tilings R_{12}, R_{30}, $TR_{14}^{6/7}$, and gTR_{14}. See Section 2 for the constructions of the tilings, and see Theorem 2 for the cyclic list of inner angles, the Schönflies symbol and the list of vertex types of each tiling.

the other spherical monohedral, non-isohedral rhombus-faced tiling gTR_{4n+2} is congruent to the tile of a spherical isohedral rhombus-faced tiling $TR_{4n+2}^{2n/(2n+1)}$ (see the image next to the image of gTR_{4n+2} in Figure 1). The definitions and the symbols for the tilings are given in the next section.

The consequences of the classification of spherical monohedral (kite/dart/rhombus)-faced tilings are: (1) none of spherical kites, spherical darts, and spherical rhombi is anisohedral. This is a slight refinement of the affirmative solution (Theorem 1) of Hilbert's question (†) on an anisohedral tile in his eighteenth problem adapted for the two-dimensional sphere. (2) In order to classify all spherical monohedral quadrangular tilings, it is enough to classify those consisting of congruent quadrangles such that the cyclic list of edge-lengths of the tile is *aaab* or *aabc* (*a*, *b*, *c* are mutually unequal), because of Ueno-Agaoka [15]:

Proposition 1. *In any spherical tilings consisting of congruent quadrangles, the tile necessarily has at least two equilateral* adjacent *edges.*

For an attempt to classify such spherical tilings, see [1].

The organization of this paper is as follows: In the next section, for each spherical monohedral (kite/dart/rhombus)-faced tilings appearing in Table 1, we give an explicit construction of it. Some constructions correspond to be "kis"ing [4, Chapter 21] for spherical tilings. This helps to show the isohedrality of spherical monohedral triangular tilings in Section 4. In Section 3, we state our main classification result (Theorem 2 including Table 1 and Figure 1). In Subsection 3.1 (Subsection 3.2, resp.), we prove that (1) Table 1 lists all spherical monohedral (kite/dart)-faced (rhombus-faced, resp.) tilings, and (2) for each spherical monohedral (kite/dart)-faced (rhombus-faced, resp.) tiling $\mathcal{Q}$, we provide values to each inner angle of the graph, represent the symmetry of $\mathcal{Q}$ by a *Schönflies symbol* [5, p. 41], and decide whether the spherical (kite/dart)-faced (rhombus-faced, resp.) tiling $\mathcal{Q}$ is isohedral or not. In Subsection 3.2, we also determine the spherical coordinates of the vertices and the length of the edges of *the* spherical non-isohedral monohedral rhombus-faced tiling. In Section 4, we identify all the anisohedral, spherical triangles by using Proposition 2, and prove that there are no anisohedral, spherical (kites/darts/rhombi).

By the sphere, we mean the unit sphere $\{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Unless otherwise stated, the unit for inner angles is π radian, while that for edge-length is radian. Ueno-Agaoka's classification (Table of [16]) of all spherical monohedral triangular tilings is included and is referred to as Table 2. The subscript of the name of a tiling indicates the number of the tiles, and the names of spherical monohedral triangular tilings are from Table 2. A spherical monohedral triangular tiling H_{20} in Table 2 is the central projection of the regular icosahedron to the sphere [16, p. 485]. Let $\mathcal{O}_8$ ($\mathcal{D}_{12}$, resp.) be the central projection of the regular octahedron (the regular dodecahedron, resp.) to the sphere.

2. Explicit constructions of the spherical monohedral (kite/dart/rhombus)-faced tilings

We will explicitly construct

- three spherical monohedral *kite-faced* tilings K_{60}, K_{24}, gK_{24},
- a deformable spherical monohedral (*kite/dart/rhombus*)-*faced* tiling TR^{α}_{2n} $(n \geq 3)$ $(1/(2n) < \alpha < 1 - 1/(2n),\ \alpha \neq 1/2)$, and
- three spherical monohedral *rhombus-faced* tilings R_{12}, R_{30}, gTR_{4n+2} $(n \geq 2)$.

We will construct the two spherical monohedral kite-faced tilings K_{60} and K_{24}, from two spherical tilings F_{120} and F_{48} by congruent right scalene triangles, and will construct the two spherical monohedral rhombus-faced tilings R_{12} and R_{30}, from two spherical tilings F_{24} and F^{I}_{60} by congruent isosceles triangles. Here F_{120}, F_{48}, F_{24} and F^{I}_{60} are listed in Ueno-Agaoka's complete classification ([16, Table], included as Table 2). According to [16, Figure 17 and p. 484], F_{24} is $\mathrm{kis}(\mathcal{O}_8)$ and F^{I}_{60} is $\mathrm{kis}(H_{20})$ where kis is the following construction.

DEFINITION 1. Let $\mathcal{T}$ be a spherical tiling. The "*kis*"ing of $\mathcal{T}$ is a spherical tiling by triangles obtained from $\mathcal{T}$ by joining the center of each tile to each vertex of that tile. The resulting tiling is denoted by kis($\mathscr{T}$). If $\mathscr{T}$ is a spherical tiling by n congruent rhombi, then kis($\mathscr{T}$) is a spherical tiling by $4n$ congruent right scalene triangles. If $\mathscr{T}$ is a spherical tiling by n regular p-gons, then kis($\mathscr{T}$) is a spherical tiling by pn congruent isosceles triangles.

The "kis"ing construction of spherical monohedral triangular tilings corresponds to the "kis"ing construction [4, Chapter 21] of *Archimedean duals*. In [4, Chapter 21], each Archimedean dual obtained via the "kis"ing of a Platonic solid or another Archimedean dual is named "kis...."

According to Ueno-Agaoka [16, Figure 21], a spherical tiling F_{120} (F_{48}, resp.) by congruent right scalene triangles, listed in Table 2, is constructed from $\mathscr{D}_{12}$ ($\mathscr{O}_8$, resp.), by (H) joining each center c of each face Φ of $\mathscr{D}_{12}$ ($\mathscr{O}_8$, resp.) to each vertex v of the face Φ, and (C) joining c to the midpoint m of each edge of Φ. Each edge (H) is the hypotenuse of the right scalene triangle of F_{120} (F_{48}, resp.). F_{120} (F_{48}, resp.) is a *barycentric subdivision* [8] (or also the *chamber system*) of $\mathscr{D}_{12}$ ($\mathscr{O}_8$, resp.).

2.1. K_{60} **and** K_{24}**.** A spherical monohedral kite-faced tiling K_{60} (K_{24}, resp.) is constructed from the spherical monohedral triangular tiling F_{120} (F_{48}, resp.), by deleting *the* hypotenuse (cf. (H)) of each right scalene triangular tile. In other words, K_{60} (K_{24}, resp.) is constructed from $\mathscr{D}_{12}$ ($\mathscr{O}_8$, resp.) by joining each center to the midpoints of the surrounding edges.

K_{24} is indeed a spherical monohedral *kite*-faced tiling, because of the following argument: All hypotenuses of the tiles of F_{48} are of equal length and bisect the inner angles $2/3$ between two adjacent edges obtained by (C), as well as an inner angle $1/2$ of $\mathscr{O}_8$. Hence, deleting the hypotenuses of the tiles of F_{48} results in a spherical monohedral tiling. The tile of K_{24} is a kite, because the tile has a pair $(2/3, 1/2)$ as opposite inner angles on the two vertices of the edge (H). However, the other pair of opposite inner angles is $(1/2, 1/2)$.

K_{60} is a spherical monohedral *kite*-faced tiling, by a similar argument.

We will prove that the symmetry of K_{24} is that of the regular octahedron, in order to assure that a construction (Subsection 2.2) of a spherical monohedral kite-faced tiling gK_{24} is well-defined. The symmetry of the regular octahedron (the regular icosahedron, resp.) is represented by a Schönflies symbol O_h (I_h, resp.) [5, p. 48 (p. 49, resp.)].

Lemma 1. *The Schönflies symbol of* K_{24} *(*K_{60}*, resp.) is* O_h *(*I_h*, resp.).*

Proof. Every symmetry operation σ of $\mathscr{O}_8$ ($\mathscr{D}_{12}$, resp.) preserves the triangular (pentagonal, resp.) tiles, the centers of the triangular (pentagonal, resp.) tiles and the midpoints of the edges of $\mathscr{O}_8$ ($\mathscr{D}_{12}$, resp.). Hence the Schönflies symbol of K_{24} (K_{60}, resp.) is O_h (I_h, resp.), by [5, p. 50]. $\square$

2.2. gK_{24}**.** A spherical monohedral tiling gK_{24} is constructed as follows: K_{24} has three mutually perpendicular, great circles. Each great circle consists of eight edges of K_{24} and is the intersection of K_{24} and some mirror plane of K_{24}. Choose one of the three great circles, and then a hemisphere $\mathscr{N}$ of K_{24} determined by the chosen great circle. Any choice of $\mathscr{N}$ results in a congruent spherical figure. It is because any great circle of K_{24} is transformed to any other great circle of K_{24} by some symmetry operation of K_{24}, by Lemma 1. Glide the hemisphere $\mathscr{N}$ by $\pi/4$ radian against the other hemisphere. Let the resulting spherical figure be gK_{24}. Because the boundary of the hemisphere $\mathscr{N}$ is equally subdivided into eight edges of K_{24}, the $\pi/4$ radian rotation keeps the tiling being edge-to-edge. See Figure 2. gK_{24} is a spherical monohedral *kite*-faced tiling, since the tiles are not changed by this gliding.

2.3. TR^{α}_{2n}**.** A spherical monohedral tiling TR^{α}_{2n} ($n \geq 3$) is constructed as in Figure 3 so that each pole is shared by n tiles. TR^{α}_{2n} is topologically the so-called *n-gonal trapezohedron of* $2n$ *faces* [11]. Here an n-gonal trapezohedron is the dual of an n-gonal antiprism. Let the cyclic list of four inner angles of the tile of TR^{α}_{2n} be $(\alpha, \beta, \alpha, \delta)$ such that $\beta = 2/n$ and $\delta = 2 - 2\alpha$, as in Figure 3 (left, right). Note $\alpha \neq 1/2$. Otherwise, $\delta = 1$ and thus the tile of TR^{α}_{2n} is a triangle.

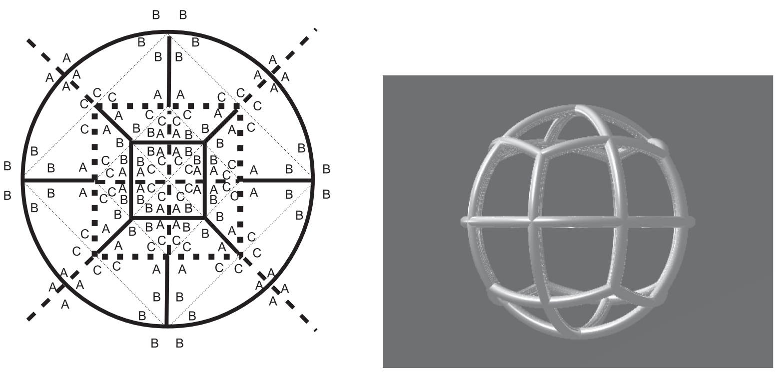

Fig. 2. The left is a graph of the spherical monohedral kite-faced tiling gK_{24} by 24 congruent kites (thick and dotted). The set of the thick, dotted, and thin edges represents a graph of the spherical monohedral triangular tiling TF_{48}. Cf. [16, Figure 88]. In gK_{24}, the inner angles between thick edges (thick edge and dotted edge, dotted edges, resp.) are $\beta = 2B$ ($\alpha = A$, $\delta = 2C$, resp.). The right is the view of gK_{24} from a two-fold axis of rotation. See Subsection 2.2.

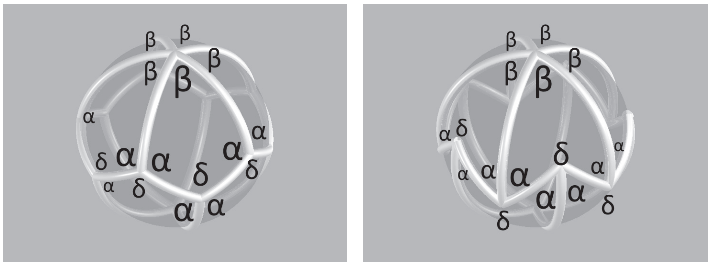

Fig. 3. The left figure is an explicit construction of a spherical monohedral kite-faced tiling TR_{2n}^{α} ($1/2 < \alpha < 1 - 1/(2n)$, $\alpha \neq 1 - 1/n$, $n \geq 3$), and the right figure is an explicit construction of a spherical monohedral dart-faced tiling TR_{2n}^{α} ($1/(2n) < \alpha < 1/2$, $n \geq 3$). The number of the tile is $2n$. The inner angle β is $2/n$.

Lemma 2. *Let n be an integer greater than or equal to three. Then*

$$TR_{2n}^{\alpha} \text{ exists and the tile is a rhombus} \Leftrightarrow \alpha = 1 - \frac{1}{n}. \tag{1}$$

$$TR_{2n}^{\alpha} \text{ exists and the tile is a dart} \Leftrightarrow \frac{1}{2n} < \alpha < \frac{1}{2}. \tag{2}$$

$$TR_{2n}^{\alpha} \text{ exists and the tile is a kite} \Leftrightarrow \frac{1}{2} < \alpha < 1 - \frac{1}{2n} \text{ and } \alpha \neq 1 - \frac{1}{n}. \tag{3}$$

Proof. First we prove that TR_{2n}^{α} exists as a spherical monohedral quadrangular tiling if and only if $1/(2n) < \alpha < 1 - 1/(2n)$ and $\alpha \neq 1/2$. Let $\mathscr{T}$ be obtained from TR_{2n}^{α} by drawing the meridian diagonal segment of each quadrangular tile Q of TR_{2n}^{α}. Then TR_{2n}^{α} exists as a spherical monohedral quadrangular tiling, if and only if $\mathscr{T}$ exists as a spherical monohedral triangular tiling. By Ueno-Agaoka's classification ([16, Table], included as Table 2) of all spherical monohedral triangular tilings and [16, Figure 4], the tiling $\mathscr{T}$ exists as a spherical monohedral triangular tiling, if and only if $\mathscr{T}$ is a spherical monohedral triangular tiling G_{4n} of Table 2 such that the list of inner angles of the triangular tile of G_{4n} is $(A, B, C) = (\alpha, \delta/2, \beta/2)$, if and only if $1/(2n) < \alpha < 1 - 1/(2n)$. Thus we have the desired consequence.

The equivalence (1) follows because $\beta = \delta$ if and only if the tile is a rhombus. The equivalence (3) follows because the tile is a kite if and only if all the inner angles α, β and δ are less than 1 and $\beta \neq \delta$. The equivalence (2) follows because $\delta = 2 - 2\alpha > 1$ if and only if the tile is a dart. $\square$

Fact 1. *$TR_6^{2/3}$ is the central projection of the regular hexahedron to the sphere.*

PROOF. In $TR_6^{2/3}$, the four inner angles of a tile Q are equal, and the horizontal diagonal segment of a tile and the vertical diagonal segment of the tile both bisect the inner angles of Q. Hence, the horizontal diagonal segment of Q subdivides Q into two congruent isosceles triangles T and the vertical diagonal segment of Q subdivides the tile into two isosceles triangles congruent to T. Thus the vertical diagonal segment and the horizontal diagonal segment are of the same length. Therefore Q is a regular square. □

2.4. R_{12} **and** R_{30}**.** A spherical monohedral rhombus-faced tiling R_{12} (R_{30}, resp.) is constructed from the spherical monohedral triangular tiling F_{24} (F_{60}^I, resp.) mentioned in the beginning of this Section, by deleting the base edge of each isosceles triangular tile. Roughly speaking, R_{12} (R_{30}, resp.) is obtained from $\mathcal{O}_8$ (H_{20}, resp.) by replacing each edge with a rhombus.

2.5. gTR_{4n+2} $(n \geq 2)$**.** We construct a spherical monohedral rhombus-faced tiling gTR_{4n+2} $(n \geq 2)$, as follows: Consider $TR_{4n+2}^{2n/(2n+1)}$ (white edges, the top left image of Figure 4), and choose a hemisphere $\mathscr{W}$ ("western hemisphere") such that on the boundary of the hemisphere $\mathscr{W}$, from the north pole N, we see a meridian edge, a meridian diagonal segment, another meridian edge and another meridian diagonal segment of $TR_{4n+2}^{2n/(2n+1)}$ (the second image of Figure 4). Every such choice results in a congruent hemisphere, because the polar axis of the tiling $TR_{4n+2}^{2n/(2n+1)}$ is a $(2n+1)$-fold axis of rotation. Consider the copy of the hemisphere $\mathscr{W}$, rotate it around the polar axis by π radian (the third image of Figure 4). Let the resulting hemisphere be $\mathscr{W}'$.

Let η be the horizontal axis which passes through the center of the boundary of $\mathscr{W}'$ and is normal to the plane spanned by the boundary of $\mathscr{W}'$. Rotate $\mathscr{W}'$ around the axis η in the length of a meridian edge of $TR_{4n+2}^{2n/(2n+1)}$. Let $\mathscr{E}$ be the resulting hemisphere ("eastern hemisphere"). On the boundary of $\mathscr{E}$, we see meridian edges, and meridian diagonal segments, alternatingly. So the boundary of $\mathscr{E}$ corresponds to that of $\mathscr{W}$. Hence gluing $\mathscr{W}$ to $\mathscr{E}$ produces a spherical monohedral rhombus-faced tiling. Let the resulting tiling be gTR_{4n+2} (the white edges of the fourth image of Figure 4).

gTR_{4n+2} with $n = 1$ is $TR_6^{2/3}$ (cf. Fact 1).

REMARK 1. The K_{24}, K_{60}, R_{12} and R_{30} we have constructed are exactly the quadrangle-faced *Archimedean duals*, topologically. In the terminology of [4, Chapter 21], the corresponding quadrangle-faced Archimedean duals are *the tetragonal icosikaitetrahedron*, *the tetragonal hexacontahedron*, *the rhombic dodecahedron* and *the rhombic triacontahedron*. The last two Archimedean duals are studied deeply in [6, Sect. 4.7].

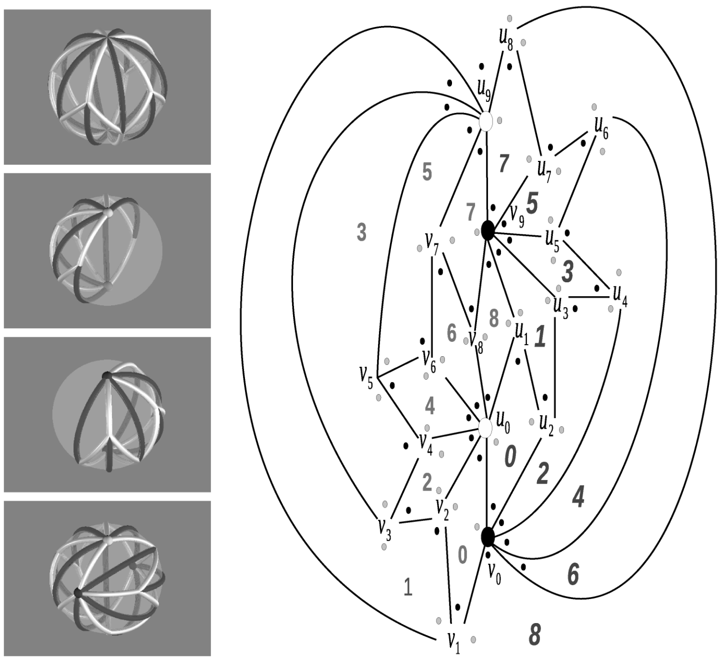

Fig. 4. The first column explains an explicit construction of a spherical monohedral rhombus-faced tiling gTR_{4n+2} (the white edges of the bottom image) from $TR_{4n+2}^{2n/(2n+1)}$ (the white edges of the top image), for $n = 2$. The western hemisphere $\mathscr{W}$ and the hemisphere $\mathscr{W}'$ mentioned in Subsection 2.5 are the second and the third image of the first column. The second column is a graph of gTR_{4n+2} $(n \geq 2)$ for $n = 4$, where the inner angles designated by black circles are $\beta = 2/(2n+1)$, and the other inner angles are $\alpha = 2n/(2n+1)$. In the bottom image, we see two pairs of antipodal vertices of type $\alpha + (n+1)\beta$. One is the pair of the north pole u_0 and the south pole u_{2n+1}, and the other is the pair of the vertex v_0 (the front black vertex) and the vertex v_{2n+1} (the back black vertex).

The spherical monohedral kite-faced tiling gK_{24} has the same skeleton as the dual of the *elongated square gyrobicupola* (or *pseudorhombicuboctahedron*). The pseudorhombicuboctahedron is the 37-th Johnson solids [13], [18] consisting of the eight regular triangles and the 18 regular squares. The vertex figures of this solid are the same, but the action of the symmetry group of the solid on the vertices is not transitive [14, p. 114][1].

gTR_{4n+2} $(n \geq 2)$ appears in [16, Figure 15]. gTR_{10} is the rightmost figure of the first line of [16, Figure 15].

[1] For every Archimedean solid, the faces are regular and the symmetry group acts transitively on the vertices. Prof. M. Deza let the second author know that a nickname of the pseudorhombicuboctahedron is the "14-th" Archimedean solid and he suggested that the nickname is due to Grünbaum [9].

3. Classification of spherical monohedral (kite/dart/rhombus)-faced tilings

We say a vertex of the tiling has *type* $n_1\alpha + n_2\beta + n_3\delta$, if around the vertex there are n_1 copies of the inner angle α, n_2 copies of the inner angle β, and n_3 copies of the inner angle δ.

THEOREM 2. (1) *The first column of Table* 1 *is* the *list of spherical monohedral kite-faced tilings, spherical monohedral dart-faced tilings, and spherical monohedral rhombus-faced tilings.*

(2) *The non-isohedral tilings in Table* 1 *are* gK_{24} *and* gTR_{4n+2} $(n \geq 2)$.
For each spherical tiling listed in the first column of Table 1, *the cyclic list of inner angles of the tile, the symmetry (Schönflies symbol) and the vertex types of the tiling are described in the second, the third, and the fourth columns, respectively.*

(3) *The tilings appearing in Table* 1 *are non-congruent to each other.*

Name of spherical tiling	The cyclic list $(\alpha,\beta,\alpha,\delta)$ of inner angles of the tile [π radian]	Symmetry (Schönflies symbol)	Vertex types [the number]
K_{24}	$\alpha=\delta=\frac{1}{2},\ \beta=\frac{2}{3}$	O_h	4α [12], 4δ [6], 3β [8]
gK_{24}	$\alpha=\delta=\frac{1}{2},\ \beta=\frac{2}{3}$	D_{4d}	$\begin{cases} 4\alpha\ [8],\ 3\beta\ [8], \\ 2\alpha+2\delta\ [8],\ 4\delta\ [2] \end{cases}$
K_{60}	$\alpha=\frac{1}{2},\ \beta=\frac{2}{3},\ \delta=\frac{2}{5}$	I_h	$\begin{cases} 4\alpha\ [30],\ 3\beta\ [20], \\ 5\delta\ [12] \end{cases}$
TR^{α}_{2n} $\left(\frac{1}{2}<\alpha<1-\frac{1}{2n}\right.$ $\left.\alpha\neq 1-\frac{1}{n}\right)$ $(n\geq 3)$	$\beta=\frac{2}{n},\ 2\alpha+\delta=2$	D_{nd}	$\begin{cases} 2\alpha+\delta\ [2n], \\ n\beta\ [2] \end{cases}$
TR^{α}_{2n} $\left(\frac{1}{2n}<\alpha<\frac{1}{2}\right)$ $(n\geq 3)$	$\beta=\frac{2}{n},\ 2\alpha+\delta=2$	D_{nd}	$\begin{cases} 2\alpha+\delta\ [2n], \\ n\beta\ [2] \end{cases}$
R_{12}	$\alpha=\frac{1}{2},\ \beta=\frac{2}{3}$	O_h	4α [6], 3β [8]
R_{30}	$\alpha=\frac{2}{5},\ \beta=\frac{2}{3}$	I_h	3β [20], 5α [12]
$TR^{1-1/n}_{2n}$ $(n\geq 3)$	$\begin{cases} \alpha=\frac{n-1}{n}, \\ \beta=\frac{2}{n} \end{cases}$	$\begin{cases} O_h \\ (n=3) \\ D_{nd} \\ (n>3) \end{cases}$	$\begin{cases} 2\alpha+\beta\ [2n], \\ n\beta\ [2] \end{cases}$
gTR_{4n+2} $(n\geq 2)$	$\begin{cases} \alpha=\frac{2n}{2n+1}, \\ \beta=\frac{2}{2n+1} \end{cases}$	D_2	$\begin{cases} \alpha+(n+1)\beta\ [4] \\ 2\alpha+\beta\ [4n] \end{cases}$

Table 1. A complete table of the spherical monohedral tilings consisting of (kites/darts/rhombi) (from top to bottom). The three parts are separated by double horizontal lines. The subscript of each tiling indicates the number of tiles. The spherical monohedral kite-faced tiling TR^{α}_{2n} and the spherical monohedral dart-faced tiling TR^{α}_{2n} are continuously deformable, and the tilings gK_{24} and gTR_{4n+2} are non-isohedral. For the Schönflies symbols D_2 (D_{nd}, O_h, I_h, resp.), see pp. 41–42 (p. 43, p. 48, p. 49, resp.) in [5]. The tilings appearing in the table are non-congruent to each other. See Theorem 2.

The rest of this section is devoted to the proof of Theorem 2.

As for Theorem 2 (1), the first column of Table 1 are exactly spherical monohedral (kite/dart/rhombus)-faced tilings which are constructed in Section 2. To show that Table 1 is *the* list of spherical monohedral (kite/dart/rhombus)-faced tilings, we combine Lemma 3, with Ueno-Agaoka's complete classification ([16, Table], included as Table 2) of spherical monohedral triangular tilings.

Definition 2. We say a spherical monohedral triangular tiling $\mathcal{T}$ *yields* a spherical monohedral quadrangular tiling $\mathcal{Q}$, if $\mathcal{T}$ has twice as many tiles as $\mathcal{Q}$, and the deletion of some edges of $\mathcal{T}$ results in $\mathcal{Q}$.

Then we have the following:

Lemma 3. (1) *For every spherical tiling $\mathcal{Q}$ by congruent kites or by congruent darts, there is a spherical tiling $\mathcal{T}$ by F congruent non-equilateral isosceles triangles or by F congruent scalene triangles, such that $\mathcal{T}$ yields $\mathcal{Q}$ and $F \geq 12$.*

(2) *For every spherical monohedral rhombus-faced tiling $\mathcal{R}$, there is some spherical tiling $\mathcal{T}$ by $F \geq 12$ congruent isosceles triangles T, such that $\mathcal{T}$ yields $\mathcal{R}$ and* the two base inner angles of the triangle T is smaller than the other inner angle of T.

Proof. The number F of tiles of $\mathcal{T}$ is an even number greater than or equal to 12, because $\mathcal{Q}$ has more than six tiles by Euler's theorem (see the first part of the proof of [15, Proposition 1]).

(1) Each tile Q of the tiling $\mathcal{Q}$ has a vertex u incident to two edges of length a and a vertex v incident to two edges of length b such that $a \neq b$, because the tile Q is a kite or a dart. The diagonal segment between u and v subdivides Q into two congruent triangles T. T is a non-equilateral isosceles triangle or a scalene triangle, because T has an edge of length a and an edge of length b such that $a \neq b$. Hence, subdividing each tile Q of the tiling $\mathcal{Q}$ results in a spherical tiling by congruent non-equilateral isosceles triangles T or by congruent scalene triangles T. (2) Assume otherwise. Then there is a spherical monohedral rhombus-faced tiling $\mathcal{R}$ with the following condition: whenever a spherical tiling $\mathcal{T}$ by congruent isosceles triangles T yields $\mathcal{R}$, the two base angles of the isosceles triangle T is greater than or equal to the other inner angle. Now let the cyclic list of four inner angles of the rhombic tile R be $(\alpha, \beta, \alpha, \beta)$. A subdivision of R results in an isosceles triangle such that the list of inner angles is $(\alpha, \beta/2, \beta/2)$ or $(\beta, \alpha/2, \alpha/2)$. Moreover, $\beta/2 \geq \alpha$ and $\alpha/2 \geq \beta$. This implies $\alpha = \beta = 0$, which is a contradiction. □

Name	V	E	inner angles A, B, C [π rad]	vertex type [number]
$\square$ F_4	4	6	$A+B+C=2$, $1/2<A,B,C<1$	$A+B+C$ [4]
F_{12}^{I}	8	18	$A=2/3,\ B=C=1/3$	$3A$ [4], $6B$ [4]
F_{12}^{II}	8	18	$A=2/3,\ B=C=1/3$	$\begin{cases} 3A\ [2],\ 2A+2B\ [2] \\ A+4B\ [2],\ 6B\ [2] \end{cases}$
F_{12}^{III}	8	18	$A=2/3,\ B=C=1/3$	$\begin{cases} 3A\ [1],\ 2A+2B\ [3] \\ A+4B\ [3],\ 6B\ [1] \end{cases}$
F_{24}	14	36	$A=2/3,\ B=C=1/4$	$3A$ [8], $8B$ [6]
F_{48}	26	72	$A=1/2,\ B=1/3,\ C=1/4$	$\begin{cases} 4A\ [12],\ 6B\ [8] \\ 8C\ [6] \end{cases}$
TF_{48}	26	72	$A=1/2,\ B=1/3,\ C=1/4$	$\begin{cases} 4A\ [8],\ 6B\ [8] \\ 8C\ [2],\ 2A+4C\ [8] \end{cases}$
F_{60}^{I}	32	90	$A=2/3,\ B=C=1/5$	$3A$ [20], $10B$ [12]
F_{60}^{II}	32	90	$A=2/5,\ B=C=1/3$	$5A$ [12], $6B$ [20]
F_{120}	62	180	$A=1/2,\ B=1/3,\ C=1/5$	$\begin{cases} 4A\ [30],\ 6B\ [20] \\ 10C\ [12] \end{cases}$
$\square$ $G_{4n}\ (n\geq 2)$	$2n+2$	$6n$	$A+B=1,\ C=1/n$, $\frac{1}{2n}<A,B<\frac{2n-1}{2n}$	$\begin{cases} 2A+2B\ [2n] \\ 2nC\ [2] \end{cases}$
$G_{4n+2}\ (n\geq 1)$	$2n+3$	$6n+3$	$A=B=1/2,\ C=\frac{2}{2n+1}$	$\begin{cases} 4A\ [2n+1] \\ (2n+1)C\ [2] \end{cases}$
$TG_{8n}\ (n\geq 2)$	$4n+2$	$12n$	$A=B=1/2,\ C=\frac{1}{2n}$	$\begin{cases} 4A\ [4n-2] \\ 2A+2nC\ [4] \end{cases}$
$\square$ $TG_{8n+4}\ (n\geq 1)$	$4n+4$	$12n+6$	$A+B=1,\ C=\frac{1}{2n+1}$, $\frac{1}{4n+2}<A,B<\frac{4n+1}{4n+2}$	$\begin{cases} A+B+(2n+1)C\ [4] \\ 2A+2B\ [4n] \end{cases}$
$MTG_{8n+4}^{I}\ (n\geq 1)$	$4n+4$	$12n+6$	$A=\frac{n+1}{2n+1},\ B=\frac{n}{2n+1},\ C=\frac{1}{2n+1}$	$\begin{cases} A+B+(2n+1)C\ [2] \\ A+3B+C\ [2] \\ 2A+2B\ [4n-2] \\ 2A+2nC\ [2] \end{cases}$
$MTG_{8n+4}^{II}\ (n\geq 2)$	$4n+4$	$12n+6$	$A=\frac{n+1}{2n+1},\ B=\frac{n}{2n+1},\ C=\frac{1}{2n+1}$	$\begin{cases} A+3B+C\ [4] \\ 2A+2B\ [4n-4] \\ 2A+2nC\ [4] \end{cases}$
$H_{4n}\ (n\geq 3)$	$2n+2$	$6n$	$A=B=\frac{n-1}{2n},\ C=2/n$	$4A+C$ [$2n$], nC [2]
$TH_{8n+4}\ (n\geq 3)$	$4n+4$	$12n+6$	$A=B=\frac{n}{2n+1},\ C=\frac{2}{2n+1}$	$\begin{cases} 4A+C\ [4n] \\ 2A+(n+1)C\ [4] \end{cases}$
$I_{8n}\ (n\geq 3)$	$4n+2$	$12n$	$A=1/2,\ B=\frac{n-1}{2n},\ C=1/n$	$\begin{cases} 4A\ [2n] \\ 4B+2C\ [2n] \\ 2nC\ [2] \end{cases}$
$TI_{16n+8}\ (n\geq 2)$	$8n+6$	$24n+12$	$A=1/2,\ B=\frac{n}{2n+1},\ C=\frac{1}{2n+1}$	$\begin{cases} 4A\ [4n+2] \\ 4B+2C\ [4n] \\ 2B+(2n+2)C\ [4] \end{cases}$

Table 2. The classification of the spherical monohedral triangular tilings [16, Table]. The first column lists the names. The explicit constructions of the tilings are explained in [16, Section 2]. There are ten sporadic tilings (above the horizontal line) and ten series (below the horizontal line). The mark $\square$ indicates that the tiling is continuously deformable. None of these tilings are isomorphic to each other except the trivial case given by the exchange of $A+B+C=2$ (F_4), $A+B=1$ (G_{4n}, TG_{8n+4}).

We will prove all spherical monohedral (kite/dart)-faced (rhombus-faced, resp.) tilings appear in Table 1, by exhausting the pairs $(\mathscr{T}, \mathscr{Q})$ of Lemma 3 (1) (Lemma 3 (2), resp.) in Lemma 5 (Lemma 7, resp.), with Lemma 4 (Lemma 4 (I), resp.).

Lemma 4. *Let $F \geq 12$ be an even number. Let $\mathscr{T}$ be a spherical tiling by F congruent isosceles triangles T. In Ueno-Agaoka's complete classification (*[16, Table]*, included as Table* 2*) of spherical monohedral triangular tilings, the following two assertions hold: Inside "$\{\cdots\}$", the greatest and the smallest inner angles of the triangular tile T are given.*

(I) *If the two base angles of the spherical isosceles triangle T are smaller than the other inner angle of T, then the spherical monohedral triangular tiling $\mathscr{T}$ is one the following:*
- F_{12}^{I}, F_{12}^{II}, F_{12}^{III}, F_{24}, F_{60}^{I}, F_{60}^{II},
- G_{4n} $(n \geq 3)$ $\{1 - 1/n, 1/n\}$,
- TG_{8n+4} $(n \geq 1)$ $\{2n/(2n+1), 1/(2n+1)\}$,
- MTG_{12}^{I} $\{2/3, 1/3\}$,
- I_{24} $\{1/2, 1/3\}$,
- H_{12} $\{2/3, 1/3\}$, H_{16} $\{1/2, 3/8\}$.

(II) *If the two base angles of the spherical isosceles triangle T are greater than the other inner angle, then the spherical monohedral triangular tiling $\mathscr{T}$ is one of the following:*
- G_{4n+2} $(n \geq 3)$ $\{1/2, 2/(2n+1)\}$,
- TG_{8n} $(n \geq 2)$ $\{1/2, 1/(2n)\}$,
- H_{4n} $(n \geq 6)$, TH_{8n+4} $(n \geq 3)$,
- G_{4n} $(n \geq 3)$ $\{1/2, 1/n\}$,
- TG_{8n+4} $(n \geq 1)$ $\{1/2, 1/(2n+1)\}$.

Proof. The lemma is [16, Proposition 6] where the number F is greater than or equal to 12. □

In Subsubsection 3.1.2, to prove Theorem 2 (2), we will use the following:

Fact 2. *Let $\mathscr{T}$ be a spherical tiling by congruent triangles such that the list of three inner angles is (A, B, C), and let $\mathscr{Q}$ be a spherical monohedral (kite/dart)-faced (rhombus-faced, resp.) tiling such that the cyclic list of four inner angles of the tile is*

$$(\alpha, \beta, \alpha, \delta) \qquad ((\alpha, \beta, \alpha, \beta), \text{ resp.}).$$

If $\mathscr{T}$ yields $\mathscr{Q}$ and $(\alpha, \beta, \alpha, \delta) = (A, 2B, A, 2C)$ $((\alpha, \beta, \alpha, \beta) = (2B, A, 2B, A)$, resp.), then the vertex types of $\mathscr{Q}$ are exactly obtained from those of $\mathscr{T}$ by applying a substitution $(A := \alpha, B = \beta/2, C = \delta/2)$ $((A := \beta, B := \alpha/2)$, resp.).

Theorem 2 (3) follows from Theorem 2 (2), because spherical monohedral tilings in Table 1 are not congruent to each other by the form of the tiles (i.e., kites, darts, rhombi), by the symmetry, or by the number of the tiles.

3.1. A complete classification of spherical monohedral (kite/dart)-faced tilings.

3.1.1. *The completeness (Theorem* 2 (1)*).*

LEMMA 5. *Let $\mathscr{T}$ be a spherical tiling by congruent non-equilateral triangles T, and $\mathscr{Q}$ be a spherical kite-faced or dart-faced monohedral tiling. Then if $\mathscr{T}$ yields $\mathscr{Q}$, then $(\mathscr{T}, \mathscr{Q})$ is (F_{48}, K_{24}), (TF_{48}, gK_{24}), (F_{120}, K_{60}) or $(G_{4n}, TR^{\alpha}_{2n})$ for some integer $n \geq 3$ and some $\alpha \in (1/(2n), 1 - 1/(2n)) \setminus \{1/2, 1 - 1/n\}$.*

To prove the previous lemma, we will use the following "uniqueness" lemma:

LEMMA 6. *Assume that $\mathscr{T}$ is a spherical tiling by congruent triangles T, $\mathscr{Q}$ is a spherical tiling by congruent quadrangles Q, and $\mathscr{T}$ yields $\mathscr{Q}$ in a manner that Q is a kite or a dart. Then the following two assertions hold:*

(1) *$\mathscr{T}$ is none of entries TG_{8n+4} $(n \geq 1)$, MTG^{I}_{8n+4} $(n \geq 2)$, and MTG^{II}_{8n+4} $(n \geq 2)$ of Table* 2.

(2) *When T is a non-equilateral isosceles triangle, T is not rectangular. Moreover, the quadrangular tile Q is unique modulo congruence for the given spherical monohedral triangular tiling $\mathscr{T}$.*

PROOF. From the assumption, $\mathscr{T}$ yields $\mathscr{Q}$ in a manner that Q is a kite or a dart. So, the tile Q of $\mathscr{Q}$ has a pair $(2X, 2Y)$ of opposite inner angles for some inner angles X, Y of the triangular tile T. It is because we must glue two copies of the triangular tile T of $\mathscr{T}$ at the edges of the same length, to obtain the quadrangular tile Q of $\mathscr{Q}$. Thus there is an inner angle X of T such that the number of occurrences of X is even for any vertex type of $\mathscr{T}$.

According to Table 2, however, if $\mathscr{T}$ is TG_{8n+4} $(n \geq 1)$ or MTG^{I}_{8n+4} $(n \geq 2)$, then for every inner angle $X \in \{A, B, C\}$, the number of occurrences of X is odd in a vertex type $A + B + (2n+1)C$ of $\mathscr{T}$. So TG_{8n+4} $(n \geq 1)$ and MTG^{I}_{8n+4} $(n \geq 2)$ are impossible. Similarly, for MTG^{II}_{8n+4} $(n \geq 2)$, for every $X \in \{A, B, C\}$, the number of occurrences of X is odd in a vertex type $A + 3B + C$ of MTG^{II}_{8n+4}. So MTG^{II}_{8n+4} $(n \geq 2)$ is impossible too. Thus the assertion (1) follows.

The first part of the assertion (2) is proved as follows. The list of three inner angles of the isosceles triangular tile T is (X, X, Y) without loss of generality. Because the quadrangular tile Q is not a rhombus, the cyclic list of the four inner angles is $(X, 2Y, X, 2Y)$. If X or Y is $\pi/2$ radian, then some inner angle of Q is π radian, which is a contradiction. Hence, the triangular tile of $\mathscr{T}$ is not rectangular. This argument proves the second part of the assertion (2), as well. This completes the proof of the assertion (2) and thus Lemma 6. □

Proof of Lemma 5. We use Lemma 3 (1). The non-equilateral triangular tile T is either a scalene triangle (Case 1) or a non-equilateral isosceles triangle (Case 2).

Case 1. The tile T is a scalene triangle.

By Lemma 6 (1), the spherical tiling $\mathscr{T}$ by $F \geq 12$ congruent *scalene* triangles is one of the following entries of Table 2 (Ueno-Agaoka's complete classification [16, Table] of the spherical monohedral triangular tilings):

$$F_{48}, TF_{48}, F_{120}, G_{4n}\ (n \geq 3),\ I_{8n}\ (n \geq 4),\ \text{or},\ TI_{16n+8}\ (n \geq 2).$$

The case $\mathscr{T} = I_{8n}$ $(n \geq 4)$ or TI_{16n+8} $(n \geq 2)$ is impossible, by Table 2. Indeed, $\mathscr{T}$ has a vertex of type $4A$ and $4B + 2C$. Here (A, B, C) is the list of three inner angles of the triangular tile of $\mathscr{T}$. By $A = 1/2$ and Proposition 1, the cyclic list of four inner angles of the tile Q of $\mathscr{Q}$ is $(\alpha, \beta, \alpha, \delta) = (A, 2B, A, 2C)$. Hence, some $\mathscr{T}$-vertex of type $4B + 2C$ causes some $\mathscr{Q}$-vertex v of type $2\beta + \delta$. So the $\mathscr{T}$-edge CA and the $\mathscr{T}$-edge BA are both $\mathscr{Q}$-edges and should match, but this is *impossible* because the triangular tile ABC of $\mathscr{T}$ is scalene.

Therefore the remaining cases are $\mathscr{T} = F_{48}, TF_{48}, F_{120}, G_{4n}$ $(n \geq 3)$.

Case 1.1. $\mathscr{T} = F_{48}, F_{120}$ or TF_{48}.

If a spherical tiling $\mathscr{T} = F_{48}$ (F_{120}, resp.) by congruent right scalene triangles yields a spherical monohedral quadrangular tiling $\mathscr{Q}$, then $\mathscr{Q}$ is K_{24} (K_{60}, resp.), by the construction (Subsection 2.1) of K_{24} (K_{60}, resp.) and the following:

Fact 3. *Every spherical tiling $\mathscr{T}$ by congruent right scalene triangles T yields* at most one *spherical monohedral quadrangular tiling $\mathscr{Q}$, modulo congruence.*

Proof. By Proposition 1, the tile Q of $\mathscr{Q}$ cannot be a parallelogram. If $\mathscr{T}$ yields another spherical monohedral quadrangular tiling, we must remove a cathetus of T, so some inner angle of Q should be 1. This is a contradiction. □

A spherical monohedral triangular tiling TF_{48} yields gK_{24}. It is proved as follows: For each tile of gK_{24}, draw a diagonal segment between opposite vertices having unequal inner angles. This subdivision of the tiles results in a spherical tiling $\mathscr{Q}'$ consisting of 48 congruent right scalene triangles having inner angles $1/3$ and $1/4$. It is because K_{24} has the same tile by the construction of gK_{24}. According to Table 2, $\mathscr{Q}'$ is either F_{48} or TF_{48}. F_{48} yields K_{24} but not gK_{24} by Fact 3. So TF_{48} yields gK_{24}.

Case 1.2. $\mathscr{T} = G_{4n}$ $(n \geq 3)$.

Let the three inner angles A, B, C of the triangular tile T of $\mathscr{T} = G_{4n}$ satisfy that $A + B = 1$, $C = 1/n$, and $1/(2n) < A, B < (2n-1)/(2n)$ as in Table 2. Since the triangular tile T is scalene, none of A and B is $C = 1/n$, and we can assume $A > B$, without loss of generality. Hence

$$1 - \frac{1}{2n} > 1 - B = A > \frac{1}{2} > B > \frac{1}{2n} \qquad \text{and} \qquad B \neq \frac{1}{n}. \tag{4}$$

The Mollweide-like projection of G_{4n} is given by the thick edges and the thin edges of Figure 5, according to [16, Figure 4, p. 471]. Let the spherical monohedral triangular tiling $\mathscr{T} = G_{4n}$ yield a tiling $\mathscr{Q}$ consisting of $2n$ congruent quadrangles Q. If Q is obtained by gluing two copies of the triangular

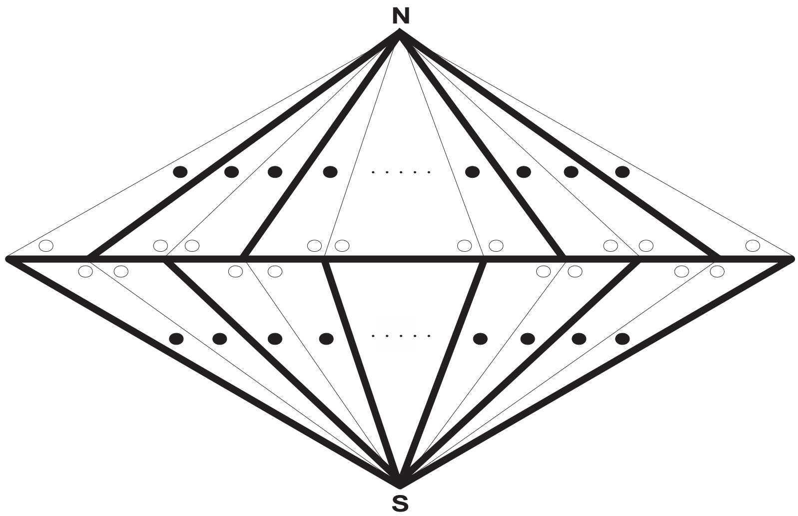

Fig. 5. The set of the thick edges and thin edges represents the Mollweide-like projection of G_{4n}, according to [16, Figure 4, p. 471]. The marks $\circ$, $\bullet$ in the figure stand for the angles A, C, while the other inner angles are B. Identify the leftmost thin (thick, resp.) edge with the rightmost thin (thick, resp.) edge. The tiling TR_{2n}^{A} by congruent kites is determined by the set of the thin edges (NA's and SA's) and the horizontal thick edges (AB's), and the tiling TR_{2n}^{B} by congruent darts is determined by the thick edges (NB's, SB's, and AB's). Here we suppose the inner angles satisfy $A + B = 1$, $C = 1/n, A > B$. See Case 1.2.

tile T of $\mathscr{T}$ at their edges AB, then the tile Q becomes a lune (i.e, a digon) but not a quadrangle because $A+B=1$. Hence the tile Q is obtained either by gluing two copies of tile T (Case (i)) at the edges AC (thin edges in Figure 5) or (Case (ii)) at the edges BC.

Case (i). The graph of $\mathscr{Q}$ is given by the thick edges of Figure 5, and is of the form Figure 3 (right). The cyclic list of four inner angles of Q is $(B, 2A, B, 2C)$. The tile Q is a dart, because Lemma 2 (2) and the above (4). Thus the tiling $\mathscr{Q}$ is listed as TR_{2n}^{B} $(1/(2n) < B < 1/2, B \neq 1/n)$ in the fifth entry of Table 1 with $(\alpha, \beta, \alpha, \delta) = (B, 2C, B, 2A)$.

Case (ii). The graph of $\mathscr{Q}$ is given by the thin edges of Figure 5 and the horizontal thick edges (the "equator.") The cyclic list of four inner angles of Q is $(A, 2B, A, 2C)$. By above (4), the angle A ranges over $(1/2, 1-1/(2n))\backslash\{1-1/n\}$. By Lemma 2 (3), the tiling $\mathscr{Q}$ is TR_{2n}^{A} of the form Figure 3 (left).

Applying the same argument of Case (i) and Case (ii) to the case $A < B$, we obtain TR_{2n}^{A} where $A \in (1/(2n), 1-1/(2n))\backslash\{1/2, 1-1/n, 1/n\}$.

Case 2. $\mathscr{T}$ is a spherical tiling by $F \geq 12$ congruent non-equilateral isosceles triangles T.

Let B and C be the two base angles of T and A $(\neq B = C)$ be the other inner angle of the triangular tile T. If a kite or a dart is obtained from gluing two copies of the isosceles triangle T, it has inner angle $2A$. Actually, the cyclic list of the four inner angles is $(2A, B, 2C, B)$, $(2A, C, 2B, C)$, or $(2A, B, C+B, C)$. Thus $\mathscr{T}$ cannot have a vertex type where A occurs odd times. So $\mathscr{T}$ is none of F_{12}^{I}, F_{12}^{II}, F_{12}^{III}, F_{24}, F_{60}^{I}, F_{60}^{II}, H_{4n} $(5 \neq n \geq 3)$ and TH_{8n+4} $(n \geq 3)$. (In Table 2, the non-base inner angle of the non-equilateral isosceles triangular tile of H_{4n} $(5 \neq n \geq 3)$ $(TH_{8n+4}$ $(n \geq 3))$ is C.) Thus if $B = C > A$, then no tiling is possible, by Lemma 4 (II) and Lemma 6 (2). Hence, the tiling $\mathscr{T}$ amounts to be G_{4n} $(n \geq 3)$ where $A = 1 - 1/n$ and $B = C = 1/n$, by the previous two sentences, Lemma 4 (I), and Lemma 6 (1), since Lemma 6 (2) implies $\mathscr{T} \neq I_{24}$.

From [16, Figure 11], we observe that the Mollweide-like projection of G_{4n} $(A = 1 - 1/n > B = C = 1/n,\ n \geq 3)$ is given by the thick edges (NC's, SC's, and AC's) and the thin edges (NA's and SA's) of Figure 6. Here the upper leftmost thin (thick, resp.) edge is identified with the upper rightmost thin (thick, resp.) edge. In Figure 6, the Mollweide-like projection of the spherical monohedral dart-faced tiling $\mathscr{Q}$ is determined by the thick edges (NC's, SC's, and AC's). Note that the set of the thin edges (NA's and SA's) and the horizontal thick edges (CA's) determines the Mollweide-like projection of a spherical monohedral *rhombus-faced* tiling by $B = C$.

The cyclic list $(\alpha, \beta, \alpha, \delta)$ of inner angles of the quadrangular tile of $\mathscr{Q}$ is $(C, 2B, C, 2A) = (1/n, 2/n, 1/n, 2-2/n)$. So the tiling is indeed a dart-

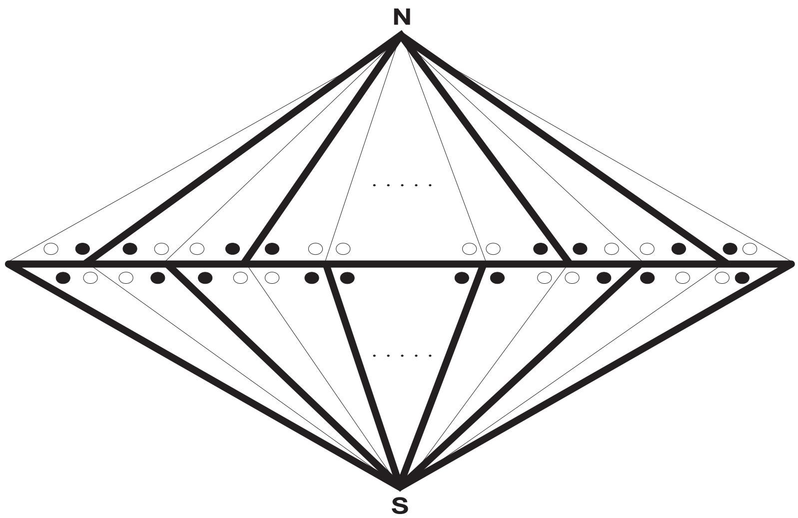

Fig. 6. The *Mollweide-like* projection of G_{4n} is given by the thick and the thin edges, and that of $TR_{2n}^{1/n}$ is given by the thick edges. The marks $\circ$, $\bullet$ stand for the inner angles A, C, while the other inner angles are B. See Case 2 of the proof of Lemma 5 and Subsubsection 3.1.2 (4).

faced $TR_{2n}^{1/n}$ by Lemma 2 (2) and $n \geq 3$. This completes the proof of Lemma 5. □

By Lemma 5 and Lemma 3 (1), we have verified Theorem 2 (1) for the case the tile is a kite or a dart. □

3.1.2. *Theorem* 2 (2). By using a procedure [5, p. 55 and Figure 3.10 (p. 56)], we will prove Theorem 2 (2) for spherical monohedral kite-faced tilings (1) K_{24} and K_{60}, (2) gK_{24} and (3) TR_{2n}^{α} $(1/2 < \alpha < 1 - 1/(2n),\ \alpha \neq 1 - 1/n,\ n \geq 3)$. Then we will prove Theorem 2 (2) for the spherical monohedral dart-faced tiling (4) TR_{2n}^{α} $(1/(2n) < \alpha < 1/2,\ n \geq 3)$.

(1) K_{24} (K_{60}, resp.) is isohedral. It is proved by Lemma 1 as follows: Each triangular (pentagonal, resp.) tile Φ of $\mathcal{O}_8$ ($\mathcal{D}_{12}$, resp.) can be transformed to another tile Φ' of $\mathcal{O}_8$ ($\mathcal{D}_{12}$, resp.) by the symmetry of $\mathcal{O}_8$ ($\mathcal{D}_{12}$, resp.). Within the tile Φ, any tile of K_{24} (K_{60}, resp.) can be transformed to any other tile of K_{24} (K_{60}, resp.), due to the following:

FACT 4. *Let $\mathcal{P}$ be the central projection of a Platonic solid with the faces being p-gons to the sphere. Then for each tile T of $\mathcal{P}$, a p-fold axis of rotation of $\mathcal{P}$ is through the center of T.*

PROOF. See [5, p. 47] for $\mathcal{P}$ being the central projection of the regular tetrahedron, [5, p. 49] for $\mathcal{P} = \mathcal{D}_{12}$ and H_{20}, and [5, p. 48] for the other $\mathcal{P}$. □

Hence any tile of K_{24} (K_{60}, resp.) can be transformed to any other tile of K_{24} (K_{60}, resp.), as desired.

The cyclic list of inner angles of the tile and the list of vertex types of K_{24} (K_{60}, resp.) are as in Table 1. It is due to Fact 2, since according to Lemma 5, the graph of K_{24} (K_{60}, resp.) is obtained from that of F_{48} (F_{120}, resp.), by deleting all the hypotenuses of the triangular tiles joining the vertex B and vertex C.

(2) The list of inner angles of the tile of gK_{24} is the list of inner angles of the tile of K_{24}, by the construction (Subsection 2.2).

The Schönflies symbol of gK_{24} is D_{4d}. We prove it according to the procedure [5, p. 55 and Figure 3.10 (p. 56)], as follows:

The polar axis is the only four-fold rotation axis of gK_{24}. The polar axis of gK_{24} is a four-fold axis of rotation, because gK_{24} is obtained from K_{24} by gliding the northern hemisphere around the four-fold, polar axis of K_{24}. Assume ρ is another four-fold axis of rotation. Then ρ is not through a tile, as the tile is not regular. The four-fold axis ρ of rotation is not through a vertex, because every four-valent vertex other than the pole is incident to edges of different length.

gK_{24} has no three-fold axis of rotation. Assume ρ is a three-fold axis of rotation. Then ρ is through neither a quadrangular tile nor a four-valent vertex. If ρ is through a three-valent vertex, then the antipodal point is strictly between edges (or is a four-valent vertex), according to the image (Figure 2 (right)).

By the same image, the tiling gK_{24} has (i) four two-fold axes of rotation perpendicular to the four-fold axis of rotation, and (ii) four mirror planes passing between two-fold axes of rotation. Each mirror plane contains exactly six consecutive edges. Thus the Schönflies symbol of gK_{24} is D_{4d}, as desired.

The tiling gK_{24} is not isohedral, because the number 24 of the tiles does not divide the order 16 of the symmetry group D_{4d}.

The cyclic list of inner angles of the tile and the list of vertex types of gK_{24} are as in Table 1. It is due to Fact 2 and Lemma 5, because, as in Figure 2 (left), we can obtain the graph (thick edges and dotted edges) of gK_{24} from the graph (thin edges, thick edges and dotted edges) of TF_{48} by deleting each hypotenuse joining a vertex of the inner angle B and a vertex of the inner angle C $(\neq B)$.

(3) In Figure 3 (left), for the cyclic list of inner angles $(\alpha, \beta, \alpha, \delta)$ of TR_{2n}^{α}, we see the type of each pole is $n\beta$ and the type of each non-pole vertex is $2\alpha + \delta$. Hence, the inner angles and the vertex types are as in Table 1.

The tile is a *kite*, by Lemma 2 (3) and the condition on the inner angle α. Thus the length a of each meridian edge is not equal to the length b of each non-meridian edge, in the tiling.

We prove that the Schönflies symbol of TR_{2n}^{α} is D_{nd}, by the procedure [5, p. 55 and Figure 3.10 (p. 56)], as follows:

If TR_{2n}^{α} has a three-fold or four-fold axis of rotation passing through a vertex, then the axis is the polar axis. It is verified as follows: Every non-pole vertex is three-valent and incident to an edge of length a and to two edges of length b $(\neq a)$. A three-fold axis of rotation is not through an inner point of a tile T, because T is a non-regular quadrangle. Obviously, a three-fold or four-fold axis of rotation cannot pass through a point of an edge of TR_{2n}^{α}.

Clearly, TR_{2n}^{α} has an n-fold axis ρ of rotation through the poles.

TR_{2n}^{α} has $2n$ two-fold axes of rotation, perpendicular to ρ. It is because TR_{2n}^{α} has $2n$ points where the equator and the non-meridian edges intersect, and has $2n$ vertical mirror planes which pass between the horizontal two-fold axes of rotation. So the Schönflies symbol of TR_{2n}^{α} is D_{nd}, as desired.

TR_{2n}^{α} is isohedral. Indeed, if two quadrangular tiles T and T' of the tiling share a meridian edge of the tiling, then T is rotated to T' around the vertical n-fold axis of rotation. If T and T' share a non-meridian edge of the tiling TR_{2n}^{α}, then T is rotated to T' around a horizontal two-fold axis of rotation.

(4) It is proved as in the previous assertion (3).

This completes the proof of Theorem 2 (1) and (2) for the case where the tile is a kite or a dart. □

3.2. A complete classification of spherical monohedral rhombus-faced tilings.

3.2.1. *The completeness (Theorem 2 (1)).*

LEMMA 7. *Suppose $\mathscr{R}$ is a spherical monohedral rhombus-faced tiling and suppose that $\mathscr{T}$ is a spherical tiling by congruent non-equilateral isosceles triangles T such that the list of inner angles A, B, C of T satisfy $A > B = C$. Then if $\mathscr{T}$ yields $\mathscr{R}$, then one of the following holds:*

(1) *$\mathscr{R}$ is $TR_6^{2/3}$ while $\mathscr{T}$ is F_{12}^{I}, F_{12}^{II}, F_{12}^{III}, MTG_{12}^{I}, H_{12}, TG_{12} $(A = 2/3, B = C = 1/3)$, or G_{12} $(A = 2/3, B = C = 1/3)$;*

(2) *$\mathscr{R}$ is R_{12} while $\mathscr{T}$ is F_{24} or I_{24};*

(3) *$\mathscr{R}$ is R_{30} while $\mathscr{T}$ is F_{60}^{I} or F_{60}^{II};*

(4) *$\mathscr{R}$ is $TR_8^{3/4}$ while $\mathscr{T}$ is H_{16} $(A = 1/2,\ B = C = 3/8$. In the entry of H_{16} of Table 2, we read $A = B = 3/8 < C = 1/2)$ or $\mathscr{T}$ is G_{16} $(A = 3/4, B = C = 1/4)$;*

(5) *$\mathscr{R}$ is $TR_{2n}^{1-1/n}$ and $\mathscr{T}$ is G_{4n} with $A = 1 - 1/n$, $B = C = 1/n$ for $n \geq 5$; or*

(6) *$\mathscr{R}$ is gTR_{4n+2} and $\mathscr{T}$ is TG_{8n+4} with $A = 2n/(2n+1)$, $B = C = 1/(2n+1)$ for $n \geq 2$.*

We will prove the previous lemma by using Lemma 3 (2) and the following "uniqueness" lemma:

Lemma 8. *Every spherical tiling $\mathcal{T}$ by congruent non-equilateral isosceles triangles T yields a spherical monohedral rhombus-faced tiling $\mathcal{R}$* uniquely modulo congruence. *If the list of inner angles of T is (A, B, C) and $A \neq B = C$, then the cyclic list of inner angles of the rhombic tile of $\mathcal{R}$ is $(A, 2B, A, 2C)$.*

Proof. By removing the base edges of $\mathcal{T}$, we obtain a spherical monohedral rhombus-faced tiling. If $\mathcal{T}$ yields another spherical monohedral quadrangular tiling, then the cyclic list of inner angles of Q should be $(2B, C, 2A, C)$. Since $A \neq B$, the tile Q is a kite or a dart. □

Proposition 2. (1) *For any $n \geq 3$, the spherical monohedral triangular tiling G_{4n} $(A = 1 - 1/n, B = C = 1/n)$ (H_{4n}, resp.) is obtained from the spherical monohedral rhombus-faced tiling $TR_{2n}^{1-1/n}$ by drawing the meridian (the "horizontal," resp.) diagonal segment of each tile. $I_{8n} = \mathrm{kis}(TR_{2n}^{1-1/n})$ $(n \geq 3)$.*

(2) *The spherical monohedral triangular tiling TH_{8n+4} $(n \geq 3)$ is obtained from the spherical monohedral rhombus-faced tiling gTR_{4n+2} by drawing the shorter diagonal segment of each tile. $TI_{16n+8} = \mathrm{kis}(gTR_{4n+2})$ $(n \geq 2)$.*

Proof. (1) See Figure 11, Figure 12, Figure 14, Figure 20, and Figure 21 of [16]. (2) See [16, Figure 21]. □

The proof of Lemma 7. We use Lemma 3 (2). Let $\mathcal{T}$ be one of spherical monohedral triangular tilings listed in Lemma 4 (I).

Case 1. $\mathcal{T}$ is F_{12}^{I}, F_{12}^{II}, F_{12}^{III}, MTG_{12}^{I}, H_{12}, G_{12} $(A = 2/3, B = C = 1/3)$, or TG_{12} $(A = 2/3, B = C = 1/3)$.

We draw one diagonal segment in each square tile of $TR_6^{2/3}$. Then the resulting tiling is exactly a tiling $\mathcal{T}$. Combinatorially, there exist just seven tilings, which are exactly the tilings $\mathcal{T}$ of this case, according to [16, pp. 481–482]. So $\mathcal{T}$ yields $TR_6^{2/3}$.

Case 2. $\mathcal{T}$ is F_{24} or I_{24}.

By the construction (Subsection 2.4) of R_{12}, the spherical tiling F_{24} by congruent right, non-equilateral isosceles triangles yields R_{12}.

The spherical tiling I_{24} by congruent non-equilateral isosceles triangles yields R_{12}, too. It is verified as follows: According to Subsection 2.4, R_{12} is constructed from $F_{24} = \mathrm{kis}(\mathcal{O}_8)$ by deleting the 12 edges of $\mathcal{O}_8$. By recalling the image [16, Figure 20] of R_{12} and the duality between $TR_6^{2/3}$ and $\mathcal{O}_8$, we can construct R_{12} from $I_{24} = \mathrm{kis}(TR_6^{2/3})$ by deleting the 12 edges of $TR_6^{2/3}$.

Case 3. $\mathscr{T}$ is F^{I}_{60} or F^{II}_{60}.

By the construction (Subsection 2.4) of R_{30}, spherical monohedral triangular tiling F^{I}_{60} yields R_{30}. Moreover, F^{II}_{60} yields R_{30}, by [16, Figure 36].

Case 4. $\mathscr{T}$ is H_{16} or G_{4n} $(A = 1 - 1/n, B = C = 1/n, n \geq 4)$.

Proposition 2 (1) implies the assertion (4) and the assertion (5) of Lemma 7.

Case 5. $\mathscr{T}$ is TG_{8n+4} $(A = 2n/(2n+1), B = C = 1/(2n+1), n \geq 2)$.

Then $\mathscr{T}$ yields gTR_{4n+2}. It is proved as follows: subdivide each rhombic tile of gTR_{4n+2} into two congruent non-equilateral isosceles triangles, by drawing the *longer* diagonal segment. This results in a spherical monohedral tilings $\mathscr{T}'$ consisting of $8n+4$ non-equilateral isosceles triangles. The cyclic list of inner angles of the rhombic tile of gTR_{4n+2} and the cyclic list of inner angles of the rhombic tile of $TR^{2n/(2n+1)}_{4n+2}$ are both $(2n/(2n+1), 2/(2n+1), 2n/(2n+1), 2/(2n+1))$, by the construction of gTR_{4n+2}. So, the cyclic list of inner angles of the isosceles triangular tile of $\mathscr{T}'$ are $(2n/(2n+1), 1/(2n+1), 1/(2n+1))$.

Owing to Ueno-Agaoka's classification (Table 2) of all spherical monohedral triangular tilings, the spherical monohedral triangular tiling $\mathscr{T}'$ is either G_{8n+4} or TG_{8n+4}. Here we have verified that G_{8n+4} yields $TR^{2n/(2n+1)}_{4n+2}$, in the previous Case 4. By Lemma 8, a spherical monohedral, isosceles triangular tiling TG_{8n+4} yields gTR_{4n+2}. This completes the proof of Lemma 7. □

By Lemma 7 and Lemma 3 (2), we established Theorem 2 (1) for the case the tile is a rhombus. □

3.2.2. *Theorem* 2 (2).

LEMMA 9. *gTR_{4n+2} $(n \geq 2)$ is the spherical monohedral,* non-isohedral *rhombus-faced tiling by the tiles of spherical monohedral tiling $TR^{2n/(2n+1)}_{4n+2}$. The smallest inner angle β and the largest inner angle α of the rhombic tile are*

$$\beta = \frac{2}{2n+1}, \qquad \alpha = \frac{2n}{2n+1}.$$

The Schönflies symbol of gTR_{4n+2} is D_2. The vertex types of gTR_{4n+2} are as in Table 1.

Actually, the edges of gTR_{4n+2} have length

$$a = \arccos\left(\cos\frac{\pi\beta}{2} \Big/ \left(1 + \cos\frac{\pi\beta}{2}\right)\right) \neq \frac{\pi}{2}. \tag{5}$$

The vertices v_i, u_i $(0 \leq i \leq 2n+1)$ have the following cartesian coordinates:

$$v_i = \left(\sin a \cos \frac{i\pi\beta}{2}, -\sin a \sin \frac{i\pi\beta}{2}, (-1)^i \cos a\right). \tag{6}$$

$$u_i = \left(\frac{(-1)^i - \cos \frac{i\pi\beta}{2}}{2} \sin 2a, \sin a \sin \frac{i\pi\beta}{2}, \sin^2 a \cos \frac{i\pi\beta}{2} + (-1)^i \cos^2 a\right). \tag{7}$$

Proof. By Lemma 8 and Fact 2, the vertex types of gTR_{4n+2} are those of TG_{8n+4} applied by a substitution $(A := \alpha, B := \beta/2, C := \beta/2)$.

(5) is proved by applying *spherical cosine law for angles* [17, Proposition 2.8] to the following isosceles triangle T: The rhombic tile of gTR_{4n+2} is subdivided into two copies of T such that the base edge has length $\pi - a$, the non-base inner angle is $\pi\alpha$ radian, and the two base inner angles are $\pi\beta/2$ radian.

Recall the construction (Subsection 2.5) of the tiling gTR_{4n+2}. The western hemisphere $\mathscr{W}$ of gTR_{4n+2} is that of a spherical monohedral rhombus-faced tiling $TR^{2n/(2n+1)}_{4n+2}$ (see the first and the second images of Figure 4). We set xyz-coordinate system so that the north pole N is $(0, 0, 1)$, the part of the unit sphere with positive x-coordinates is the front hemisphere, and the hemisphere $\mathscr{W}$ is the part of the unit sphere with negative y-coordinates.

The vertices of the western hemisphere $\mathscr{W}$ are: the north pole u_0, the south pole u_{2n+1}, and v_i $(0 \le i \le 2n+1)$ where the angle from u_0 to each v_i is $\pi/2 - (-1)^i(\pi/2 - a)$ and the longitude of each v_i from the boundary of $\mathscr{W}$ is $-i\pi\beta/2$ radian. So, the cartesian coordinates of the vertices on $\mathscr{W}$ are as in (6).

The edges of the hemisphere $\mathscr{W}$ are:

$$\{u_0, v_k\} \qquad (k = 0, 2, 4, \ldots, 2n), \tag{8}$$

$$\{u_{2n+1}, v_l\} \qquad (l = 1, 3, 5, \ldots, 2n+1), \tag{9}$$

$$\{v_j, v_{j+1}\} \qquad (j = 0, 1, 2, \ldots, 2n). \tag{10}$$

The eastern hemisphere $\mathscr{E}$ of gTR_{4n+2} is the image of the western hemisphere $\mathscr{W}$ by the orthogonal transformation T_aK. Here K is the π-radian rotation operation around the z-axis, and T_a is the rotation in a radian around the y-axis from the z-axis toward the x-axis.

So the vertices of gTR_{4n+2} on the hemisphere $\mathscr{E}$ are

$$u_i := T_aKv_i \qquad (0 \le i \le 2n+1). \tag{11}$$

Hence the cartesian coordinate of u_i is (7). By this, u_0 and u_{2n+1} are indeed the north pole and the south pole, and thus $T_aKN = v_0$ and $T_aKS = v_{2n+1}$.

The edges of the eastern hemisphere $\mathscr{E}$ are the edges (8), (9), and (10) with u and v swapped.

This is exactly the graph of Figure 4 (right).

We will prove the angle assignment of gTR_{4n+2} is as in Figure 4 (right), as follows: If an inner angle of gTR_{4n+2} corresponds to some inner angle of the western hemisphere of $TR_{4n+2}^{2n/(2n+1)}$, then the type of the inner angle is α or β as in Figure 4 (right). The two newly arisen inner angles $\angle v_1v_0u_{2n}$ and $\angle u_1u_0v_{2n}$ of gTR_{4n+2} near the boundary of $\mathscr{W}$ have the desired values. For example, $\angle v_1v_0u_{2n} = \angle v_1v_0u_{2n+1} + \angle u_{2n+1}v_0u_{2n} = \angle v_1v_0u_{2n+1} + \angle v_{2n}v_0v_{2n+1} = \angle v_0u_0v_2$ and so on.

We will prove that the Schönflies symbol of gTR_{4n+2} is D_2, by the procedure [5, p. 55 and Figure 3.10 (p. 56)], as follows:

If gTR_{4n+2} has a $k(\geq 2)$-fold axis ρ of rotation, then $k = 2$. To see it, let p be a point where ρ and gTR_{4n+2} intersects. If p is an inner point of a tile of gTR_{4n+2}, then $k = 2$ because the tile is a rhombus. If $k > 2$, then p is not a vertex of the tiling gTR_{4n+2}, because each vertex type of the tiling is of the form $iA + mB$ for some different inner angles A, B, some positive integer m, and some positive integer $i < 3$. If p is on a edge, then $k = 2$ clearly.

gTR_{4n+2} has three mutually perpendicular two-fold axes of rotation:

(1) The y-axis.

The π-radian rotation around it transforms the vertex v_i to the vertex v_{2n+1-i} and the vertex u_i to the vertex u_{2n+1-i}. This rotation preserves the edges of gTR_{4n+2}. Indeed, in the western hemisphere, this rotation transforms the edges (8) to the edges (9) and vise versa, and preserves the edges (10). In the eastern hemisphere, this rotation transforms the edges similarly.

(2) ℓ through the midpoint of u_0 and v_0.

The π-radian rotation around it is a transformation $T_{a/2}KT_{-a/2}$. The transformation maps the vertex $u_i = T_aKv_i$ $(0 \leq i \leq 2n+1)$ to $(T_{a/2}K)^2 v_i = v_i$, and vice versa. Clearly, this rotation preserves the edges of gTR_{4n+2}.

(3) ℓ' through the midpoint of u_0 and v_{2n+1}.

The π-radian rotation around it is the composition of the two π-radian rotations around the y-axis and ℓ.

To show that the tiling gTR_{4n+2} has no mirror plane passing between two of the three two-fold axes, it is sufficient to show that $v_0u_0v_{2n+1}u_{2n+1}$ is a proper rectangle, because the symmetry of gTR_{4n+2} fixes the proper rectangle. Here $v_0u_0v_{2n+1}u_{2n+1}$ is indeed a rectangle, because the vertex v_0 is antipodal to v_{2n+1} by (6) and (7) and the vertex u_0 is antipodal to u_{2n+1}. The rectangle is not regular, because the edge-length a of gTR_{4n+2} is not $\pi/2$ radian.

The xz-plane containing the two axes ℓ and ℓ' of rotation is not a mirror plane of gTR_{4n+2}. To see it, first observe that the y-coordinate of the vertex v_0 of gTR_{4n+2} is 0. Among the vertices adjacent to v_0, the number of vertices having positive y-coordinates is, however, greater than that of vertices having negative y-coordinates, by (7) and $\beta = 2/(2n+1)$. Therefore the Schönflies symbol of gTR_{4n+2} is D_2.

gTR_{4n+2} $(n \geq 2)$ is not isohedral, because the number $4n+2$ of tiles does not divide the order four of the symmetry group D_2. This completes the proof of Lemma 9. □

We finish the proof of Theorem 2 (2), by verifying the data of the following spherical monohedral rhombus-faced tilings given in Table 1: (1) $TR_{2n}^{1-1/n}$ and (2) R_{12} and R_{30}.

(1) By Fact 1, the Schönflies symbol of $TR_6^{2/3}$ is O_h, and the symmetry group transitively acts on the tiling $TR_6^{2/3}$.

Consider a spherical monohedral rhombus-faced tiling $TR_{2n}^{1-1/n}$ with $n > 3$. Then, the list of three inner angles around any three-valent vertex of $TR_{2n}^{1-1/n}$ is $(\alpha, \alpha, \delta) = (1-1/n, 1-1/n, 2/n)$, where $\alpha \neq \delta$ because $n > 3$. Hence, we can prove that the Schönflies symbol of $TR_{2n}^{1-1/n}$ is D_{nd} and the tiling $TR_{2n}^{1-1/n}$ is isohedral, as in the proof for the spherical monohedral kite-faced tiling TR_{2n}^{α} $(1/2 < \alpha < 1 - 1/(2n),\ \alpha \neq 1 - 1/n,\ n \geq 3)$ of Theorem 2 (2).

(2) The Schönflies symbol of the spherical tiling F_{24} (F_{60}^{I}, resp.) by congruent isosceles triangles is O_h (I_h, resp), because of [5, p. 50] and the following:

Lemma 10. *If $\mathscr{P}$ is the central projection of some Platonic solid to the sphere, then every symmetry operation of $\mathscr{P}$ is that of* kis($\mathscr{P}$), *and* kis($\mathscr{P}$) *is isohedral.*

Proof. Every symmetry operation of $\mathscr{P}$ is that of kis($\mathscr{P}$), because it transforms a tile of $\mathscr{P}$ to a tile of $\mathscr{P}$, a vertex of $\mathscr{P}$ to a vertex of $\mathscr{P}$, and the center of a tile of $\mathscr{P}$ to the center of a tile of $\mathscr{P}$.

Suppose a tile T of $\mathscr{P}$ is a regular p-gon. By Fact 4, within the tile T, the p tiles of kis($\mathscr{P}$) rotated to each other around a p-fold axis. Moreover T is transformed to another tile of $\mathscr{P}$, by a symmetry operation of $\mathscr{P}$. Thus, in kis($\mathscr{P}$), any tile is transformed to any other tile by a symmetry operation of kis($\mathscr{P}$). □

The Schönflies symbol of R_{12} (R_{30}, resp.) is that of F_{24} (F_{60}^{I}, resp.), because each symmetry operation of F_{24} (F_{60}^{I}, resp.) preserves the tiles and the edges of F_{24} (F_{60}^{I}, resp.).

By the construction (Subsection 2.4) of R_{12} (R_{30}, resp.) and Fact 4, if T, T' are tiles of R_{12} (R_{30}, resp.) sharing the center c of a tile of $\mathscr{O}_8$ (H_{20}, resp.), then T is rotated to T' around the three-fold (five-fold, resp.) axis ρ of the rotation such that ρ passes through the center c. If T and T' does not share any vertex, then some symmetry operation of $\mathscr{O}_8$ (H_{20}, resp.) rotates T to another tile T'' of R_{12} (R_{30}, resp.) such that T'' and T' share the center of some tile of $\mathcal{O}_8$ (H_{20}, resp.).

By Table 2, the list of inner angles of the isosceles triangular tile T of F_{24} (F^I_{60}, resp.) is $(A, B, C) = (2/3, 1/4, 1/4)$ ($(2/3, 1/5, 1/5)$, resp.). As we glue two copies of T at the base edges, the cyclic list of inner angles of the rhombic tile of R_{12} (R_{30}, resp.) is $(\alpha, \beta, \alpha, \beta) = (2B, A, 2B, A)$. So, the vertex types of R_{12} (R_{30}, resp.) are as in Table 1, by Fact 2. □

4. The anisohedral spherical triangles

We will prove that the tiles of the following spherical monohedral triangular tilings are *not* anisohedral.

LEMMA 11. *The followings are spherical isohedral triangular tilings:*
(1) *Every instance of a continuously deformable tiling* F_4, F^I_{12}, F_{24}, F_{48}, F^I_{60}, F^{II}_{60}, *and* F_{120}.
(2) *Every instance of a continuously deformable tiling* G_{4n} $(n \geq 2)$, *and* G_{4n+2} $(n \geq 1)$.
(3) *An instance* H_{20} *of an infinite series* H_{4n} $(n \geq 3)$.
(4) *An instance* I_{24} *of an infinite series* I_{8n} $(n \geq 3)$.

PROOF. (1) Let $\mathcal{T}$ be an instance of F_4. $\mathcal{T}$ has two two-fold axes of rotation, each of which is through the midpoint of an edge and that of the opposite edge, in view of the development map of F_4 [16, Figure 17]. We can transform any tile T of $\mathcal{T}$ to any other tile T', by π-rotation around an axis ρ which is through the midpoint of the edge shared by the two tiles T and T'. Hence F_4 is isohedral.

Next, we prove that F_{48} is isohedral. Indeed, every symmetry operation of $\mathcal{O}_8$ is that of F_{48}, as in the proof of Lemma 1.

For each tile T of $\mathcal{O}_8$ and each vertex A of T, there is a mirror plane σ passing through A and bisecting the edge of T opposite to A, because of [5, (ix) of p. 48]. Let M be the midpoint of the edge. The mirror plane σ passes through M and the center c of T.

Within the tile T of $\mathcal{O}_8$, a pair of right scalene triangular tiles of F_{48} adjacent at the edge cM are reflected to each other by the mirror plane σ, and another pair of right scalene triangular tiles of F_{48} adjacent at their hypotenuses cA are reflected to each other by another mirror plane σ. Therefore within the tile T of $\mathcal{O}_8$, the six right scalene triangular tiles of F_{48} are transformed to each other by repeated reflection operations with appropriate mirror planes σ's of $\mathcal{O}_8$ where all such σ's pass through the center c of T.

Consider that tiles S and S' of F_{48} are in different tiles T and T' of $\mathcal{O}_8$. Then, to S apply a symmetry operation ρ that maps T to T', and then to the resulting tile of F_{48} apply a symmetry operation mentioned above in order to get S'. Hence, F_{48} is isohedral.

The proof of the isohedrality of F_{120} is obtained from the proof of the isohedrality of F_{48}, by replacing F_{48} with F_{120}, $\mathcal{O}_8$ with H_{20}, and "(ix) of p. 48" with "(v) and (vi) of p. 49."

The isohedrality of F_{12}^{I}, F_{24}, F_{60}^{I}, and F_{60}^{II} is by Lemma 10 and their constructions (cf. $F_{60}^{II} = \mathrm{kis}(\mathscr{D}_{12})$ by [16, Figure 17 and p. 484]).

(2) An instance $\mathscr{T}$ of G_{4n} $(n \geq 2)$ has n horizontal two-fold axes of rotation and n vertical mirror planes passing between two horizontal two-fold axes, according to [16, Figure 4]. Let T and T' be any tiles of $\mathscr{T}$. If T' shares a meridian edge e of T, then T is reflected to T' by a vertical mirror plane σ of $\mathscr{T}$ such that σ contains e. Moreover, if T' shares a non-meridian edge e' of T, then there is a horizontal two-fold axis ρ of rotation of $\mathscr{T}$ such that ρ passes through the midpoint of e' and T is rotated to T' around ρ. Hence, any tile of $\mathscr{T}$ is transformed to any other tile of $\mathscr{T}$ by a symmetry operation of $\mathscr{T}$. Thus, $\mathscr{T}$ is isohedral.

G_{4n+2} $(n \geq 1)$ is isohedral by the vertical $(2n+1)$-fold axis of rotation and the horizontal mirror plane, according to [16, Figure 4].

(3) An instance H_{20} is the central projection of the regular icosahedron to the sphere, and thus the Schönflies symbol of H_{20} is I_h, and H_{20} is isohedral.

(4) This is due to Proposition 2 (1) and Lemma 10. $\square$

The other spherical triangular tilings are not isohedral.

Lemma 12. *Spherical monohedral triangular tilings* H_{4n} $(5 \neq n \geq 3)$, I_{8n} $(n \geq 4)$, TH_{8m+4} $(m \geq 3)$, *and* TI_{16m+8} $(m \geq 2)$ *are non-isohedral.*

Proof. For $\mathscr{T} = H_{4n}$ $(5 \neq n \geq 3)$ or I_{8n} $(n \geq 4)$, the Schönflies symbol of $\mathscr{T}$ is D_{nd}. It is due to Proposition 2, because every symmetry operation of $TR_{2n}^{1-1/n}$ preserves the diagonal segments of the tile and the Schönflies symbol of $TR_{2n}^{1-1/n}$ is D_{nd} by Theorem 2.

I_{8n} $(n \geq 4)$ is not isohedral, since the order of the symmetry group D_{nd} of I_{8n} is $4n$ according to [5, p. 44] but the number $8n$ of the tiles of I_{8n} does not divide $4n$.

H_{4n} $(5 \neq n \geq 3)$ is not isohedral, because no symmetry operation transforms a tile around the pole to a tile on the equator (See the first line of Figure 7).

We can prove that the Schönflies symbol of TH_{8m+4} $(m \geq 3)$ (TI_{16m+8} $(m \geq 2)$, resp.) is D_2, in a similar manner as we proved that the Schönflies symbol of gTR_{4n+2} is D_2 in Lemma 9, in view of the construction (Proposition 2 (2)) of TH_{8m+4} (TI_{16m+8}, resp.).

TH_{8m+4} ($m \geq 3$) (TI_{16m+8} ($m \geq 2$), resp.) is not isohedral, because the number $8m+4$ ($m \geq 3$) ($16m+8$ ($m \geq 2$), resp.) of tiles does not divide the order 4 of the symmetry group D_2. □

Although H_{12} is not isohedral, the tile of H_{12} is congruent to the tile of the tiling G_{12} with $A = 2/3, B = C = 1/3$. G_{12} is isohedral by Lemma 11 (2). The other spherical monohedral, *non-isohedral* triangular tilings listed in the previous lemma are spherical tilings by anisohedral triangles, owing to the following lemma:

LEMMA 13. (1) *If the tile of* H_{4n} $(5 \neq n \geq 4)$ *is the tile of another spherical monohedral triangular tiling* $\mathscr{T}$, *then* $\mathscr{T}$ *is* TH_{8m+4} *for some* $m \geq 3$ *with* $4n = 8m+4$.

(2) *If the tile of* TH_{8m+4} $(m \geq 3)$ *is the tile of another spherical monohedral triangular tiling* $\mathscr{T}$, *then* $\mathscr{T}$ *is* H_{8m+4}.

(3) *If the tile of* I_{8n} $(n \geq 4)$ *is the tile of another spherical monohedral triangular tiling* $\mathscr{T}$, *then* $\mathscr{T}$ *is* TI_{16m+8} *for some* $m \geq 2$ *with* $8n = 16m+8$.

(4) *If the tile of* TI_{16m+8} $(m \geq 2)$ *is the tile of another spherical monohedral triangular tiling* $\mathscr{T}$, *then* $\mathscr{T}$ *is* I_{16m+8}.

PROOF. (1) The number of tile of $\mathscr{T}$ is $4n$ $(5 \neq n \geq 4)$ and the tile of H_{4n} is an *isosceles* triangle. So, if $\mathscr{T}$ is sporadic, then $n \geq 6$ and $\mathscr{T}$ is F_{24}, F_{60}^{I} or F_{60}^{II}. However, the non-base inner angle of $\mathscr{T}$ is greater than the non-base inner angle $2/n$ of H_{4n} for each case. Thus $\mathscr{T}$ is a non-sporadic spherical tiling by congruent *isosceles* triangles. Hence, $\mathscr{T}$ is G_{4n}, TG_{4n}, or TH_{8m+4} for some $m \geq 3$ such that $8m+4 = 4n$. By [16, Table] (included as Table 2), the tile of G_{4n} and that of TG_{4n} have an inner angle $1/n$ but the tile of H_{4n} does not, as $5 \neq n \geq 4$. Thus $\mathscr{T}$ is TH_{8m+4}, as desired.

(3) The number of tile of $\mathscr{T}$ is $8n \geq 32$ and the tile of I_{8n} is a right scalene triangle. So, if $\mathscr{T}$ is sporadic, then (1) $n = 6$ and $\mathscr{T}$ is F_{48} or TF_{48}, or (2) $n = 15$ and $\mathscr{T} = F_{120}$. Then the smallest inner angle of $\mathscr{T}$ is greater than the smallest inner angle $1/n$ of I_{8n}. Thus $\mathscr{T}$ is a non-sporadic spherical tiling by congruent right scalene triangles. So $\mathscr{T}$ is G_{8n}, or TI_{8n}. By [16, Table] (included as Table 2), an inner angle $1/(2n)$ of the tile of G_{8n} is smaller than the smallest angle $1/n$ of the tile of I_{8n} for each case. So the tile of $\mathscr{T}$ is not congruent to the tile of G_{8n}. Thus $\mathscr{T} = TI_{8n}$, as desired.

Assertion (2) (assertion (4), resp.) follows from assertion (1) (assertion (3), resp.), because the tile of TH_{8m+4} ($m \geq 3$) (TI_{16m+8} ($m \geq 2$), resp.) is also the tile of H_{8m+4} (I_{16m+8}, resp.). □

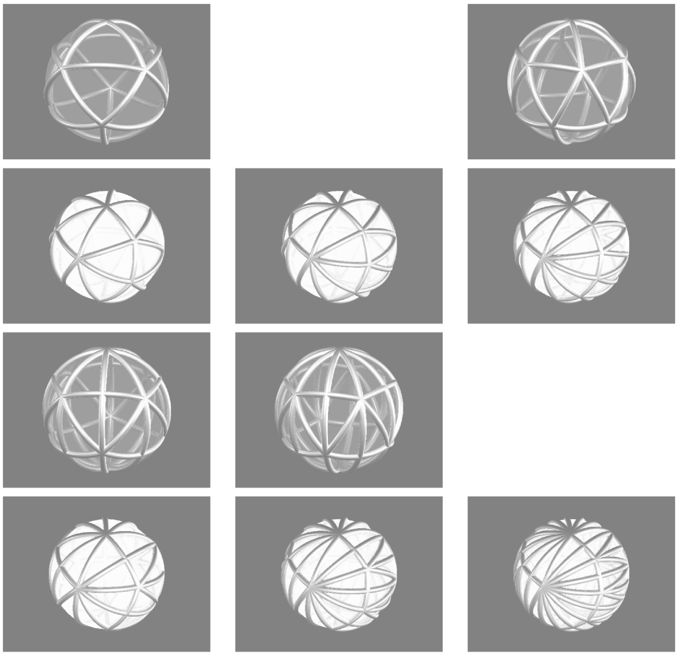

Fig. 7. The first (the second, resp.) line consists of spherical tilings H_{4n} $(n = 4, 6)$ (TH_{8m+4} $(m = 3, 4, 5)$, resp.) by $4n$ ($8m + 4$, resp.) congruent anisohedral isosceles triangles AI_n (AI_{2m+1}, resp.). The third (the fourth, resp.) line consists of spherical tilings I_{8n} $(n = 4, 5)$ (TI_{16m+8} $(m = 2, 3, 4)$, resp.) by $8n$ ($16m + 8$, resp.) congruent anisohedral right scalene triangles ARS_n (ARS_{2m+1}, resp.). See Theorem 1 and [16, Table] (included as Table 2 in this paper).

Lemma 14. *The spherical tilings by congruent* anisohedral *triangles are* exactly H_{4n} $(5 \neq n \geq 4)$, TH_{8m+4} $(m \geq 3)$, I_{8n} $(n \geq 4)$, *and* TI_{16m+8} $(m \geq 2)$.

Proof. H_{4n} $(5 \neq n \geq 4)$, I_{8n} $(n \geq 4)$, TH_{8m+4} $(m \geq 3)$ and TI_{16m+8} $(m \geq 2)$ are spherical tilings by congruent anisohedral triangles, because of Lemma 12 and Lemma 13.

We will prove that for any *other* $\mathscr{T}$ listed in Ueno-Agaoka's complete classification ([16, Table], included as Table 2) of spherical monohedral triangular tilings but not listed in Lemma 11, the tile T of $\mathscr{T}$ is congruent to some spherical isohedral triangular tiling. We will argue such $\mathscr{T}$ from top to bottom of Table 2. If $\mathscr{T}$ is F_{12}^{II}, or F_{12}^{III}, then the tile T of $\mathscr{T}$ admits F_{12}^{I}, because the list of three inner angles of the tile T is the list of three inner angles of the tile of F_{12}^{I}. As F_{12}^{I} is isohedral by Lemma 11 (1), the tile of $\mathscr{T}$ is not anisohedral. The tile of TF_{48} admits F_{48}, because the list of three inner angles of T is the list of three inner angles of the tile of F_{48}. F_{48} is isohedral by Lemma 11 (1). Hence, the tile of TF_{48} is not anisohedral. If $\mathscr{T}$ is TG_{8n}

($n \geq 2$), TG_{8n+4} ($n \geq 1$), MTG^{I}_{8n+4} ($n \geq 1$), MTG^{II}_{8n+4} ($n \geq 2$), or H_{12}, then the tile T of $\mathscr{T}$ admits an instance of G_m where m is the number of tiles of $\mathscr{T}$. $\mathscr{T}$ is isohedral by Lemma 11 (2). Hence, the tile of $\mathscr{T}$ is not anisohedral. $\square$

Because Ueno-Agaoka's complete classification of spherical monohedral triangular tilings describes the inner angles of the tiles, Theorem 1 follows from the previous Lemma (see Figure 7).

THEOREM 3. *No spherical (kite/dart/rhombus) is anisohedral.*

PROOF. By Theorem 2 (2) and Table 1, the tiles of the only spherical monohedral, non-isohedral (kite/dart/rhombus)-faced tilings gK_{24} and gTR_{4n+2} ($n \geq 2$) are those of spherical *isohedral* rhombus-faced tilings K_{24} and $TR^{2n/(2n+1)}_{4n+2}$. $\square$

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References

[1] Y. Akama. Classification of spherical tilings by congruent quadrangles over pseudo-double wheels (I)—a special tiling by congruent concave quadrangles. Hiroshima Math. J., Vol. 43, No. 3, pp. 285–304, 2013.

[2] D. V. Alekseevskij, È. B. Vinberg, and A. S. Solodovnikov. Geometry of Spaces of Constant Curvature. In Geometry, II, Vol. 29 of Encyclopaedia Math. Sci., pp. 1–138. Springer, Berlin, 1993.

[3] B. Bollobás. Filling the plane with congruent convex hexagons without overlapping. Ann. Univ. Sci. Budapest. Eötvös Sect. Math., Vol. 6, pp. 117–123, 1963.

[4] J. H. Conway, H. Burgiel, and C. Goodman-Strauss. The Symmetries of Things. A K Peters Ltd., Wellesley, MA, 2008.

[5] F. A. Cotton. Chemical Applications of Group Theory. John Wiley & Sons, third edition, 1990.

[6] H. S. M. Coxeter. Regular Polytopes. Dover Publications Inc., New York, third edition, 1973.

[7] C. Davis, B. Grünbaum, and F. A. Sherk, editors. The Geometric Vein. The Coxeter Festschrift. Springer-Verlag, New York, 1981.

[8] O. Delgado-Friedrichs. Data structures and algorithms for tilings. I. Theoret. Comput. Sci., Vol. 303, No. 2–3, pp. 431–445, 2003.

[9] B. Grünbaum. An enduring error. Elem. Math., Vol. 64, No. 3, pp. 89–101, 2009.

[10] B. Grünbaum and G. C. Shephard. Tilings with congruent tiles. Bull. Amer. Math. Soc. (N.S.), Vol. 3, No. 3, pp. 951–973, 1980.

[11] B. Grünbaum and G. C. Shephard. Spherical tilings with transitivity properties. In Davis, et al. [7], pp. 65–98.
[12] H. Heesch and O. Kienzle. Flächenschluss. System der Formen lückenlos aneinanderschliessender Flachteile. Springer-Verlag, Berlin, 1963.
[13] N. W. Johnson. Convex polyhedra with regular faces. Canad. J. Math., Vol. 18, pp. 169–200, 1966.
[14] S. K. Lando and A. K. Zvonkin. Graphs on Surfaces and Their Applications, volume 141 of Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin, 2004.
[15] Y. Ueno and Y. Agaoka. Examples of spherical tilings by congruent quadrangles. Mem. Fac. Integrated Arts and Sci., Hiroshima Univ. Ser. IV, Vol. 27, pp. 135–144, 2001.
[16] Y. Ueno and Y. Agaoka. Classification of tilings of the 2-dimensional sphere by congruent triangles. Hiroshima Math. J., Vol. 32, No. 3, pp. 463–540, 2002.
[17] P. M. H. Wilson. Curved Spaces. From Classical Geometries to Elementary Differential Geometry. Cambridge University Press, Cambridge, 2008.
[18] V. A. Zalgaller. Convex Polyhedra with Regular Faces. Translated from Russian. Seminars in Mathematics, V. A. Steklov Mathematical Institute, Leningrad, Vol. 2. Consultants Bureau, New York, 1969.

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