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Nonautonomous differential equations and Lipschitz evolution operators in Banach spaces

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Abstract. A new class of Lipschitz evolution operators is introduced and a characterization of continuous infinitesimal generators of such evolution operators is given. It is shown that a continuous mapping A from a subset Ω of $[a,b) \times X$ into X, where $[a,b)$ is a real half-open interval and X is a real Banach space, is the infinitesimal generator of a Lipschitz evolution operator if and only if it satisfies a sub-tangential condition, a general type of quasi-dissipative condition with respect to a metric-like functional and a connectedness condition. An application of the results to the initial value problem for the quasilinear wave equation with dissipation is also given.

1. Introduction and main theorems

Throughout this paper, $\mathbf{R}$ denotes the set of all real numbers. Let X be a real Banach space with norm $\|\cdot\|$. For a subset Q of $\mathbf{R} \times X$, $Q(t)$ denotes the section of Q at $t \in \mathbf{R}$, that is, $Q(t) = \{x \in X; (t,x) \in Q\}$.

Let $[a,b)$ be a subinterval of $\mathbf{R}$ and Ω a subset of $[a,b) \times X$ such that $-\infty < a < b \leq \infty$ and $\Omega(t) \neq \emptyset$ for $t \in [a,b)$. Let A be a continuous mapping from Ω to X. Given $(\tau,z) \in \Omega$, we consider the following initial value problem:

$$(\mathrm{IVP}; \tau, z) \qquad \begin{cases} u'(t) = A(t,u(t)) & \text{for } \tau \leq t < b, \\ u(\tau) = z. \end{cases}$$

Suppose that the problem $(\mathrm{IVP}; \tau, z)$ has a unique solution $u(\cdot)$ on $[\tau,b)$ for every $(\tau,z) \in \Omega$. Defining $U(t,\tau)z = u(t)$, we have the following properties from the uniqueness of solutions:

(E1) $U(\tau,\tau)z = z$ and $U(t,s)U(s,\tau)z = U(t,\tau)z$ for $z \in \Omega(\tau)$ and $a \leq \tau \leq s \leq t < b$.

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Set $\Delta = \{(t,\tau); a \le \tau \le t < b\}$. Usually, we have also the following properties from the continuous dependence of solutions on the initial data $(\tau, z) \in \Omega$:

(E2) Let $(t,\tau) \in \Delta$, $z \in \Omega(\tau)$, $(t_n, \tau_n) \in \Delta$ and $z_n \in \Omega(\tau_n)$ for $n = 1, 2, \dots$. If $(t_n, \tau_n) \to (t, \tau)$ and $z_n \to z$ as $n \to \infty$, then $U(t_n, \tau_n)z_n \to U(t,\tau)z$ as $n \to \infty$.

By an *evolution operator* on Ω, we mean a family $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$ of operators $U(t,\tau) : \Omega(\tau) \to \Omega(t)$ satisfying (E1) and (E2). Such a family $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$ is called a *Lipschitz evolution operator* on Ω, if the following additional condition is satisfied:

(E3) There exist a number $L \ge 1$ and a continuous function $\omega : [a,b) \to [0,\infty)$ such that

$$\|U(t,\tau)x - U(t,\tau)y\| \le L \exp\left(\int_\tau^t \omega(\theta)d\theta\right)\|x - y\|$$

for $x, y \in \Omega(\tau)$ and $(t,\tau) \in \Delta$.

The main purpose of this paper is to establish the conditions on the continuous mapping A which are necessary and sufficient to guarantee the existence of the Lipschitz evolution operator associated with A. The obtained results extend that of Kobayashi and Tanaka in [8] concerning the autonomous case where A is independent of t. In particular, a type of generalized quasi-dissipativity condition on A with respect to a metric-like functional is shown to be necessary for the existence of the Lipschitz evolution operator. Sufficient conditions on A for the existence of evolution operators have been studied by many authors and this paper is related with the works of Iwamiya [4], Kato [5], [6], Kenmochi and Takahashi [7], Lakshmikantham, Mitchell and Mitchell [10], Martin [11], [12], [13], Murakami [15], Pavel and Vrabie [19], Pavel [18] and Cârjă, Necula and Vrabie [22]. Several types of generalized quasi-dissipativity conditions on A are introduced and investigated in [15], [12], [10], [6], [20] and [2]. Such a kind of generalized quasi-dissipativity conditions was first found by Okamura [17] as a uniqueness criteria for ordinary differential equations. See [1] or [24]. Our results extend the most of them. As in [7], [6] and [4], the domain Ω is allowed to be genuinely noncylindrical and the subtangential condition, which was first found by Nagumo [16], is used to construct approximate solutions to (IVP; τ, z). The advantage of these assumptions is illustrated by an application of the results to the initial value problems for nonlinear wave equations.

Let $J \subset [a,b)$ be a subinterval of the form $[\tau, c]$ or $[\tau, c)$. An X-valued continuous function $u : J \to X$ is called a *solution to* (IVP; τ, z) *on* J, if $u(\tau) = z$, $(t, u(t)) \in \Omega$ for $t \in J, u$ is differentiable on J and $u'(t) = A(t, u(t))$ for $t \in J$. A solution to (IVP; τ, z) on $[\tau, b)$ is called a *global solution*.

Let $d(x, D)$ denote the distance from $x \in X$ to $D \subset X$, i.e., $d(x, D) = \inf\{\|x - y\|; y \in D\}$. We consider the following conditions.

($\Omega 1$) A is continuous on Ω.

($\Omega 2$) If $(t_n, x_n) \in \Omega$, $t_n \uparrow t \in [a, b)$ in $\mathbf{R}$ and $x_n \to x$ in X as $n \to \infty$, then $(t, x) \in \Omega$.

($\Omega 3$) $\liminf_{h \downarrow 0} h^{-1} d(x + hA(t, x), \Omega(t + h)) = 0$ for $(t, x) \in \Omega$.

($\Omega 4$) There exists a functional $V : [a, b) \times X \times X \to [0, \infty)$ satisfying the following properties ($V1$)–($V4$) and a continuous function $\omega : [a, b) \to [0, \infty)$ such that

$$D_+ V(t, x, y)(A(t, x), A(t, y)) \leq \omega(t) V(t, x, y)$$

for $x, y \in \Omega(t)$ and $t \in [a, b)$. Here, for $(t, x, y) \in [a, b) \times X \times X$ and $(\xi, \eta) \in X \times X$,

$$D_+ V(t, x, y)(\xi, \eta) = \liminf_{h \downarrow 0} \frac{1}{h} (V(t + h, x + h\xi, y + h\eta) - V(t, x, y)),$$

where the values ∞ and $-\infty$ are not excluded.

($V1$) There exists a number $L > 0$ such that $|V(t, x, y) - V(t, \hat{x}, \hat{y})| \leq L(\|x - \hat{x}\| + \|y - \hat{y}\|)$ for $(x, y), (\hat{x}, \hat{y}) \in X \times X$ and $t \in [a, b)$.

($V2$) $V(t, x, x) = 0$ for $t \in [a, b)$ and $x \in \Omega(t)$.

($V3$) If $\{t_n\}$ is a sequence in $[a, b)$ and $\{(x_n, y_n)\}$ is a sequence in $X \times X$ such that $(x_n, y_n) \in \Omega(t_n) \times \Omega(t_n)$ for $n \geq 1$, $t_n \to t \in [a, b)$ and $(x_n, y_n) \to (x, y) \in \Omega(t) \times \Omega(t)$ as $n \to \infty$, then $V(t, x, y) \leq \liminf_{n \to \infty} V(t_n, x_n, y_n)$.

($V4$) If $\{t_n\}$ is a sequence in $[a, b)$ and $\{(x_n, y_n)\}$ is a sequence in $X \times X$ such that $(x_n, y_n) \in \Omega(t_n) \times \Omega(t_n)$ for $n \geq 1$, $t_n \to t \in [a, b)$ and $V(t_n, x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.

($\Omega 5$) For any $(\tau, z) \in \Omega$, there exists a connected component C of Ω such that $(\tau, z) \in C$ and $C(t) \neq \emptyset$ for $t \in (\tau, b)$.

Remark 1. *Condition* ($V1$) *with* ($V2$) *implies the following:*

$$|V(t, x, y)| \leq L\|x - y\| \qquad \text{for } (x, y) \in \Omega(t) \times \Omega(t) \text{ and } t \in [a, b).$$

The following are our main theorems.

Theorem 1. *Let A be a mapping from Ω into X such that conditions* ($\Omega 1$)–($\Omega 4$) *are satisfied. Let C be a connected component of Ω and set $d = \sup\{t \in [a, b); C(t) \neq \emptyset\}$. Then the following assertions hold true:*

(i) *For $(\tau, z) \in C$, (IVP; τ, z) has a unique solution $u(t; \tau, z)$ on $[\tau, d)$ and the interval $[\tau, d)$ is the maximal interval of existence of solution.*

(ii) *For* $z, \hat{z} \in C(\tau)$ *and* $t \in [\tau, d)$,

$$V(t, u(t;\tau,z), u(t;\tau,\hat{z})) \le \exp\left(\int_\tau^t \omega(\theta)d\theta\right) V(\tau, z, \hat{z}).$$

Theorem 2. *Let A be a mapping from Ω into X such that $(\Omega1)$ and $(\Omega2)$ are satisfied. Then there exists a Lipschitz evolution operator $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$ on Ω such that $u(t) := U(t,\tau)z$ is a global solution to* (IVP; τ, z) *for any $(\tau, z) \in \Omega$ if and only if conditions $(\Omega3)$–$(\Omega5)$ are satisfied, where condition $(V4)$ is replaced by the following condition:*

$(V4)'$ *For any $t \in [a, b)$ and $x, y \in \Omega(t)$, $\|x - y\| \le V(t, x, y)$.*

Theorem 1 consists of the uniqueness and local existence of solutions to initial value problems (IVP; τ, z) and the global existence theorem as well as the continuous dependence of solutions on initial data. They are discussed in Sections 2 and 3 respectively. The proof of Theorem 2 is given in Section 4. An application of our results to the initial value problem for quasi-linear wave equations is given in Section 5.

2. Uniqueness and local existence of solutions

In this section, we construct the solutions to the initial value problem (IVP; τ, z). We assume that conditions $(\Omega1)$–$(\Omega4)$. The following proposition ensures the uniqueness of solutions.

Proposition 1. *Let $[\tau, c) \subset [a, b)$ and $z_i \in \Omega(\tau)$ for $i = 1, 2$. Let u_i be solutions to* (IVP; τ, z_i) *on $[\tau, c)$, for $i = 1, 2$, respectively. Then*

$$V(t, u_1(t), u_2(t)) \le \exp\left(\int_\tau^t \omega(s)ds\right) V(\tau, z_1, z_2)$$

for $t \in [\tau, c)$. In particular, if $z_1 = z_2$, then $u_1(t) = u_2(t)$ for $t \in [\tau, c)$.

Proof. Set $w(t) = V(t, u_1(t), u_2(t))$ for $t \in [\tau, c)$. From $(V3)$ we see that w is lower semi-continuous on $[\tau, c)$. Let $t \in [\tau, c)$ and $h \in (0, c - t)$. From $(V1)$ it follows that

$$\begin{aligned}
&(w(t+h) - w(t))/h - (V(t+h, u_1(t) + hA(t, u_1(t)), u_2(t) \\
&\qquad\qquad + hA(t, u_2(t))) - V(t, u_1(t), u_2(t)))/h \\
&\le |V(t+h, u_1(t+h), u_2(t+h)) \\
&\quad - V(t+h, u_1(t) + hA(t, u_1(t)), u_2(t) + hA(t, u_2(t)))|/h
\end{aligned}$$

$$\leq L(\|u_1(t+h) - u_1(t) - hA(t, u_1(t))\|/h$$
$$+ \|u_2(t+h) - u_2(t) - hA(t, u_2(t))\|/h).$$

Taking the inferior limit as $h \downarrow 0$ yields

$$\liminf_{h\downarrow 0}(w(t+h) - w(t))/h \leq D_+V(t, u_1(t), u_2(t))(A(t, u_1(t)), A(t, u_2(t))).$$

From (Ω4) we have $D_+w(t) \leq \omega(t)w(t)$, where $D_+w(t)$ denotes the lower right derivative of $w(t)$. Therefore, we see that the function

$$t \to \exp\left(-\int_\tau^t \omega(s)ds\right)w(t)$$

is lower semicontinuous on $[\tau, c)$ and $D_+(\exp(-\int_\tau^t \omega(s)ds)w(t)) \leq 0$ for $t \in [\tau, c)$. By [3, Lemma 6.3], we have $w(t) \leq \exp(\int_\tau^t \omega(s)ds)w(\tau)$ for $t \in [\tau, c)$. Refer to [9] or [21] for the same kind of differential inequalities. □

For each $(t, x) \in \mathbf{R} \times X$ and $r > 0$, we define $S_r(t, x) = \{(s, y) \in \mathbf{R} \times X;\ |s - t| < r, \|y - x\| < r\}$. We need the following lemmas which are proved in [7] without using condition (Ω4).

Lemma 1 ([7, Lemma 1]). *Let $(t, x) \in \Omega$ and $\eta > 0$. Let $r > 0$ be a number such that $\|A(s, y) - A(t, x)\| \leq \eta$ for $(s, y) \in \Omega \cap S_r(t, x)$. Let $M > 0$ be a number such that $\|A(s, y)\| \leq M$ for $(s, y) \in \Omega \cap S_r(t, x)$. Set $h_0 = \min\{r, r/M, b - t\}$. Then*

$$d(x + hA(t, x), \Omega(t+h)) \leq h\eta \qquad \text{for } h \in (0, h_0).$$

Lemma 2 ([7, Lemma 2]). *Let $(t, x) \in \Omega$ and $\varepsilon \in (0, 1)$. Let $r > 0$ and $M > 0$ be numbers such that $t + r < b$ and such that $\|A(s, y) - A(t, x)\| \leq \varepsilon/3$ and $\|A(s, y)\| \leq M$ for $(s, y) \in \Omega \cap S_r(t, x)$. Let $h \in (0, r/(M+1)]$. Let $\{s_k\}_{k=0}^n$ be a partition of $[t, t+h] : t = s_0 < s_1 < \cdots < s_n = t + h$. Then there exists a sequence $\{y_k\}_{k=0}^n$ of elements in X such that*

(i) *$y_0 = x$ and $(s_k, y_k) \in \Omega$ for $0 \leq k \leq n$;*
(ii) *$\|y_k - x\| \leq (M + \varepsilon)(s_k - t)$ for $0 \leq k \leq n$;*
(iii) *$\|y_{k-1} + (s_k - s_{k-1})A(s_{k-1}, y_{k-1}) - y_k\| \leq \varepsilon(s_k - s_{k-1})$ for $1 \leq k \leq n$.*

We also need the following lemma.

Lemma 3. *Let $(t, x) \in \Omega$ and $\varepsilon \in (0, 1)$. Let $r > 0$ and $M > 0$ be numbers such that $t + r < b$ and $\|A(s, y)\| \leq M$ for $(s, y) \in \Omega \cap S_r(t, x)$. Let $\sigma \in (0, r/(M+1)]$. Then the following assertions hold true:*

(i) *If a sequence* $\{(s_i, y_i)\}_{i=0}^n$ *in* Ω *satisfies*

$$t = s_0 < s_1 < \cdots < s_n \le t + \sigma, \tag{2.1}$$

$$\|y_{i-1} + (s_i - s_{i-1})A(s_{i-1}, y_{i-1}) - y_i\| \le \varepsilon(s_i - s_{i-1})$$
$$\text{for } 1 \le i \le n, \text{ where } y_0 = x, \tag{2.2}$$

then

$$\|y_i - y_j\| \le (M + \varepsilon)(s_i - s_j) \qquad \text{for } 0 \le j \le i \le n,$$
$$\|A(s_i, y_i)\| \le M \qquad \text{for } 0 \le i \le n.$$

Moreover, if $\eta > 0$ *and* $\|A(s, y) - A(t, x)\| \le \eta$ *for* $(s, y) \in \Omega \cap S_r(t, x)$, *then*

$$\|x + (s_n - t)A(t, x) - y_n\| \le (\varepsilon + \eta)(s_n - t). \tag{2.3}$$

(ii) *Let* $\eta > 0$ *and* $\|A(s, y) - A(t, x)\| \le \eta$ *for* $(s, y) \in \Omega \cap S_r(t, x)$. *If a sequence* $\{(s_i, y_i)\}_{i=0}^\infty$ *in* Ω *satisfies*

$$t = s_0 < s_1 < \cdots < s_i < \cdots < t + \sigma \qquad \text{and} \qquad \lim_{i\to\infty} s_i = t + \sigma, \tag{2.4}$$

$$\|y_{i-1} + (s_i - s_{i-1})A(s_{i-1}, y_{i-1}) - y_i\| \le \varepsilon(s_i - s_{i-1})$$
$$\text{for } i \ge 1, \text{ where } y_0 = x, \tag{2.5}$$

then $\hat{y} = \lim_{i\to\infty} y_i$ *exists in* X, $\hat{y} \in \Omega(t + \sigma)$ *and*

$$\|x + \sigma A(t, x) - \hat{y}\| \le (\varepsilon + \eta)\sigma. \tag{2.6}$$

PROOF. To prove (i), let $\{(s_i, y_i)\}_{i=0}^n$ be a sequence in Ω satisfying (2.1) and (2.2). We first show inductively that $(s_i, y_i) \in S_r(t, x)$ for $0 \le i \le n$. It is obvious that $(s_0, y_0) \in S_r(t, x)$. Let k be a nonnegative integer such that $k < n$ and assume that $(s_i, y_i) \in S_r(t, x)$ for $0 \le i \le k$. From (2.2) we obtain

$$\|y_{i-1} - y_i\| \le (s_i - s_{i-1})\|A(s_{i-1}, y_{i-1})\| + \varepsilon(s_i - s_{i-1})$$

for $1 \le i \le n$. Since $\|A(s_i, x_i)\| \le M$ for $0 \le i \le k$ by assumption, we have

$$\|y_i - y_{i-1}\| \le (M + \varepsilon)(s_i - s_{i-1})$$

for $1 \le i \le k + 1$. Summing up this inequality from $i = 1$ to $i = k + 1$, we find that

$$\|y_{k+1} - x\| \le (M + \varepsilon)(s_{k+1} - t) < (M + 1)\sigma \le r.$$

It is obvious that $s_{k+1} - t \le \sigma < \sigma(M + 1) \le r$. These mean that $(s_{k+1}, y_{k+1}) \in S_r(t, x)$. Thus, we inductively prove that $(s_i, y_i) \in S_r(t, x)$ for $0 \le i \le n$.

Since $(s_k, y_k) \in S_r(t,x)$ for $0 \le k \le n$, we have $\|A(s_k, y_k)\| \le M$ for $0 \le k \le n$ and $\|y_k - y_{k-1}\| \le (M+\varepsilon)(s_k - s_{k-1})$ for $1 \le k \le n$. Therefore, we find that

$$\|y_i - y_j\| \le (M+\varepsilon)(s_i - s_j)$$

for $0 \le j \le i \le n$. To prove (2.3), let $\eta > 0$ and assume that $\|A(s,y) - A(t,x)\| \le \eta$ for $(s,y) \in \Omega \cap S_r(t,x)$. Since $\{(s_i, y_i); 0 \le i \le n\} \subset \Omega \cap S_r(t,x)$, we have $\|A(s_i, y_i) - A(t,x)\| \le \eta$ for $0 \le i \le n$. From (2.2) we see that

$$\begin{aligned}
&\|y_{i-1} + (s_i - s_{i-1})A(t,x) - y_i\| \\
&\quad \le \|y_{i-1} + (s_i - s_{i-1})A(s_{i-1}, y_{i-1}) - y_i\| \\
&\qquad + \|(s_i - s_{i-1})(A(t,x) - A(s_{i-1}, y_{i-1}))\| \\
&\quad \le \varepsilon(s_i - s_{i-1}) + \eta(s_i - s_{i-1}) = (\varepsilon + \eta)(s_i - s_{i-1})
\end{aligned}$$

for $1 \le i \le n$. Hence

$$\begin{aligned}
\|x + (s_n - t)A(t,x) - y_n\| &\le \sum_{i=1}^{n} \|y_{i-1} + (s_i - s_{i-1})A(t,x) - y_i\| \\
&\le (\varepsilon + \eta)(s_n - t).
\end{aligned}$$

To prove (ii), let $\{(s_i, y_i)\}_{i=0}^{\infty}$ be a sequence in Ω satisfying (2.4) and (2.5). From (i) we obtain $\|y_i - y_j\| \le (M+\varepsilon)(s_i - s_j)$ for $0 \le j \le i$. This implies that $\hat{y} = \lim_{i\to\infty} y_i$ exists in X and is in $\Omega(t+\sigma)$ by (Ω2). By (i) again, we note that the inequality (2.3) holds for $n \ge 0$. Passing to the limit in (2.3) as $n \to \infty$, we obtain

$$\begin{aligned}
\|x + \sigma A(t,x) - \hat{y}\| &= \lim_{n\to\infty} \|x + (s_n - t)A(t,x) - y_n\| \\
&\le \lim_{n\to\infty} (\varepsilon + \eta)(s_n - t) = (\varepsilon + \eta)\sigma,
\end{aligned}$$

namely, the desired inequality (2.6) is proved. □

The local existence of approximation solutions to (IVP; τ, z) is given by the following proposition, which is essentially shown in [7] and [4]. We give the proof for completeness.

Proposition 2. *Let $(t,x) \in \Omega$ and $\varepsilon \in (0,1)$. Let $r > 0$ and $M > 0$ be numbers such that $t + r < b$ and $\|A(s,y)\| \le M$ for $(s,y) \in \Omega \cap S_r(t,x)$. Let $\sigma \in (0, r/(M+1)]$. Then there exists a sequence $\{(s_i, y_i)\}_{i=0}^{\infty}$ in Ω such that*

(i) *$t = s_0 < s_1 < \cdots < s_i < \cdots < t + \sigma$ and $\lim_{i\to\infty} s_i = t + \sigma$;*

(ii) *$s_i - s_{i-1} \le \varepsilon$ for $i \ge 1$;*

(iii) $\|y_{i-1} + (s_i - s_{i-1})A(s_{i-1}, y_{i-1}) - y_i\| \le \varepsilon(s_i - s_{i-1})/2$ *for* $i \ge 1$, *where* $y_0 = x$;

(iv) *if* $(s, y) \in \Omega \cap S_{(M+1)(s_i - s_{i-1})}(s_{i-1}, y_{i-1})$, *then*

$$\|A(s, y) - A(s_{i-1}, y_{i-1})\| \le \varepsilon/4 \qquad \text{for } i \ge 1.$$

Proof. Set $(s_0, y_0) = (t, x)$. Let k be a positive integer and assume that there exists a sequence $\{(s_i, y_i)\}_{i=0}^{k-1}$ in Ω which satisfies the first half of (i) and (ii)–(iv) for $1 \le i \le k-1$. We consider a nonnegative number $\hat{h}_k$ defined by the supremum of $h \in [0, \varepsilon]$ such that $h < t + \sigma - s_{k-1}$ and

$$\|A(s, y) - A(s_{k-1}, y_{k-1})\| \le \varepsilon/4 \qquad \text{for } (s, y) \in \Omega \cap S_{h(M+1)}(s_{k-1}, y_{k-1}).$$

By the continuity of A, we have $\hat{h}_k > 0$. Thus there exists a number $h_k \in (0, \varepsilon]$ such that $\hat{h}_k/2 < h_k < t + \sigma - s_{k-1}$ and

$$\|A(s, y) - A(s_{k-1}, y_{k-1})\| \le \varepsilon/4 \qquad \text{for } (s, y) \in \Omega \cap S_{r_k}(s_{k-1}, y_{k-1}), \tag{2.7}$$

where $r_k = h_k(M+1)$. Set $s_k = s_{k-1} + h_k$. Then $s_{k-1} < s_k < t + \sigma$ and conditions (ii) and (iv) with $i = k$ are satisfied. By Lemma 3, $\|A(s_i, y_i)\| \le M$ for $0 \le i \le k-1$. The inequality (2.7) implies that $\|A(s, y)\| \le M + \varepsilon/4$ for $(s, y) \in \Omega \cap S_{r_k}(s_{k-1}, y_{k-1})$. Hence, Lemma 1, with (t, x), r, M and η replaced by (s_{k-1}, y_{k-1}), r_k, $M + \varepsilon/4$ and $\varepsilon/4$ respectively, implies that

$$d(y_{k-1} + h_k A(s_{k-1}, y_{k-1}), \Omega(s_k)) \le \varepsilon h_k/4.$$

Thus there exists an element $y_k \in \Omega(s_k)$ satisfying (iii) with $i = k$.

We shall show that $\lim_{i\to\infty} s_i = t + \sigma$. Assume to the contrary that $\hat{s} = \lim_{i\to\infty} s_i < t + \sigma$. By Lemma 3 (i) we obtain $\|y_i - y_j\| \le (M + \varepsilon/2)(s_i - s_j)$ for $0 \le j \le i$. Hence, $\lim_{i\to\infty} y_i$ exists in X, and we denote its limit by $\hat{y}$. Since $(\hat{s}, \hat{y}) = \lim_{i\to\infty}(s_i, y_i)$ in $\mathbf{R} \times X$ and $(s_i, y_i) \in \Omega$ for $i \ge 1$, we have $(\hat{s}, \hat{y}) \in \Omega$ by (Ω2). The continuity of A enables us to choose $\eta \in (0, \varepsilon]$ such that

$$\eta \le t + \sigma - \hat{s} \quad \text{and} \quad \|A(s, y) - A(\hat{s}, \hat{y})\| \le \varepsilon/8 \qquad \text{for } (s, y) \in \Omega \cap S_{\hat{r}}(\hat{s}, \hat{y}),$$

where $\hat{r} = 2(M+1)\eta$. Choose an integer $i_0 \ge 1$ so that $\hat{s} - s_{i-1} \le \eta$ and $\|\hat{y} - y_{i-1}\| \le (M+1)\eta$ for $i \ge i_0$. Then, for $i \ge i_0$ and $(s, y) \in S_{(M+1)\eta}(s_{i-1}, y_{i-1})$, we have

$$|s - \hat{s}| \le |s - s_{i-1}| + |s_{i-1} - \hat{s}| < (M+1)\eta + \eta \le 2(M+1)\eta,$$

$$\|y - \hat{y}\| \le \|y - y_{i-1}\| + \|y_{i-1} - \hat{y}\| < 2(M+1)\eta.$$

Hence $S_{(M+1)\eta}(s_{i-1}, y_{i-1}) \subset S_{\hat{r}}(\hat{s}, \hat{y})$ for $i \ge i_0$. By the choice of η, we see that if $i \ge i_0$, then

$$\|A(s, y) - A(s_{i-1}, y_{i-1})\| \le \|A(s, y) - A(\hat{s}, \hat{y})\| + \|A(\hat{s}, \hat{y}) - A(s_{i-1}, y_{i-1})\|$$
$$\le \varepsilon/8 + \varepsilon/8 = \varepsilon/4$$

for $(s, y) \in \Omega \cap S_{(M+1)\eta}(s_{i-1}, y_{i-1})$. Since $\eta < t + \sigma - s_{i-1}$ for $i \ge 1$, the definition of $\hat{h}_i$ implies that $\eta \le \hat{h}_i < 2h_i = 2(s_i - s_{i-1})$ for $i \ge i_0$ and the right-hand side tends to zero as $i \to \infty$. This contradicts the fact that η is positive. □

In what follows, we write $\bar{\omega}([\hat{a}, \hat{b}]) = \sup_{s \in [\hat{a}, \hat{b}]} \omega(s)$ for $[\hat{a}, \hat{b}] \subset [a, b)$. To prove the convergence of the approximate solutions, we need the following Propositions, which are the refinements of the results in [11], [10], [6] and [8].

Proposition 3. *Let $t \in [a, b)$, $(x, \hat{x}) \in \Omega(t) \times \Omega(t)$ and $\eta, \hat{\eta} \in (0, 1)$. Let $r > 0$ and $M > 0$ be numbers such that $t + r < b$,*

$$\|A(s, z)\| \le M \quad \text{and} \quad \|A(s, z) - A(t, x)\| \le \eta/4 \qquad \text{for } (s, z) \in \Omega \cap S_r(t, x),$$
$$\|A(s, \hat{z})\| \le M \quad \text{and} \quad \|A(s, \hat{z}) - A(t, \hat{x})\| \le \hat{\eta}/4 \qquad \text{for } (s, \hat{z}) \in \Omega \cap S_r(t, \hat{x}).$$

Let $\sigma \in (0, r/(M + 1)]$. Then there exists a pair $(y, \hat{y}) \in \Omega(t + \sigma) \times \Omega(t + \sigma)$ such that

$$\|x + \sigma A(t, x) - y\| \le \eta\sigma, \tag{2.8}$$
$$\|\hat{x} + \sigma A(t, \hat{x}) - \hat{y}\| \le \hat{\eta}\sigma, \tag{2.9}$$
$$V(t + \sigma, y, \hat{y}) \le \exp(\sigma\bar{\omega}([t, t + \sigma]))(V(t, x, \hat{x}) + L(\eta + \hat{\eta})\sigma). \tag{2.10}$$

Proof. We shall show that there exist two sequences $\{(s_j, z_j)\}_{j=0}^{\infty}$ and $\{(s_j, \hat{z}_j)\}_{j=0}^{\infty}$ in Ω such that

$$t = s_0 < s_1 < \cdots < s_j < \cdots < t + \sigma \qquad \text{and} \qquad \lim_{j \to \infty} s_j = t + \sigma, \tag{2.11}$$

$$\|z_{j-1} + (s_j - s_{j-1})A(s_{j-1}, z_{j-1}) - z_j\| \le 3\eta(s_j - s_{j-1})/4$$
$$\text{for } j \ge 1, \text{ where } z_0 = x, \tag{2.12}$$

$$\|\hat{z}_{j-1} + (s_j - s_{j-1})A(s_{j-1}, \hat{z}_{j-1}) - \hat{z}_j\| \le 3\hat{\eta}(s_j - s_{j-1})/4$$
$$\text{for } j \ge 1, \text{ where } \hat{z}_0 = \hat{x}, \tag{2.13}$$

$$(V(s_j, z_j, \hat{z}_j) - V(s_{j-1}, z_{j-1}, \hat{z}_{j-1}))/(s_j - s_{j-1})$$
$$\le \omega(s_{j-1})V(s_{j-1}, z_{j-1}, \hat{z}_{j-1}) + L(\eta + \hat{\eta}) \qquad \text{for } j \ge 1. \tag{2.14}$$

Set $(s_0, z_0, \hat{z}_0) = (t, x, \hat{x})$ and assume that sequences $\{(s_j, z_j)\}_{j=0}^{i-1}$ and $\{(s_j, \hat{z}_j)\}_{j=0}^{i-1}$ in Ω with $i \ge 1$ satisfy the first half of (2.11) and (2.12)–(2.14) for $1 \le j \le i - 1$. Then we need to show that there exist $s_i \in \mathbf{R}$, $z_i \in \Omega(s_i)$ and $\hat{z}_i \in \Omega(s_i)$

such that $s_{i-1} < s_i < t + \sigma$ and (2.12)–(2.14) with $j = i$ are satisfied. Let $\hat{h}_i$ denote the supremum of all $h \geq 0$ such that $h < t + \sigma - s_{i-1}$ and

$$V(s_{i-1} + h, z_{i-1} + hA(s_{i-1}, z_{i-1}), \hat{z}_{i-1} + hA(s_{i-1}, \hat{z}_{i-1})) - V(s_{i-1}, z_{i-1}, \hat{z}_{i-1})$$
$$\leq h(\omega(s_{i-1})V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) + (\eta + \hat{\eta})L/4).$$

Since $\hat{h}_i > 0$ by (Ω4), there exists a number $h_i > 0$ such that $\hat{h}_i/2 < h_i < t + \sigma - s_{i-1}$ and

$$V(s_{i-1} + h, z_{i-1} + hA(s_{i-1}, z_{i-1}), \hat{z}_{i-1} + hA(s_{i-1}, \hat{z}_{i-1})) - V(s_{i-1}, z_{i-1}, \hat{z}_{i-1})$$
$$\leq h(\omega(s_{i-1})V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) + (\eta + \hat{\eta})L/4). \tag{2.15}$$

Set $s_i = s_{i-1} + h_i$. It is obvious that $s_{i-1} < s_i < t + \sigma$. To prove that $S_{(M+1)h_i}(s_{i-1}, z_{i-1}) \subset S_r(t, x)$, we note by Lemma 3 (i) with $\varepsilon = 3\eta/4$ that

$$\|z_{i-1} - x\| \leq (M + 3\eta/4)(s_{i-1} - t) < (M + 1)(s_{i-1} - t).$$

If $(s, z) \in S_{(M+1)h_i}(s_{i-1}, z_{i-1})$, then

$$|s - t| \leq |s - s_{i-1}| + |s_{i-1} - t| < (M + 1)(h_i + s_{i-1} - t)$$
$$= (M + 1)(s_i - t) \leq (M + 1)\sigma \leq r$$

and

$$\|z - x\| \leq \|z - z_{i-1}\| + \|z_{i-1} - x\| < (M + 1)(h_i + s_{i-1} - t) \leq r.$$

This means that $S_{(M+1)h_i}(s_{i-1}, z_{i-1}) \subset S_r(t, x)$. By assumption, we have

$$\|A(s, z)\| \leq M \qquad \text{and} \qquad \|A(s, z) - A(t, x)\| \leq \eta/4 \tag{2.16}$$

for $(s, z) \in \Omega \cap S_{(M+1)h_i}(s_{i-1}, z_{i-1})$. From the second inequality of (2.16), we see that if $(s, z) \in \Omega \cap S_{(M+1)h_i}(s_{i-1}, z_{i-1})$, then

$$\|A(s, z) - A(s_{i-1}, z_{i-1})\| \leq \|A(s, z) - A(t, x)\| + \|A(s_{i-1}, z_{i-1}) - A(t, x)\|$$
$$\leq \eta/4 + \eta/4 = \eta/2.$$

Hence, by Lemma 1 with $r = (M + 1)h_i$, $(t, x) = (s_{i-1}, z_{i-1})$ and $h = h_i$, we find that

$$d(z_{i-1} + h_iA(s_{i-1}, z_{i-1}), \Omega(s_i)) \leq h_i\eta/2 = \eta(s_i - s_{i-1})/2.$$

This implies that there exists $z_i \in \Omega(s_i)$ such that (2.12) holds true for $j = i$. Similarly, we can show that there exists $\hat{z}_i \in \Omega(s_i)$ satisfying (2.13) with $j = i$.

By (V1) we obtain (2.14) with $j = i$ by the inequality (2.15) combined with (2.12) and (2.13) with $j = i$. Indeed, we have

$$
\begin{aligned}
&(V(s_i, z_i, \hat{z}_i) - V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}))/h_i \\
&\quad = (V(s_i, z_i, \hat{z}_i) - V(s_i, z_{i-1} + h_i A(s_{i-1}, z_{i-1}), \hat{z}_{i-1} + h_i A(s_{i-1}, \hat{z}_{i-1})))/h_i \\
&\qquad + (V(s_i, z_{i-1} + h_i A(s_{i-1}, z_{i-1}), \hat{z}_{i-1} + h_i A(s_{i-1}, \hat{z}_{i-1})) \\
&\qquad\quad - V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}))/h_i \\
&\quad \le L(\|z_i - (z_{i-1} + h_i A(s_{i-1}, z_{i-1}))\| + \|\hat{z}_i - (\hat{z}_{i-1} + h_i A(s_{i-1}, \hat{z}_{i-1}))\|)/h_i \\
&\qquad + \omega(s_{i-1}) V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) + (\eta + \hat{\eta})L/4 \\
&\quad \le 3(\eta + \hat{\eta})L/4 + \omega(s_{i-1}) V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) + (\eta + \hat{\eta})L/4 \\
&\quad \le \omega(s_{i-1}) V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) + L(\eta + \hat{\eta}).
\end{aligned}
$$

It remains to prove the second half of (2.11). Assume to the contrary that $s_\infty = \lim_{j\to\infty} s_j < t + \sigma$. Lemma 3 (i) asserts that $\{z_j\}$ and $\{\hat{z}_j\}$ are Cauchy sequences in X, since

$$\limsup_{i,j\to\infty} \|z_i - z_j\| \le \limsup_{i,j\to\infty} (M + 3\eta/4)(s_i - s_j) = 0,$$

$$\limsup_{i,j\to\infty} \|\hat{z}_i - \hat{z}_j\| \le \limsup_{i,j\to\infty} (M + 3\hat{\eta}/4)(s_i - s_j) = 0.$$

This implies that $z_\infty = \lim_{j\to\infty} z_j$ and $\hat{z}_\infty = \lim_{j\to\infty} \hat{z}_j$ exist in X and are in $\Omega(s_\infty)$ by $(\Omega 2)$. By $(\Omega 4)$, we choose a number $h > 0$ so that $h < t + \sigma - s_\infty$ and

$$
\begin{aligned}
&\{V(s_\infty + h, z_\infty + hA(s_\infty, z_\infty), \hat{z}_\infty + hA(s_\infty, \hat{z}_\infty)) - V(s_\infty, z_\infty, \hat{z}_\infty)\}/h \\
&\qquad \le \omega(s_\infty) V(s_\infty, z_\infty, \hat{z}_\infty) + (\eta + \hat{\eta})L/8. \qquad (2.17)
\end{aligned}
$$

Let $r_j = s_\infty + h - s_{j-1}$ for $j \ge 1$. Then we have $r_j < t + \sigma - s_{j-1}$ for $j \ge 1$ and $r_j \to h$ as $j \to \infty$. Since $\hat{h}_j < 2h_j = 2(s_j - s_{j-1}) \to 0$ as $j \to \infty$, there exists an integer $j_0 \ge 1$ such that $\hat{h}_j < r_j$ for $j \ge j_0$. By the definition of $\hat{h}_j$, we have

$$
\begin{aligned}
&\{V(s_{j-1} + r_j, z_{j-1} + r_j A(s_{j-1}, z_{j-1}), \hat{z}_{j-1} + r_j A(s_{j-1}, \hat{z}_{j-1})) - V(s_{j-1}, z_{j-1}, \hat{z}_{j-1})\}/r_j \\
&\qquad > \omega(s_{j-1}) V(s_{j-1}, z_{j-1}, \hat{z}_{j-1}) + (\eta + \hat{\eta})L/4
\end{aligned}
$$

for $j \ge j_0$. Since $s_{j-1} \to s_\infty$, $z_{j-1} \to z_\infty$, $\hat{z}_{j-1} \to \hat{z}_\infty$ and $r_j \to h$ as $j \to \infty$ and $s_{j-1} + r_j = s_\infty + h$ for $j \ge 1$, from $(V1)$ and $(V3)$ we obtain

$$
\begin{aligned}
&\{V(s_\infty + h, z_\infty + hA(s_\infty, z_\infty), \hat{z}_\infty + hA(s_\infty, \hat{z}_\infty)) - V(s_\infty, z_\infty, \hat{z}_\infty)\}/h \\
&\qquad \ge \omega(s_\infty) V(s_\infty, z_\infty, \hat{z}_\infty) + (\eta + \hat{\eta})L/4,
\end{aligned}
$$

which contradicts to (2.17).

We now turn to the proof of the existence of pair $(y, \hat{y}) \in \Omega(t) \times \Omega(t)$ satisfying (2.8)–(2.10). We apply Lemma 3 (ii) to show that $y = \lim_{j\to\infty} z_j$ and $\hat{y} = \lim_{j\to\infty} \hat{z}_j$ exist in X and are in $\Omega(t+\sigma)$ and that they satisfy (2.8) and (2.9), that is,

$$\|x + \sigma A(t,x) - y\| \le (3\eta/4 + \eta/4)\sigma \le \eta\sigma,$$

$$\|\hat{x} + \sigma A(t,\hat{x}) - \hat{y}\| \le (3\hat{\eta}/4 + \hat{\eta}/4)\sigma \le \hat{\eta}\sigma.$$

We note here that $1 + t \le e^t$ for $t \ge 0$. We deduce from (2.14) that

$$V(s_j, z_j, \hat{z}_j) \le \exp(h_j\varpi([t, t+\sigma]))(V(s_{j-1}, z_{j-1}, \hat{z}_{j-1}) + h_j L(\eta + \hat{\eta}))$$

for $j \ge 1$. Hence, we inductively show that

$$V(s_j, z_j, \hat{z}_j) \le \exp((s_j - t)\varpi([t, t+\sigma]))(V(t, x, \hat{x}) + L(\eta + \hat{\eta})(s_j - t))$$

for $j \ge 0$. Thus we obtain (2.10) by letting $j \to \infty$. □

Proposition 4. *Let $(\tau, z) \in \Omega$ and $\lambda, \mu \in (0, 1/2)$. Let $R > 0$ and $M > 0$ be numbers such that $\tau + R < b$ and $\|A(s, y)\| \le M$ for $(s, y) \in \Omega \cap S_R(\tau, z)$. Let $\sigma \in (0, R/(M+1)]$. For each $\varepsilon \in \{\lambda, \mu\}$, let $\{(t_i^\varepsilon, x_i^\varepsilon)\}_{i=0}^\infty$ be a sequence in Ω satisfying the following conditions:*

(i) *$\tau = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_i^\varepsilon < \cdots < \tau + \sigma$ and $\lim_{i\to\infty} t_i^\varepsilon = \tau + \sigma$;*

(ii) *$t_i^\varepsilon - t_{i-1}^\varepsilon \le \varepsilon$ for $i \ge 1$;*

(iii) *$\|x_{i-1}^\varepsilon + (t_i^\varepsilon - t_{i-1}^\varepsilon)A(t_{i-1}^\varepsilon, x_{i-1}^\varepsilon) - x_i^\varepsilon\| \le \varepsilon(t_i^\varepsilon - t_{i-1}^\varepsilon)/2$ for $i \ge 1$, where $x_0^\varepsilon = z$;*

(iv) *if $(s, y) \in \Omega \cap S_{(M+1)(t_i^\varepsilon - t_{i-1}^\varepsilon)}(t_{i-1}^\varepsilon, x_{i-1}^\varepsilon)$, then*

$$\|A(s, y) - A(t_{i-1}^\varepsilon, x_{i-1}^\varepsilon)\| \le \varepsilon/4 \qquad \text{for } i \ge 1.$$

Let $\{s_k\}_{k=0}^\infty$ be a sequence such that $s_k < s_{k+1}$ for $k \ge 0$ and

$$\{s_k; k = 0, 1, 2, \ldots\} = \{t_i^\lambda; i = 0, 1, 2, \ldots\} \cup \{t_j^\mu; j = 0, 1, 2, \ldots\}.$$

Then there exists a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^\infty$ in $X \times X$ such that $(z_k^\lambda, z_k^\mu) \in \Omega(s_k) \times \Omega(s_k)$ for each $k \ge 0$ and the following three properties are satisfied:

(a) *if $s_k = t_i^\lambda$, then $z_k^\lambda = x_i^\lambda$; if $s_k = t_j^\mu$, then $z_k^\mu = x_j^\mu$;*

(b) *for each $\varepsilon = \lambda, \mu$, we have*

$$\sum_{j=q}^{k} \|z_{j-1}^\varepsilon + (s_j - s_{j-1})A(s_{j-1}, z_{j-1}^\varepsilon) - z_j^\varepsilon\| \le 2\varepsilon(s_k - s_{q-1}) + 3\varepsilon \sum_{t_i^\varepsilon \in \{s_q, \ldots, s_k\}} (t_i^\varepsilon - t_{i-1}^\varepsilon)$$

for $1 \le q \le k$ and $k \ge 1$;

(c) *for* $k \geq 0$,

$$V(s_k, z_k^\lambda, z_k^\mu) \leq \exp((s_k - \tau)\bar{\omega}([\tau, s_k]))\{2L(\lambda + \mu)(s_k - \tau) + \eta_k(\lambda, \mu)\},$$

where

$$\eta_k(\lambda, \mu) = 3L\left(\lambda \sum_{t_i^\lambda \in \{s_1, \dots, s_k\}} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu \in \{s_1, \dots, s_k\}} (t_j^\mu - t_{j-1}^\mu)\right).$$

Proof. Set $z_0^\varepsilon = z$ for each $\varepsilon = \lambda, \mu$. Assume that sequences $\{(s_k, z_k^\lambda)\}_{k=0}^{l-1}$ and $\{(s_k, z_k^\mu)\}_{k=0}^{l-1}$ in Ω with $l \geq 1$ satisfy properties (a)–(c) for $0 \leq k \leq l-1$. Let i and j be positive integers such that $t_{i-1}^\lambda < s_l \leq t_i^\lambda$ and $t_{j-1}^\mu < s_l \leq t_j^\mu$, respectively. By Lemma 3 (i) with $\varepsilon = \lambda/2$ we obtain $\|x_{i-1}^\lambda - z\| \leq (M + \lambda/2)(t_{i-1}^\lambda - \tau)$. If $(s, y) \in S_{(M+1)(t_i^\lambda - t_{i-1}^\lambda)}(t_{i-1}^\lambda, x_{i-1}^\lambda)$, then we get

$$\begin{aligned}|s - \tau| &\leq |s - t_{i-1}^\lambda| + |t_{i-1}^\lambda - \tau| < (M+1)(t_i^\lambda - t_{i-1}^\lambda) + (t_{i-1}^\lambda - \tau)\\ &\leq (M+1)\sigma \leq R\end{aligned}$$

and

$$\begin{aligned}\|y - z\| &\leq \|y - x_{i-1}^\lambda\| + \|x_{i-1}^\lambda - z\|\\ &< (M+1)(t_i^\lambda - t_{i-1}^\lambda) + (M + \lambda/2)(t_{i-1}^\lambda - \tau) < (M+1)\sigma \leq R.\end{aligned}$$

Hence $S_{(M+1)(t_i^\lambda - t_{i-1}^\lambda)}(t_{i-1}^\lambda, x_{i-1}^\lambda) \subset S_R(\tau, z)$. This implies that

$$\|A(s, y)\| \leq M \qquad \text{for } (s, y) \in \Omega \cap S_{(M+1)(t_i^\lambda - t_{i-1}^\lambda)}(t_{i-1}^\lambda, x_{i-1}^\lambda). \tag{2.18}$$

We shall show that for each $\varepsilon = \lambda, \mu$,

$$\|A(s, y)\| \leq M \qquad \text{and} \qquad \|A(s, y) - A(s_{l-1}, z_{l-1}^\varepsilon)\| \leq \varepsilon/2 \tag{2.19}$$

for $(s, y) \in \Omega \cap S_{(M+1)(s_l - s_{l-1})}(s_{l-1}, z_{l-1}^\varepsilon)$. By the definition of $\{s_k\}$ we observe that

$$\begin{aligned}&t_{i-1}^\lambda \leq s_{l-1} < s_l \leq t_i^\lambda, \qquad t_{j-1}^\mu \leq s_{l-1} < s_l \leq t_j^\mu,\\ &t_{i-1}^\lambda = s_p \quad \text{for some } 0 \leq p \leq l-1, \qquad \text{and}\\ &t_{j-1}^\mu = s_q \quad \text{for some } 0 \leq q \leq l-1.\end{aligned}$$

By the hypothesis (a) of induction, we have $z_p^\lambda = x_{i-1}^\lambda$ and $z_q^\mu = x_{j-1}^\mu$. If $0 \leq p < l-1$, then the set $\{s_{p+1}, \dots, s_{l-1}\}$ contains no points t_i^λ. By the hypothesis (b) of induction, we have

$$\|z_{k-1}^\lambda + (s_k - s_{k-1})A(s_{k-1}, z_{k-1}^\lambda) - z_k^\lambda\| \leq 2\lambda(s_k - s_{k-1}) \tag{2.20}$$

for $k = p+1, \dots, l-1$. By (2.18) and (2.20), we use Lemma 3 (i) with $(t, x) = (t_{i-1}^{\lambda}, x_{i-1}^{\lambda}) = (s_p, z_p^{\lambda})$, $\varepsilon = 2\lambda$ and $r = (M+1)(t_i^{\lambda} - t_{i-1}^{\lambda})$ to obtain $\|z_{l-1}^{\lambda} - z_p^{\lambda}\| \le (M + 2\lambda)(s_{l-1} - s_p)$. This is valid for $p = l-1$. If $(s, y) \in S_{(M+1)(s_l - s_{l-1})}(s_{l-1}, z_{l-1}^{\lambda})$, then we get

$$\begin{aligned}|s - t_{i-1}^{\lambda}| &\le |s - s_{l-1}| + |s_{l-1} - t_{i-1}^{\lambda}| \\ &< (M+1)(s_l - s_{l-1}) + (s_{l-1} - t_{i-1}^{\lambda}) \le (M+1)(t_i^{\lambda} - t_{i-1}^{\lambda}), \\ \|y - x_{i-1}^{\lambda}\| &\le \|y - z_{l-1}^{\lambda}\| + \|z_{l-1}^{\lambda} - x_{i-1}^{\lambda}\| \\ &< (M+1)(s_l - s_{l-1}) + (M + 2\lambda)(s_{l-1} - s_p) \le (M+1)(t_i^{\lambda} - t_{i-1}^{\lambda}).\end{aligned}$$

This means that

$$S_{(M+1)(s_l - s_{l-1})}(s_{l-1}, z_{l-1}^{\lambda}) \subset S_{(M+1)(t_i^{\lambda} - t_{i-1}^{\lambda})}(t_{i-1}^{\lambda}, x_{i-1}^{\lambda}). \tag{2.21}$$

Thus, the claim (2.19) with $\varepsilon = \lambda$ follows from (2.18) and condition (iv). Indeed,

$$\begin{aligned}&\|A(s, y) - A(s_{l-1}, z_{l-1}^{\lambda})\| \\ &\quad \le \|A(s, y) - A(t_{i-1}^{\lambda}, x_{i-1}^{\lambda})\| + \|A(t_{i-1}^{\lambda}, x_{i-1}^{\lambda}) - A(s_{l-1}, z_{l-1}^{\lambda})\| \\ &\quad \le \lambda/4 + \lambda/4 = \lambda/2\end{aligned}$$

for $(s, y) \in \Omega \cap S_{(M+1)(s_l - s_{l-1})}(s_{l-1}, z_{l-1}^{\lambda})$. We apply the above argument again, with p and i replaced by q and j, to show that (2.19) holds true for $\varepsilon = \mu$.

By virtue of (2.19), we deduce from Proposition 3 with $t = s_{l-1}$, $(x, \hat{x}) = (z_{l-1}^{\lambda}, z_{l-1}^{\mu})$, $\eta = 2\lambda$, $\hat{\eta} = 2\mu$ and $r = (M+1)(s_l - s_{l-1})$ that there exists a pair $(y_l^{\lambda}, y_l^{\mu}) \in \Omega(s_{l-1} + (s_l - s_{l-1})) \times \Omega(s_{l-1} + (s_l - s_{l-1})) = \Omega(s_l) \times \Omega(s_l)$ satisfying

$$\|z_{l-1}^{\varepsilon} + (s_l - s_{l-1})A(s_{l-1}, z_{l-1}^{\varepsilon}) - y_l^{\varepsilon}\| \le 2\varepsilon(s_l - s_{l-1}) \qquad \text{for } \varepsilon = \lambda, \mu, \tag{2.22}$$

$$\begin{aligned}V(s_l, y_l^{\lambda}, y_l^{\mu}) &\le \exp((s_l - s_{l-1})\varpi([s_{l-1}, s_l])) \\ &\quad \times (V(s_{l-1}, z_{l-1}^{\lambda}, z_{l-1}^{\mu}) + 2L(\lambda + \mu)(s_l - s_{l-1})).\end{aligned} \tag{2.23}$$

We define $(z_l^{\lambda}, z_l^{\mu}) \in \Omega(s_l) \times \Omega(s_l)$ by

$$z_l^{\lambda} = \begin{cases} y_l^{\lambda} & \text{for } s_l < t_i^{\lambda}, \\ x_i^{\lambda} & \text{for } s_l = t_i^{\lambda} \end{cases} \qquad \text{and} \qquad z_l^{\mu} = \begin{cases} y_l^{\mu} & \text{for } s_l < t_j^{\mu}, \\ x_j^{\mu} & \text{for } s_l = t_j^{\mu}. \end{cases}$$

If $s_l = t_i^{\lambda}$, then by condition (iii) we have

$$\|x_{i-1}^{\lambda} + (s_l - t_{i-1}^{\lambda})A(t_{i-1}^{\lambda}, x_{i-1}^{\lambda}) - z_l^{\lambda}\| \le (s_l - t_{i-1}^{\lambda})\lambda/2,$$

while in view of (2.18) and (iv) we find, by applying Lemma 3 (i), with $\varepsilon = 2\lambda$, $\eta = \lambda/4$, $r = (M+1)(t_i^\lambda - t_{i-1}^\lambda)$ and $(t,x) = (t_{i-1}^\lambda, x_{i-1}^\lambda)$, to (2.20) and (2.22), that

$$\|x_{i-1}^\lambda + (s_l - t_{i-1}^\lambda)A(t_{i-1}^\lambda, x_{i-1}^\lambda) - y_l^\lambda\| \le (2\lambda + \lambda/4)(s_l - t_{i-1}^\lambda).$$

These inequalities together yield

$$\begin{aligned}\|z_l^\lambda - y_l^\lambda\| &\le \|x_{i-1}^\lambda + (s_l - t_{i-1}^\lambda)A(t_{i-1}^\lambda, x_{i-1}^\lambda) - y_l^\lambda\| \\ &\quad + \|x_{i-1}^\lambda + (s_l - t_{i-1}^\lambda)A(t_{i-1}^\lambda, x_{i-1}^\lambda) - z_l^\lambda\| \\ &\le (9/4 + 1/2)\lambda(s_l - t_{i-1}^\lambda) \le 3\lambda \sum_{t_i^\lambda = s_l} (t_i^\lambda - t_{i-1}^\lambda). \end{aligned} \tag{2.24}$$

Similarly, we get

$$\|z_l^\mu - y_l^\mu\| \le 3\mu \sum_{t_j^\mu = s_l} (t_j^\mu - t_{j-1}^\mu). \tag{2.25}$$

Combining (2.24) and (2.25) with (2.22), and adding the resulting inequality to the inequality (b) with $k = l-1$, we conclude that the desired property (b) holds true for $k = l$.

Finally, we show that (c) is true for $k = l$. Using (2.24), (2.25) and $(V1)$ we have

$$\begin{aligned}|V(s_l, z_l^\lambda, z_l^\mu) - V(s_l, y_l^\lambda, y_l^\mu)| &\le L(\|z_l^\lambda - y_l^\lambda\| + \|z_l^\mu - y_l^\mu\|) \\ &\le 3L\left(\lambda \sum_{t_i^\lambda = s_l}(t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu = s_l}(t_j^\mu - t_{j-1}^\mu)\right).\end{aligned}$$

Combining this and (2.23), we obtain

$$\begin{aligned}V(s_l, z_l^\lambda, z_l^\mu) &\le V(s_l, y_l^\lambda, y_l^\mu) + 3L\left(\lambda \sum_{t_i^\lambda = s_l}(t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu = s_l}(t_j^\mu - t_{j-1}^\mu)\right) \\ &\le \exp((s_l - s_{l-1})\overline{\omega}([s_{l-1}, s_l]))(V(s_{l-1}, z_{l-1}^\lambda, z_{l-1}^\mu) \\ &\qquad + 2L(\lambda + \mu)(s_l - s_{l-1})) \\ &\quad + 3L\left(\lambda \sum_{t_i^\lambda = s_l}(t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu = s_l}(t_j^\mu - t_{j-1}^\mu)\right)\end{aligned}$$

$$\leq \exp((s_l - \tau)\varpi([\tau, s_l]))(2L(\lambda + \mu)(s_l - \tau) + \eta_{l-1}(\lambda, \mu))$$

$$+ 3L\left(\lambda \sum_{t_i^\lambda = s_l} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu = s_l} (t_j^\mu - t_{j-1}^\mu)\right)$$

$$\leq \exp((s_l - \tau)\varpi([\tau, s_l]))(2L(\lambda + \mu)(s_l - \tau) + \eta_l(\lambda, \mu)).$$

This means that (c) is true for $k = l$, and the proof is completed. □

The following is a local existence theorem of solutions to (IVP; τ, z).

Theorem 3. *Let $(\tau, z) \in \Omega$. Let $R > 0$ and $M > 0$ be numbers such that $\tau + R < b$ and $\|A(s, y)\| \leq M$ for $(s, y) \in \Omega \cap S_R(\tau, z)$. Let $\sigma \in (0, R/(M+1)]$. Then there exists a solution u to* (IVP; τ, z) *on $[\tau, \tau + \sigma]$ such that*

$$\|u(t) - u(s)\| \leq M|t - s| \qquad \textit{for } t, s \in [\tau, \tau + \sigma].$$

Proof. Let $\varepsilon \in (0, 1/2)$. Then, by Proposition 2, there exists a sequence $\{(t_i^\varepsilon, x_i^\varepsilon)\}_{i=0}^\infty$ in Ω satisfying (i)–(iv) of Proposition 4. Let $u^\varepsilon : [\tau, \tau + \sigma) \to X$ be the function defined by $u^\varepsilon(t) = x_i^\varepsilon$ for $t \in [t_i^\varepsilon, t_{i+1}^\varepsilon)$ and $i \geq 0$. We want to prove that the family $\{u^\varepsilon\}$ converges in X uniformly on $[\tau, \tau + \sigma)$ as $\varepsilon \downarrow 0$.

Let $\lambda, \mu \in (0, 1/2)$ and let $\{s_k\}_{k=0}^\infty$ be a sequence defined as in Proposition 4. Then there exists a sequence $\{(z_k^\lambda, z_k^\mu)\}$ in $X \times X$ satisfying $(z_k^\lambda, z_k^\mu) \in \Omega(s_k) \times \Omega(s_k)$ for $k \geq 0$ and (a)–(c) of Proposition 4. We first prove that

$$\sup_{k \geq 0} \|z_k^\lambda - z_k^\mu\| \to 0 \qquad \text{as } \lambda, \mu \downarrow 0. \tag{2.26}$$

Assume to the contrary that there exist $\varepsilon_0 > 0$, two null sequences $\{\lambda_n\}$ and $\{\mu_n\}$ of positive numbers, and a sequence $\{k_n\}$ of nonnegative integers such that

$$\|z_{k_n}^{\lambda_n} - z_{k_n}^{\mu_n}\| \geq \varepsilon_0 \qquad \text{for } n \geq 1. \tag{2.27}$$

Since the sequence $\{s_{k_n}\}$ is bounded as $n \to \infty$, it has a convergent subsequence $\{s_{k_{n_l}}\}$. Since $(z_{k_{n_l}}^{\lambda_{n_l}}, z_{k_{n_l}}^{\mu_{n_l}}) \in \Omega(s_{k_{n_l}}) \times \Omega(s_{k_{n_l}})$ for $l \geq 1$, and since

$$V(s_{k_{n_l}}, z_{k_{n_l}}^{\lambda_{n_l}}, z_{k_{n_l}}^{\mu_{n_l}}) \leq 5L \exp(\sigma\varpi([\tau, \tau + \sigma]))(\lambda_{n_l} + \mu_{n_l})\sigma \qquad \text{for } l \geq 1$$

by Proposition 4 (c), we deduce from condition ($V4$) that $\lim_{l \to \infty} \|z_{k_{n_l}}^{\lambda_{n_l}} - z_{k_{n_l}}^{\mu_{n_l}}\| = 0$. This is a contradiction to (2.27).

Let $t \in [\tau, \tau + \sigma)$. Let $k \geq 1$ be an integer such that $t \in [s_{k-1}, s_k)$. Let i and j be positive integers such that $t_{i-1}^\lambda \leq s_{k-1} < s_k \leq t_i^\lambda$ and $t_{j-1}^\mu \leq$

$s_{k-1} < s_k \le t_j^\mu$, respectively. Then we have, in a similar way to the derivation of (2.21), $\|z_{k-1}^\lambda - x_{i-1}^\lambda\| \le (M+1)(t_i^\lambda - t_{i-1}^\lambda)$ and $\|z_{k-1}^\mu - x_{j-1}^\mu\| \le (M+1)(t_j^\mu - t_{j-1}^\mu)$. Since

$$\begin{aligned}\|u^\lambda(t) - u^\mu(t)\| &\le \|x_{i-1}^\lambda - z_{k-1}^\lambda\| + \|z_{k-1}^\lambda - z_{k-1}^\mu\| + \|z_{k-1}^\mu - x_{j-1}^\mu\| \\ &\le (M+1)(\lambda+\mu) + \|z_{k-1}^\lambda - z_{k-1}^\mu\|,\end{aligned}$$

we observe from (2.26) that the family $\{u^\varepsilon(t)\}$ is uniformly Cauchy on $[\tau, \tau+\sigma)$. By Lemma 3 (i) we obtain

$$\|u^\varepsilon(t) - u^\varepsilon(s)\| \le (M + \varepsilon/2)(|t-s| + 2\varepsilon) \qquad \text{for } t, s \in [\tau, \tau+\sigma)$$

and $\varepsilon \in (0, 1/2)$. These facts imply that there exists a continuous function u defined on $[\tau, \tau+\sigma]$ such that $\sup_{t\in[\tau,\tau+\sigma)}\|u^\varepsilon(t) - u(t)\| \to 0$ as $\varepsilon \downarrow 0$. It is clear that $u(\tau) = z$ and $\|u(t) - u(s)\| \le M|t-s|$ for $t, s \in [\tau, \tau+\sigma]$. Let $\tau^\varepsilon : [\tau, \tau+\sigma) \to \mathbf{R}$ be the function defined by $\tau^\varepsilon(t) = t_i^\varepsilon$ for $t \in [t_i^\varepsilon, t_{i+1}^\varepsilon)$ and $i \ge 0$. Then $\tau \le \tau^\varepsilon(t) \le t < \tau+\sigma$ and $\lim_{\varepsilon\downarrow 0} \tau^\varepsilon(t) = t$ for $t \in [\tau, \tau+\sigma)$. From Proposition 4 (iii) we deduce that

$$\left\| u^\varepsilon(t_i^\varepsilon) - u^\varepsilon(0) - \int_\tau^{t_i^\varepsilon} A(\tau^\varepsilon(s), u^\varepsilon(s))ds \right\| \le \varepsilon(t_i^\varepsilon - \tau)/2 \le \varepsilon\sigma/2 \qquad (2.28)$$

for $i \ge 0$. Since $(\tau^\varepsilon(t), u^\varepsilon(t)) \in \Omega$ and $\|A(\tau^\varepsilon(t), u^\varepsilon(t))\| \le M$ for $t \in [\tau, \tau+\sigma)$ and since $(\tau^\varepsilon(t), u^\varepsilon(t)) \to (t, u(t))$, we have $(t, u(t)) \in \Omega$ and $A(\tau^\varepsilon(t), u^\varepsilon(t)) \to A(t, u(t))$ for $t \in [\tau, \tau+\sigma)$ as $\varepsilon \downarrow 0$, by (Ω2) and (Ω1) respectively. From (2.28) we obtain

$$u(t) - u(0) = \int_\tau^t A(s, u(s))ds$$

for $t \in [\tau, \tau+\sigma)$. Since $t \to A(t, u(t))$ is continuous on $[\tau, \tau+\sigma]$, u is a solution to (IVP; τ, z) on $[\tau, \tau+\sigma]$. Since the uniqueness follows from Proposition 1, the proof is completed. □

3. Global existence of solutions

In this section we investigate the intervals where the solutions to (IVP; τ, z) exist under assumptions (Ω1)–(Ω4). We follow the arguments in [4], [6] and [7].

PROPOSITION 5. *Let $(\tau, z) \in \Omega$. Then there exists $c_0 \in (\tau, b)$ such that for any $c \in (\tau, c_0)$, the following properties are satisfied:*

(i) *(IVP; τ, z) has a solution u on $[\tau, c]$.*

(ii) *For any* $\varepsilon > 0$, *there exists a number* $r \in (0, c - \tau)$ *which satisfies the following:*

(a) $(\mathrm{IVP}; t, x)$ *has a solution* v *on* $[t, c]$ *for any* $(t, x) \in \Omega \cap S_r(\tau, z)$,

(b) *if* $(t, x), (\hat{t}, \hat{x}) \in \Omega \cap S_r(\tau, z)$, v *and* $\hat{v}$ *are solutions to* $(\mathrm{IVP}; t, x)$ *on* $[t, c]$ *and* $(\mathrm{IVP}; \hat{t}, \hat{x})$ *on* $[\hat{t}, c]$ *respectively, then* $V(s, v(s), \hat{v}(s)) < \varepsilon$ *for* $s \in [t, c] \cap [\hat{t}, c]$.

Proof. Let $R > 0$ and $M > 0$ be numbers such that $\tau + R < b$ and $\|A(t, x)\| \le M$ for $(t, x) \in \Omega \cap S_R(\tau, z)$, and set $c_0 = \tau + R/(M + 1)$. We shall show that for any number $c \in (\tau, c_0)$, the desired properties are satisfied. The first property (i) follows from Theorem 3.

We shall show that such a number c has the second property (ii). Let $\varepsilon > 0$. We take $\delta > 0$ so that $\exp(\int_\tau^s \omega(\theta) d\theta)\delta < \varepsilon$ for any $s \in [a, c]$. Next, we choose $r > 0$ so small that $\tau + r < c \le \tau + (R - r)/(M + 1) - r$ and

$$2L(M + 1)r \le \exp\left(\int_\tau^s \omega(\theta) d\theta\right)\delta \tag{3.1}$$

for $s \in [\tau - r, \tau + r] \cap [a, b)$. To prove (a), let $(t, x) \in \Omega \cap S_r(\tau, z)$. Set $\hat{r} = R - r$. Since $\tau + r < c < \tau + R/(M + 1) < \tau + R$, we have $\hat{r} > 0$. Moreover, we have $t + \hat{r} = (t - \tau) + \tau + \hat{r} \le r + \tau + \hat{r} = \tau + R < b$. For $(s, y) \in S_{\hat{r}}(t, x)$, we have

$$|s - \tau| \le |s - t| + |t - \tau| < \hat{r} + r = R$$

and

$$\|y - z\| \le \|y - x\| + \|x - z\| < \hat{r} + r = R.$$

Thus $S_{\hat{r}}(t, x) \subset S_R(\tau, z)$. Since $\|A(s, y)\| \le M$ for $(s, y) \in \Omega \cap S_{\hat{r}}(t, x)$ and $t + \hat{r} < b$, $(\mathrm{IVP}; t, x)$ has a solution v on $[t, t + \hat{r}/(M + 1)]$ by Theorem 3. Since $t + \hat{r}/(M + 1) > \tau - r + (R - r)/(M + 1) \ge c$, we certainly infer that v is defined on $[t, c]$.

To prove (b), let $\hat{v}$ be a solution to $(\mathrm{IVP}; \hat{t}, \hat{x})$ on $[\hat{t}, c]$ with $(\hat{t}, \hat{x}) \in \Omega \cap S_r(\tau, z)$. Assume that $\hat{t} \le t$ without loss of generality. Then

$$\begin{aligned} \|\hat{v}(t) - v(t)\| &= \|\hat{v}(t) - x\| \le \|\hat{v}(t) - \hat{x}\| + \|\hat{x} - z\| + \|z - x\| \\ &\le \|\hat{v}(t) - \hat{v}(\hat{t})\| + 2r \le M(t - \hat{t}) + 2r \\ &= M((t - \tau) + (\tau - \hat{t})) + 2r \le 2(M + 1)r. \end{aligned}$$

By Remark 1 and (3.1), we have

$$V(t, v(t), \hat{v}(t)) \le 2L(M + 1)r \le \exp\left(\int_\tau^t \omega(\theta) d\theta\right)\delta.$$

Thus, by Proposition 1, we obtain

$$V(s, v(s), \hat{v}(s)) \le \exp\left(\int_t^s \omega(\theta)d\theta\right) V(t, v(t), \hat{v}(t)) \le \exp\left(\int_\tau^s \omega(\theta)d\theta\right)\delta < \varepsilon$$

for $s \in [t, c]$. □

Let $(\tau, z) \in \Omega$ and let u be a solution to (IVP; τ, z) which is noncontinuable to the right. We denote its *final time* by $T(\tau, z)$. It is clear that $\tau < T(\tau, z) \le b$ and u is a solution to (IVP; τ, z) on $[\tau, T(\tau, z))$. Since (IVP; τ, z) has a unique solution, $T(\tau, z) \in (\tau, b]$ is well-defined for every $(\tau, z) \in \Omega$. We consider T as a function from the metric space Ω into the extended real line $\mathbf{R} \cup \{\infty\}$ endowed with the usual topology.

Proposition 6. *Let $(\tau, z) \in \Omega$ and let d be a number such that $\tau < d < T(\tau, z)$. Then there exists a number $r > 0$ with $\tau + r < b$ such that $T(t, x) > d$ for any $(t, x) \in \Omega \cap S_r(\tau, z)$.*

Proof. Let $(\tau, z) \in \Omega$ and let d be a number such that $\tau < d < T(\tau, z)$. Let u be a solution to (IVP; τ, z) on $[\tau, d]$. Since the set $\{(s, u(s)); s \in [\tau, d]\}$ is compact in Ω and A is continuous on Ω, there exists a number $M > 0$ such that $\|A(s, u(s))\| < M$ for $s \in [\tau, d]$.

We first prove that there exists a number $R > 0$ such that $\|A(s, x)\| \le M$ for any $s \in [\tau, d]$ and $x \in \Omega(s)$ satisfying $V(s, x, u(s)) < R$. Assume to the contrary that for any $n \ge 1$ there exist $s_n \in [\tau, d]$ and $x_n \in \Omega(s_n)$ such that $V(s_n, x_n, u(s_n)) < 1/n$ and $\|A(s_n, x_n)\| > M$. Since the sequence $\{s_n\}$ is bounded, there exists a convergent subsequence $\{s_{n_k}\}$ converging to some number $s \in [\tau, d]$. Since $V(s_{n_k}, x_{n_k}, u(s_{n_k})) \to 0$ as $k \to \infty$, we have $\|x_{n_k} - u(s_{n_k})\| \to 0$ as $k \to \infty$ by $(V4)$. Since $u(s_{n_k}) \to u(s)$ as $k \to \infty$, we have $(s_{n_k}, x_{n_k}) \to (s, u(s))$ as $k \to \infty$. Thus, by $(\Omega 1)$, we have $\|A(s, u(s))\| \ge M$. This contradicts to the definition of M.

By Proposition 5, we can choose a number c such that $\tau < c < d$ and properties (i) and (ii) in Proposition 5 are satisfied for (τ, z). Let $\varepsilon > 0$ be a number such that $\varepsilon \exp(\int_c^s \omega(\theta)d\theta) \le R$ for $s \in [c, d]$, and then choose $r > 0$ so that $\tau + r < c$ and Proposition 5 (ii) is satisfied for the number ε. Let $(t, x) \in \Omega \cap S_r(\tau, z)$. We want to show that $d < T(t, x)$. To this end, assume to the contrary that $T(t, x) \le d$ and let v be a noncontinuable solution to (IVP; t, x). Note by Proposition 5 (ii) that $[t, c] \subset [t, T(t, x))$ and $V(c, v(c), u(c)) < \varepsilon$. By Proposition 1, we have

$$\begin{aligned} V(s, v(s), u(s)) &\le V(c, v(c), u(c)) \exp\left(\int_c^s \omega(\theta)d\theta\right) \\ &< \varepsilon \exp\left(\int_c^s \omega(\theta)d\theta\right) \le R \end{aligned}$$

for $s \in [c, T(t,x))$. From the fact proved first, we observe that $\|A(s, v(s))\| \le M$ for $s \in [c, T(t,x))$. Thus $\|v(t) - v(s)\| \le M|t - s|$ for $t, s \in [c, T(t,x))$. Therefore, $w = \lim_{s \uparrow T(t,x)} v(s)$ exists in X and $(T(t,x), w) \in \Omega$ by $(\Omega 2)$. In view of Theorem 3, this contradicts the fact that v is noncontinuable to the right of $T(t,x)$. Hence $T(t,x) > d$. □

Proposition 7. *Let $(\tau, z) \in \Omega$ and let $\{(\tau_n, z_n)\}_{n \ge 1}$ be a sequence in Ω converging to (τ, z) as $n \to \infty$. For $n \ge 1$, let u_n be a noncontinuable solution to* (IVP; τ_n, z_n)*, and let u be a noncontinuable solution to* (IVP; τ, z). *Assume that $d \in (\tau, b)$ satisfies $d < T(\tau_n, z_n)$ for $n \ge 1$. Then the following assertions hold:*

(i) *$d < T(\tau, z)$.*

(ii) *For any $\sigma \in (\tau, d)$, the sequence $\{u_n\}$ converges to u uniformly on $[\sigma, d]$ as $n \to \infty$.*

Proof. Let $c \in (\tau, d)$ be a number with the properties (i) and (ii) in Proposition 5, and let $\tau < \sigma < c$. We may assume that $\tau_n < \sigma < c < d < T(\tau_n, z_n)$ for $n \ge 1$, because $\lim_{n \to \infty} \tau_n = \tau < d$. Let $\varepsilon > 0$. Let $r \in (0, c - \tau)$ be a number with the property (ii) in Proposition 5 for the number ε. Since $(\tau_n, z_n) \to (\tau, z)$ as $n \to \infty$, there exists an integer $n_0 \ge 1$ such that $(\tau_n, z_n) \in \Omega \cap S_r(\tau, z)$ for $n \ge n_0$. By Proposition 5 (ii-b) we observe that if $n, m \ge n_0$, then $V(s, u_m(s), u_n(s)) \le \varepsilon$ for $s \in [\sigma, c]$ and

$$
\begin{aligned}
V(t, u_m(t), u_n(t)) &\le \exp\left(\int_c^t \omega(\theta)\,d\theta\right) V(c, u_m(c), u_n(c)) \\
&\le \varepsilon \exp((d - c)\overline{\omega}([c, d]))
\end{aligned}
$$

for $t \in [c, d]$. By $(V4)$, the sequence $\{u_n\}$ is uniformly Cauchy on $[\sigma, d]$. Define $\hat{u}(t) = \lim_{n \to \infty} u_n(t)$ for $t \in [\sigma, d]$. Then we observe that $\hat{u}'(t) = A(t, \hat{u}(t))$ for $t \in [\sigma, d]$. By Proposition 5, we observe that if $n \ge n_0$, then $V(s, u_n(s), u(s)) \le \varepsilon$ for $s \in [\sigma, c]$. Thus, we have $\hat{u}(\sigma) = \lim_{n \to \infty} u_n(\sigma) = u(\sigma)$. Hence $\hat{u}$ is a solution to (IVP; $\sigma, u(\sigma)$) on $[\sigma, d]$. Note that u is a solution to (IVP; τ, z) on $[\tau, \sigma]$. Since the function $v : [\tau, d] \to X$ defined by $v(t) = u(t)$ for $t \in [\tau, \sigma]$ and $v(t) = \hat{u}(t)$ for $t \in [\sigma, d]$ is a solution to (IVP; τ, z) on $[\tau, d]$, we have $T(\tau, z) > d$. Since $v(t) = u(t)$ for $t \in [\tau, d]$ by uniqueness, we observe that the sequence $\{u_n\}$ converges to u uniformly on $[\sigma, d]$ as $n \to \infty$. □

Proposition 8. *T is a continuous function from Ω into $\mathbf{R} \cup \{\infty\}$.*

Proof. Let $(\tau, z) \in \Omega$ and let $\{(t_n, x_n)\}_{n \ge 1}$ be a sequence in Ω converging to (τ, z). Let $\tau < d < T(\tau, z)$. Since $\lim_{n \to \infty}(t_n, x_n) = (\tau, z)$, we deduce from Proposition 6 that $d < T(t_n, x_n)$ for sufficiently large integers

n. Thus $d \le \liminf_{n\to\infty} T(t_n, x_n)$. Since d is arbitrary, we obtain $T(\tau, z) \le \liminf_{n\to\infty} T(t_n, x_n)$. Note that

$$\tau < T(\tau, z) \le \liminf_{n\to\infty} T(t_n, x_n) \le \limsup_{n\to\infty} T(t_n, x_n),$$

and let d satisfy $\tau < d < \limsup_{n\to\infty} T(t_n, x_n)$. Then there exists a subsequence $\{(t_{n_k}, x_{n_k})\}_{k\ge 1}$ of $\{(t_n, x_n)\}_{n\ge 1}$ such that $d < T(t_{n_k}, x_{n_k})$ for $k \ge 1$. Since $(t_{n_k}, x_{n_k}) \to (\tau, z)$ as $k \to \infty$, it follows from Proposition 7 that $d < T(\tau, z)$. Since d is arbitrary chosen, we conclude that $\limsup_{n\to\infty} T(t_n, x_n) \le T(\tau, z)$. Hence, we obtain $\lim_{n\to\infty} T(t_n, x_n) = T(\tau, z)$. □

A global existence theorem is given as follows.

Theorem 4. *Let C be a connected component of Ω and set $d = \sup\{t \in [a, b); C(t) \neq \emptyset\}$. Then for each $(\tau, z) \in C$,* (IVP; τ, z) *has a unique solution on $[\tau, d)$ and the interval $[\tau, d)$ is the maximal interval of existence of solution. In particular, if Ω is connected, then for $(\tau, z) \in \Omega$,* (IVP; τ, z) *has a unique solution on $[\tau, b)$.*

Proof. We shall show that $T : \Omega \to \mathbf{R} \cup \{\infty\}$ takes the constant value d on C. To prove that $T(C)$ is a singleton set, let $c, \hat{c} \in T(C) = \{T(t, x); (t, x) \in C\}$. Without loss of generality, we assume that $c \le \hat{c}$, and set

$$C_1 = \{(t, x) \in C; T(t, x) \le c\} \qquad \text{and} \qquad C_2 = \{(t, x) \in C; T(t, x) > c\}.$$

If $C = C_1$, then $\hat{c} \le c$, and so $T(C)$ is a singleton set $\{c\}$. To prove that $C = C_1$, we have only to prove that $C_2 = \emptyset$ because C_1 and C_2 are disjoint. To this end, assume to the contrary that C_2 is nonempty. Since T is continuous on C by Proposition 8, C_2 is an open subset of C. Let $\{(t_n, x_n)\}_{n\ge 1}$ be a sequence in C_2 converging to $(t, x) \in C$. By the definition of C_2, we have $c < T(t_n, x_n)$ for $n \ge 1$. Proposition 7 asserts that $c < T(t, x)$. This implies that C_2 is a closed subset of C. It follows that $C = C_1 \cup C_2$, and C_1 and C_2 are disjoint, nonempty and open in C. This is impossible because C is connected, and so we conclude that $C_2 = \emptyset$.

Since $T(C)$ is a singleton set, we can write $T(C) = \{c\}$ for some $c \in \mathbf{R} \cup \{\infty\}$. Since $t < T(t, x) = c$ for $(t, x) \in C$, we obtain $d = \sup\{t; C(t) \neq \emptyset\} \le c$. On the other hand, let $s < c$. Note that $c = T(t, x)$ for some $(t, x) \in C$. If $t < s$ then a noncontinuable solution u to (IVP; t, x) satisfies $(s, u(s)) \in C$, and so $C(s) \neq \emptyset$. This implies that $s \le d$. If $s \le t$ then $s \le t \le d$ because $C(t) \neq \emptyset$. Since s is arbitrarily chosen such that $s < c$, we have $c \le d$. Consequently, we get $T(C) = \{d\}$. □

Theorem 1 is a consequence of Proposition 1 and Theorems 3 and 4.

4. Proof of Theorem 2

Proof of the necessity part. Let $(\tau, z) \in \Omega$ and $u(t) = U(t, \tau)z$ for $t \in [\tau, b)$. Let C be a connected component of Ω such that $(\tau, z) \in C$. Since $\{(t, u(t));\ t \in [\tau, b)\}$ is a connected set in Ω containing (τ, z), we have $(t, u(t)) \in C$ for $t \in [\tau, b)$ by the maximality of C; hence $C(t) \neq \emptyset$ for $t \in [\tau, b)$. This means that $(\Omega 5)$ holds true. Since $u(\tau + h) \in \Omega(\tau + h)$ for $h \in (0, b - \tau)$, we have

$$\begin{aligned} h^{-1} d(z + hA(\tau, z), \Omega(\tau + h)) &\leq h^{-1} \|z + hA(\tau, z) - u(\tau + h)\| \\ &= \|A(\tau, u(\tau)) - h^{-1}(u(\tau + h) - u(\tau))\| \\ &\to \|A(\tau, u(\tau)) - u'(\tau)\| = 0 \end{aligned}$$

as $h \downarrow 0$. Thus, $(\Omega 3)$ also holds true. It remains to show that $(\Omega 4)$ holds true. We set

$$V_0(t, x, y) = \sup_{\sigma \in [t, b)} \left\{ \exp\left(-\int_t^\sigma \omega(\theta) d\theta \right) \|U(\sigma, t)x - U(\sigma, t)y\| \right\}$$

for $t \in [a, b)$ and $x, y \in \Omega(t)$. From (E1) and (E3) we see that

$$\|x - y\| \leq V_0(t, x, y) \leq L\|x - y\| \qquad \text{for } t \in [a, b) \text{ and } x, y \in \Omega(t). \tag{4.1}$$

For any $x, y \in X$, $t \in [a, b)$ and $x', y' \in \Omega(t)$, we have

$$\begin{aligned} &V_0(t, x', y') - L(\|x - x'\| + \|y - y'\|) \\ &\qquad \leq L\|x' - y'\| - L(\|x - x'\| + \|y - y'\|) \leq L\|x - y\|. \end{aligned}$$

Thus, we can define $V : [a, b) \times X \times X \to [0, \infty)$ by

$$V(t, x, y) = \sup_{(x', y') \in \Omega(t) \times \Omega(t)} \{\max(0, V_0(t, x', y') - L(\|x - x'\| + \|y - y'\|))\}$$

for $(t, x, y) \in [a, b) \times X \times X$. Since

$$\begin{aligned} V_0(t, x', y') &\leq V_0(t, x', x) + V_0(t, x, y) + V_0(t, y, y') \\ &\leq V_0(t, x, y) + L(\|x - x'\| + \|y - y'\|) \end{aligned}$$

for $t \in [a, b)$ and $(x, y), (x', y') \in \Omega(t) \times \Omega(t)$, we have $V(t, x, y) \leq V_0(t, x, y)$ for $t \in [a, b)$ and $(x, y) \in \Omega(t) \times \Omega(t)$. The converse inequality follows readily from the definition of V. Thus $V(t, x, y) = V_0(t, x, y)$ for $t \in [a, b)$ and $(x, y) \in \Omega(t) \times \Omega(t)$. This combined with (4.1) implies that the functional V satisfies $(V4)'$ and $(V2)$.

Let $(x, y), (\hat{x}, \hat{y}) \in X \times X$ and $t \in [a, b)$. For any $(x', y') \in \Omega(t) \times \Omega(t)$, we have

$$
\begin{aligned}
&V_0(t,x',y') - L(\|x-x'\| + \|y-y'\|) \\
&\quad - (V_0(t,x',y') - L(\|\hat{x}-x'\| + \|\hat{y}-y'\|)) \\
&\quad = L(\|\hat{x}-x'\| + \|\hat{y}-y'\|) - L(\|x-x'\| + \|y-y'\|) \\
&\quad \le L(\|\hat{x}-x\| + \|\hat{y}-y\|),
\end{aligned}
$$

which implies that

$$V_0(t,x',y') - L(\|x-x'\| + \|y-y'\|) \le V(t,\hat{x},\hat{y}) + L(\|\hat{x}-x\| + \|\hat{y}-y\|)$$

and

$$V(t,x,y) \le V(t,\hat{x},\hat{y}) + L(\|\hat{x}-x\| + \|\hat{y}-y\|).$$

Thus, we obtain $(V1)$.

To prove $(V3)$, let $t_n \in [a,b)$ with $t_n \to t \in [a,b)$ as $n \to \infty$ and let $(x_n, y_n) \in \Omega(t_n) \times \Omega(t_n)$ with $(x_n, y_n) \to (x,y) \in \Omega(t) \times \Omega(t)$ as $n \to \infty$. Let $\sigma \in (t,b)$ and N a number such that $\sigma > t_n$ for $n \ge N$. Then we have

$$V_0(t_n,x_n,y_n) \ge \exp\left(-\int_{t_n}^{\sigma} \omega(\theta)d\theta\right)\|U(\sigma,t_n)x_n - U(\sigma,t_n)y_n\| \qquad \text{for } n \ge N.$$

Taking the inferior limit as $n \to \infty$, we have

$$\liminf_{n\to\infty} V_0(t_n,x_n,y_n) \ge \exp\left(-\int_{t}^{\sigma} \omega(\theta)d\theta\right)\|U(\sigma,t)x - U(\sigma,t)y\|.$$

By (4.1), we have $V_0(t_n,x_n,y_n) \ge \|x_n - y_n\|$ for $n \ge 1$. Taking the inferior limit as $n \to \infty$, we see that the above inequality is also valid for $\sigma = t$. Thus, we have

$$\liminf_{n\to\infty} V_0(t_n,x_n,y_n) \ge V_0(t,x,y).$$

Finally, we prove the dissipativity condition

$$D_+V(t,x,y)(A(t,x),A(t,y)) \le \omega(t)V(t,x,y) \qquad \text{for } x,y \in \Omega(t) \text{ and } t \in [a,b).$$

For this purpose, let $t \in [a,b)$ and $x,y \in \Omega(t)$. Since

$$
\begin{aligned}
&\|U(\sigma,t+h)U(t+h,t)x - U(\sigma,t+h)U(t+h,t)y\| \\
&\quad = \exp\left(\int_t^{\sigma} \omega(\theta)d\theta\right) \cdot \exp\left(-\int_t^{\sigma} \omega(\theta)d\theta\right)\|U(\sigma,t)x - U(\sigma,t)y\| \\
&\quad \le \exp\left(\int_t^{\sigma} \omega(\theta)d\theta\right) V_0(t,x,y) \\
&\quad = \exp\left(\int_t^{t+h} \omega(\theta)d\theta\right) \cdot \exp\left(\int_{t+h}^{\sigma} \omega(\theta)d\theta\right) V_0(t,x,y)
\end{aligned}
$$

for $h \in (0, b-t)$ and $\sigma \in [t+h, b)$, we have

$$V_0(t+h, U(t+h,t)x, U(t+h,t)y) \le \exp\left(\int_t^{t+h} \omega(\theta)d\theta\right) V_0(t,x,y) \tag{4.2}$$

for $h \in (0, b-t)$. Since $V(t,x,y) = V_0(t,x,y)$ for $t \in [a,b)$ and $x, y \in \Omega(t)$ and since $V(t,\cdot,\cdot)$ is Lipschitz continuous on $X \times X$ with Lipschitz constant L, by (4.2) we have

$$\begin{aligned}
&(V(t+h, x+hA(t,x), y+hA(t,y)) - V(t,x,y))/h \\
&\quad \le (V(t+h, U(t+h,t)x, U(t+h,t)y) - V(t,x,y))/h \\
&\qquad + L(\|x+hA(t,x) - U(t+h,t)x\| + \|y+hA(t,y) - U(t+h,t)y\|)/h \\
&\quad \le \frac{1}{h}\left(\exp\left(\int_t^{t+h} \omega(\theta)d\theta\right) - 1\right) V(t,x,y) \\
&\qquad + L(\|x+hA(t,x) - U(t+h,t)x\| + \|y+hA(t,y) - U(t+h,t)y\|)/h \\
&\quad \to \omega(t)V(t,x,y) \qquad \text{as } h \downarrow 0.
\end{aligned}$$

This means that the desired dissipativity condition holds true. □

Proof of the sufficiency part. By condition (Ω5), Theorem 4 asserts that for any $(\tau, z) \in \Omega$, there exists a unique global solution $u = u(\cdot; \tau, z)$ to (IVP; τ, z) on $[\tau, b)$. Define $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$ by $U(t,\tau)z = u(t;\tau,z)$ for $(\tau,z) \in \Omega$ and $t \in [\tau, b)$. Then we see that for each $(t,\tau) \in \Delta$, $U(t,\tau)$ maps $\Omega(\tau)$ to $\Omega(t)$. We immediately obtain (E1) from the uniqueness of solutions to initial value problem (IVP; τ, z). By Proposition 1, we find, noting $(V4)'$, that

$$\begin{aligned}
\|U(t,\tau)z - U(t,\tau)\hat{z}\| &\le V(t, U(t,\tau)z, U(t,\tau)\hat{z}) \\
&\le \exp\left(\int_\tau^t \omega(\theta)d\theta\right) V(\tau, z, \hat{z}) \le L \exp\left(\int_\tau^t \omega(\theta)d\theta\right) \|z - \hat{z}\|
\end{aligned}$$

for $z, \hat{z} \in \Omega(\tau)$ and $(t,\tau) \in \Delta$, namely, (E3) holds true.

It remains to show that (E2) holds true. Let $(t_n, \tau_n), (t,\tau) \in \Delta$, $z_n \in \Omega(\tau_n)$ and $z \in \Omega(\tau)$ and suppose that $(t_n, \tau_n) \to (t,\tau)$ and $z_n \to z$ as $n \to \infty$. We have to show that $u(t_n; \tau_n, z_n) = U(t_n, \tau_n)z_n \to u(t;\tau,z) = U(t,\tau)z$ as $n \to \infty$. First, we assume that $t > \tau$. Let $d \in (\tau, b)$ be a number such that $t < d$ and take $\sigma \in (\tau, t)$. Since $t_n \to t$ as $n \to \infty$, we may assume that $t_n \in [\sigma, d]$ for $n \ge 1$. Then, we deduce from Proposition 7 that $\lim_{n\to\infty} u(\cdot; \tau_n, z_n) = u(\cdot; \tau, z)$ uniformly on $[\sigma, d]$, and hence $u(t_n; \tau_n, z_n) \to u(t;\tau,z)$ as $n \to \infty$. Next, we assume that $t = \tau$. Since $u(t;\tau,z) = U(t,\tau)z = z$, we need to show that

$u(t_n; \tau_n, z_n) \to z$ as $n \to \infty$. To this end, let $M > 0$ and $R > 0$ be numbers such that $\tau + R < b$ and $\|A(s, y)\| \le M$ for $(s, y) \in \Omega \cap S_R(\tau, z)$. Since $(\tau_n, z_n) \to (\tau, z)$ as $n \to \infty$, there exists an integer $N \ge 1$ such that $\tau_n + R/2 < b$ and $(\tau_n, z_n) \in S_{R/2}(\tau, z)$ for $n \ge N$. Take $r = R/2$. Thus, we observe that if $n \ge N$, then $S_r(\tau_n, z_n) \subset S_R(\tau, z)$ and $\|A(s, y)\| \le M$ for $(s, y) \in \Omega \cap S_r(\tau_n, z_n)$. Let $\sigma \in (0, r/(M+1))$. Thus, we deduce from Theorem 3 that if $n \ge N$ then

$$\|u(s; \tau_n, z_n) - u(\hat{s}; \tau_n, z_n)\| \le M|s - \hat{s}|$$

for $s, \hat{s} \in [\tau_n, \tau_n + \sigma]$. Since $\tau_n \to \tau$ and $t_n \to t = \tau$ as $n \to \infty$, we find that $t_n \in [\tau_n, \tau_n + \sigma]$ for sufficient large n, and so the above inequality implies that

$$\|u(t_n; \tau_n, z_n) - z_n\| \le M|t_n - \tau_n|$$

for sufficient large n. Since $z_n \to z$ as $n \to \infty$, we conclude that $u(t_n; \tau_n, z_n) \to z$ as $n \to \infty$. □

5. Application to wave equations

In this section, we apply Theorem 1 to the initial value problem for nonlinear wave equation with dissipation:

$$\begin{cases} \partial_t u = \partial_x v, \quad \partial_t v = \partial_x \sigma(t, u) - \gamma v, \\ u(0, x) = u_0(x),\ v(0, x) = v_0(x) \quad \text{for } x \in \mathbf{R} \text{ and } t \in [0, \infty). \end{cases} \tag{5.1}$$

Here γ is a positive constant and $\sigma(\cdot\,,\cdot)$ a real-valued smooth function on $[0, \infty) \times \mathbf{R}$ satisfying $\sigma(t, 0) = 0$ for $t \in [0, \infty)$. We make the following assumptions on the function σ.

(i) There exists a positive constant δ_0 such that $\sigma_r(t, r) \ge \delta_0$ for $(t, r) \in [0, \infty) \times \mathbf{R}$.

(ii) There exists a constant $L_0 > 0$ such that

$$\|\sigma_r(t, \cdot)\|_{L^\infty} \le L_0, \quad \|\sigma_{rr}(t, \cdot)\|_{L^\infty} \le L_0$$
$$\text{and} \quad \|\sigma_{rrr}(t, \cdot)\|_{L^\infty} \le L_0 \qquad \text{for } t \in [0, \infty).$$

(iii) There exists a continuous integrable function $h : [0, \infty) \to [0, \infty)$ such that

$$\|\sigma_{tr}(t, \cdot)\|_{L^\infty} \le h(t) \qquad \text{for } t \in [0, \infty).$$

Let $X = L^2(\mathbf{R}) \times L^2(\mathbf{R})$ with the standard norm $\|(u, v)\| = (\|u\|_{L^2}^2 + \|v\|_{L^2}^2)^{1/2}$, and define $H : [0, \infty) \times H^2(\mathbf{R}) \times H^2(\mathbf{R}) \to \mathbf{R}$ by

$$\begin{aligned}H(t,u,v) &= H^{(0)}(t,u,v)+H^{(1)}(t,u,v)+H^{(2)}(t,u,v)\\ &= \int_{-\infty}^{\infty}\left(\int_0^u \sigma(t,r)dr+\frac{1}{2}v^2\right)dx\\ &\quad+\frac{1}{2}\int_{-\infty}^{\infty}(\sigma_r(t,u)(\partial_x u)^2+(\gamma u+\partial_x v)^2)dx\\ &\quad+\frac{1}{2}\int_{-\infty}^{\infty}(\sigma_r(t,u)(\partial_x^2 u)^2+(\gamma\partial_x u+\partial_x^2 v)^2)dx\end{aligned}$$

for $(u,v)\in H^2(\mathbf{R})\times H^2(\mathbf{R})$ and $t\in[0,\infty)$. The assumptions imply that there exist constants $C_0\geq c_0>0$ such that

$$c_0\|(u,v)\|^2_{H^2\times H^2}\leq H(t,u,v)\leq C_0\|(u,v)\|^2_{H^2\times H^2} \tag{5.2}$$

for $(u,v)\in H^2(\mathbf{R})\times H^2(\mathbf{R})$ and $t\in[0,\infty)$. The following proposition will be used in order to convert the problem (5.1) into the initial value problem for a continuous mapping $A:\Omega\ (\subset[0,\infty)\times X)\to X$.

Proposition 9. *Let $t\in[0,\infty)$ and $(u_0,v_0)\in H^2(\mathbf{R})\times H^2(\mathbf{R})$. Then there exists $\lambda_0>0$ such that for any $\lambda\in(0,\lambda_0]$, the problem*

$$(u_\lambda-u_0)/\lambda=\partial_x v_\lambda, \tag{5.3}$$

$$(v_\lambda-v_0)/\lambda=\sigma_r(t,u_0)\partial_x u_\lambda-\gamma v_\lambda \tag{5.4}$$

has a solution $(u_\lambda,v_\lambda)\in H^3(\mathbf{R})\times H^3(\mathbf{R})$ satisfying the following properties:

(i) *The family $\{(u_\lambda,v_\lambda)\}$ converges to (u_0,v_0) in $H^2(\mathbf{R})\times H^2(\mathbf{R})$ as $\lambda\downarrow 0$.*

(ii) *There exists a nondecreasing continuous function $g:[0,\infty)\to[0,\infty)$ with $g(0)=0$, depending only γ and $\sigma(\cdot,\cdot)$, such that*

$$\begin{aligned}&\frac{1}{\lambda}(H(t+\lambda,u_\lambda,v_\lambda)-H(t,u_0,v_0))\\ &\quad\leq\frac{1}{2\lambda}\left(\int_t^{t+\lambda}h(s)ds\right)\|u_\lambda\|^2_{H^2}-\gamma\delta_0\|\partial_x u_\lambda\|^2_{H^1}\\ &\qquad+(1+\lambda^2)g(\|(u_0,v_0)\|_{H^2\times H^2}\vee\|(u_\lambda,v_\lambda)\|_{H^2\times H^2})\\ &\qquad\times(\|\partial_x u_0\|_{H^1}\vee\|\partial_x u_\lambda\|_{H^1})^2\end{aligned} \tag{5.5}$$

for $\lambda\in(0,\lambda_0]$.

Here and subsequently, we use notation $a\vee b=\max\{a,b\}$ for $a,b\in\mathbf{R}$.

Proof. Let $t\in[0,\infty)$ and $(u_0,v_0)\in H^2(\mathbf{R})\times H^2(\mathbf{R})$. Define $D(L(t))=H^1(\mathbf{R})\times H^1(\mathbf{R})$ and

$$L(t)(u,v)=(\partial_x v,\sigma_r(t,u_0)\partial_x u-\gamma v)$$

for $(u, v) \in D(L(t))$. Let β_0 be a positive number such that $\beta_0 \geq L_0 \|\partial_x u_0\|_{L^\infty}/(2\sqrt{\delta_0})$. Since

$$\frac{\|\partial_x(\sigma_r(t, u_0))\|_{L^\infty}}{2\sqrt{\delta_0}} = \frac{\|\sigma_{rr}(t, u_0)\partial_x u_0\|_{L^\infty}}{2\sqrt{\delta_0}} \leq \beta_0,$$

we deduce from [8, Proposition 5.7] that $L(t) - \beta_0 I$ is m-dissipative in $X = L^2(\mathbf{R}) \times L^2(\mathbf{R})$ with inner product $((u, v), (\hat{u}, \hat{v})) = (\int_{-\infty}^{\infty} \sigma_r(t, u_0)u\hat{u} + v\hat{v}\, dx)^{1/2}$ for $(u, v), (\hat{u}, \hat{v}) \in X$. Choose $\lambda_0 > 0$ so that $\lambda_0\beta_0 < 1$. Then, for $\lambda \in (0, \lambda_0]$, $(u_\lambda, v_\lambda) := (I - \lambda L(t))^{-1}(u_0, v_0)$ satisfies (5.3) and (5.4). Note that $D(L(t)^k) = H^k(\mathbf{R}) \times H^k(\mathbf{R})$ for $k = 2, 3$. It follows from the proof of [8, Proposition 5.7] that $(u_\lambda, v_\lambda) \in D(L(t)^3)$ and $L(t)^k(u_\lambda, v_\lambda) = (I - \lambda L(t))^{-1}L(t)^k(u_0, v_0)$ for $k = 0, 1, 2$ and that the family $\{L(t)^k(u_\lambda, v_\lambda)\}$ converges to $L(t)^k(u_0, v_0)$ in X as $\lambda \downarrow 0$, for $k = 0, 1, 2$. Hence the family $\{(u_\lambda, v_\lambda)\}$ converges to (u_0, v_0) in $H^2(\mathbf{R}) \times H^2(\mathbf{R})$ as $\lambda \downarrow 0$.

We shall show (ii). Since $\sigma(t, 0) = 0$, we have $\sigma(t, u_\lambda) \in H^1(\mathbf{R})$ and $\partial_x\sigma(t, u_\lambda) = \sigma_r(t, u_\lambda)\partial_x u_\lambda$. By (5.4), we get

$$\frac{1}{\lambda}(v_\lambda - v_0) = \partial_x\sigma(t, u_\lambda) - \gamma v_\lambda + (\sigma_r(t, u_0) - \sigma_r(t, u_\lambda))\partial_x u_\lambda.$$

We multiply this equality and (5.3) by v_λ and $\sigma(t, u_\lambda)$, respectively. The sum of these two equations gives us

$$\begin{aligned}
&\frac{1}{\lambda}\sigma(t, u_\lambda)(u_\lambda - u_0) + \frac{1}{\lambda}v_\lambda(v_\lambda - v_0) \\
&\qquad = \partial_x(v_\lambda\sigma(t, u_\lambda)) - \gamma v_\lambda^2 + v_\lambda(\sigma_r(t, u_0) - \sigma_r(t, u_\lambda))\partial_x u_\lambda.
\end{aligned}$$

Integrating this equality, we have

$$\begin{aligned}
&\frac{1}{\lambda}\int_{-\infty}^{\infty} \sigma(t, u_\lambda)(u_\lambda - u_0)dx + \frac{1}{\lambda}\int_{-\infty}^{\infty} v_\lambda(v_\lambda - v_0)dx \\
&\quad = -\gamma\int_{-\infty}^{\infty} v_\lambda^2\, dx + \int_{-\infty}^{\infty} v_\lambda(\sigma_r(t, u_0) - \sigma_r(t, u_\lambda))\partial_x u_\lambda\, dx \\
&\quad \leq \frac{1}{4\gamma}\int_{-\infty}^{\infty} (\sigma_r(t, u_0) - \sigma_r(t, u_\lambda))^2(\partial_x u_\lambda)^2 dx \\
&\quad \leq \frac{L_0^2}{4\gamma}\int_{-\infty}^{\infty} (u_0 - u_\lambda)^2(\partial_x u_\lambda)^2 dx = \frac{\lambda^2 L_0^2}{4\gamma}\int_{-\infty}^{\infty} (\partial_x v_\lambda)^2(\partial_x u_\lambda)^2 dx \\
&\quad \leq \frac{\lambda^2 L_0^2}{4\gamma}\|\partial_x v_\lambda\|_{H^1}^2 \int_{-\infty}^{\infty} (\partial_x u_\lambda)^2 dx.
\end{aligned}$$

Since the function $r \to \sigma(t,r)$ is nondecreasing, we have

$$\frac{1}{\lambda}\int_{-\infty}^{\infty}\left(\int_{u_0}^{u_\lambda}\sigma(t,r)dr\right)dx + \frac{1}{2\lambda}\int_{-\infty}^{\infty}(v_\lambda^2 - v_0^2)dx$$
$$\leq \frac{\lambda^2 L_0^2}{4\gamma}\|\partial_x v_\lambda\|_{H^1}^2 \int_{-\infty}^{\infty}(\partial_x u_\lambda)^2 dx,$$

or

$$\frac{1}{\lambda}(H^{(0)}(t+\lambda, u_\lambda, v_\lambda) - H^{(0)}(t, u_0, v_0))$$
$$\leq \frac{1}{\lambda}\int_{-\infty}^{\infty}\left(\int_0^{u_\lambda}(\sigma(t+\lambda, r) - \sigma(t,r))dr\right)dx$$
$$+ \frac{\lambda^2 L_0^2}{4\gamma}\|\partial_x v_\lambda\|_{H^1}^2 \int_{-\infty}^{\infty}(\partial_x u_\lambda)^2 dx.$$

The first term on the right-hand side is estimated as follows:

$$\frac{1}{\lambda}\int_{-\infty}^{\infty}\left(\int_0^{u_\lambda}(\sigma(t+\lambda, r) - \sigma(t,r))dr\right)dx$$
$$= \frac{1}{\lambda}\int_t^{t+\lambda}\left(\int_{-\infty}^{\infty}\left(\int_0^{u_\lambda}\sigma_t(s,r)dr\right)dx\right)ds$$
$$= \frac{1}{\lambda}\int_t^{t+\lambda}\left(\int_{-\infty}^{\infty}\left(\int_0^{u_\lambda}\left(\int_0^1 \sigma_{tr}(s,\theta r)d\theta\right)r\,dr\right)dx\right)ds$$
$$\leq \frac{1}{2\lambda}\left(\int_t^{t+\lambda}h(s)ds\right)\|u_\lambda\|_{L^2}^2.$$

Hence

$$\frac{1}{\lambda}(H^{(0)}(t+\lambda, u_\lambda, v_\lambda) - H^{(0)}(t, u_0, v_0))$$
$$\leq \frac{1}{2\lambda}\left(\int_t^{t+\lambda}h(s)ds\right)\|u_\lambda\|_{L^2}^2 + \frac{\lambda^2}{4\gamma}L_0^2\|\partial_x v_\lambda\|_{H^1}^2\|\partial_x u_\lambda\|_{L^2}^2. \tag{5.6}$$

Differentiating (5.3) and (5.4), we have

$$\frac{1}{\lambda}(\partial_x u_\lambda - \partial_x u_0) = \partial_x(\partial_x v_\lambda), \tag{5.7}$$

$$\frac{1}{\lambda}((\gamma u_\lambda + \partial_x v_\lambda) - (\gamma u_0 + \partial_x v_0)) = \partial_x(\sigma_r(t, u_0)\partial_x u_\lambda). \tag{5.8}$$

We multiply (5.7) and (5.8) by $\sigma_r(t,u_0)\partial_x u_\lambda$ and $\gamma u_\lambda + \partial_x v_\lambda$, respectively. The sum of these two equations gives us

$$\frac{1}{2\lambda}\sigma_r(t,u_0)((\partial_x u_\lambda)^2 - (\partial_x u_0)^2) + \frac{1}{2\lambda}((\gamma u_\lambda + \partial_x v_\lambda)^2 - (\gamma u_0 + \partial_x v_0)^2)$$
$$\leq \partial_x(\sigma_r(t,u_0)\partial_x u_\lambda \partial_x v_\lambda) + \gamma u_\lambda \partial_x(\sigma_r(t,u_0)\partial_x u_\lambda).$$

Integrating this equality, we have

$$\frac{1}{2\lambda}\int_{-\infty}^{\infty}\sigma_r(t,u_0)((\partial_x u_\lambda)^2 - (\partial_x u_0)^2)dx$$
$$+\frac{1}{2\lambda}\int_{-\infty}^{\infty}((\gamma u_\lambda + \partial_x v_\lambda)^2 - (\gamma u_0 + \partial_x v_0)^2)dx$$
$$\leq -\gamma\int_{-\infty}^{\infty}(\partial_x u_\lambda)(\sigma_r(t,u_0)\partial_x u_\lambda)dx.$$

Thus

$$\frac{1}{\lambda}(H^{(1)}(t+\lambda,u_\lambda,v_\lambda) - H^{(1)}(t,u_0,v_0))$$
$$\leq \frac{1}{2\lambda}\int_{-\infty}^{\infty}(\sigma_r(t+\lambda,u_\lambda) - \sigma_r(t,u_0))(\partial_x u_\lambda)^2 dx - \gamma\int_{-\infty}^{\infty}\sigma_r(t,u_0)(\partial_x u_\lambda)^2 dx.$$

Since

$$|\sigma_r(t+\lambda,u_\lambda) - \sigma_r(t,u_0)| \leq |\sigma_r(t+\lambda,u_\lambda) - \sigma_r(t,u_\lambda)| + |\sigma_r(t,u_\lambda) - \sigma_r(t,u_0)|$$
$$\leq \left|\int_t^{t+\lambda}\sigma_{tr}(s,u_\lambda)ds\right| + L_0|u_\lambda - u_0| \leq \int_t^{t+\lambda}h(s)ds + \lambda L_0|\partial_x v_\lambda|, \tag{5.9}$$

we have

$$\frac{1}{\lambda}(H^{(1)}(t+\lambda,u_\lambda,v_\lambda) - H^{(1)}(t,u_0,v_0))$$
$$\leq \frac{1}{2\lambda}\left(\int_t^{t+\lambda}h(s)ds\right)\|\partial_x u_\lambda\|_{L^2}^2 + \frac{1}{2}L_0\|\partial_x v_\lambda\|_{H^1}\|\partial_x u_\lambda\|_{L^2}^2$$
$$-\gamma\delta_0\|\partial_x u_\lambda\|_{L^2}^2. \tag{5.10}$$

Differentiating (5.7) and (5.8), we have

$$\frac{1}{\lambda}(\partial_x^2 u_\lambda - \partial_x^2 u_0) = \partial_x(\partial_x^2 v_\lambda), \tag{5.11}$$

$$\frac{1}{\lambda}((\gamma\partial_x u_\lambda + \partial_x^2 v_\lambda) - (\gamma\partial_x u_0 + \partial_x^2 v_0))$$
$$= \partial_x(\sigma_{rr}(t, u_0)\partial_x u_0 \partial_x u_\lambda + \sigma_r(t, u_0)\partial_x^2 u_\lambda). \tag{5.12}$$

We multiply (5.11) and (5.12) by $\sigma_r(t, u_0)\partial_x^2 u_\lambda$ and $\gamma\partial_x u_\lambda + \partial_x^2 v_\lambda$, respectively. The sum of these two equations gives us

$$\frac{1}{2\lambda}\sigma_r(t, u_0)((\partial_x^2 u_\lambda)^2 - (\partial_x^2 u_0)^2) + \frac{1}{2\lambda}((\gamma\partial_x u_\lambda + \partial_x^2 v_\lambda)^2 - (\gamma\partial_x u_0 + \partial_x^2 v_0)^2)$$
$$\leq \partial_x(\sigma_r(t, u_0)\partial_x^2 u_\lambda \partial_x^2 v_\lambda) + \gamma\partial_x u_\lambda \partial_x(\sigma_r(t, u_0)\partial_x^2 u_\lambda)$$
$$+ (\partial_x^2 v_\lambda + \gamma\partial_x u_\lambda)\partial_x(\sigma_{rr}(t, u_0)\partial_x u_0 \partial_x u_\lambda).$$

Integrating this equality, we have

$$\frac{1}{2\lambda}\int_{-\infty}^{\infty} \sigma_r(t, u_0)((\partial_x^2 u_\lambda)^2 - (\partial_x^2 u_0)^2)dx$$
$$+ \frac{1}{2\lambda}\int_{-\infty}^{\infty} ((\gamma\partial_x u_\lambda + \partial_x^2 v_\lambda)^2 - (\gamma\partial_x u_0 + \partial_x^2 v_0)^2)dx$$
$$\leq -\gamma\int_{-\infty}^{\infty} \sigma_r(t, u_0)(\partial_x^2 u_\lambda)^2 dx + \int_{-\infty}^{\infty} (\gamma\partial_x u_\lambda + \partial_x^2 v_\lambda)\partial_x(\sigma_{rr}(t, u_0)\partial_x u_0 \partial_x u_\lambda)dx$$
$$= -\gamma\int_{-\infty}^{\infty} \sigma_r(t, u_0)(\partial_x^2 u_\lambda)^2 dx - \gamma\int_{-\infty}^{\infty} \partial_x^2 u_\lambda(\sigma_{rr}(t, u_0)\partial_x u_0 \partial_x u_\lambda)dx$$
$$+ \int_{-\infty}^{\infty} (\partial_x^2 v_\lambda)\partial_x(\sigma_{rr}(t, u_0)\partial_x u_0 \partial_x u_\lambda)dx.$$

Hence

$$\frac{1}{\lambda}(H^{(2)}(t + \lambda, u_\lambda, v_\lambda) - H^{(2)}(t, u_0, v_0))$$
$$\leq \frac{1}{2\lambda}\int_{-\infty}^{\infty} (\sigma_r(t + \lambda, u_\lambda) - \sigma_r(t, u_0))(\partial_x^2 u_\lambda)^2 dx - \gamma\int_{-\infty}^{\infty} \sigma_r(t, u_0)(\partial_x^2 u_\lambda)^2 dx$$
$$- \gamma\int_{-\infty}^{\infty} \partial_x^2 u_\lambda(\sigma_{rr}(t, u_0)(\partial_x u_0)\partial_x u_\lambda)dx$$
$$+ \int_{-\infty}^{\infty} (\partial_x^2 v_\lambda)\partial_x(\sigma_{rr}(t, u_0)\partial_x u_0 \partial_x u_\lambda)dx. \tag{5.13}$$

The third term on the right-hand side is estimated by

$$-\gamma \int_{-\infty}^{\infty} \partial_x^2 u_\lambda (\sigma_{rr}(t, u_0)(\partial_x u_0)\partial_x u_\lambda) dx$$
$$\leq \gamma L_0 \|\partial_x^2 u_\lambda\|_{L^2} \|\partial_x u_0\|_{L^\infty} \|\partial_x u_\lambda\|_{L^2} \leq \gamma L_0 \|u_0\|_{H^2} \|\partial_x u_\lambda\|_{H^1}^2.$$

Since

$$\partial_x(\sigma_{rr}(t, u_0)\partial_x u_0 \partial_x u_\lambda)$$
$$= \sigma_{rrr}(t, u_0)(\partial_x u_0)^2 \partial_x u_\lambda + \sigma_{rr}(t, u_0)\partial_x^2 u_0 \partial_x u_\lambda + \sigma_{rr}(t, u_0)\partial_x u_0 \partial_x^2 u_\lambda,$$

we have

$$\int_{-\infty}^{\infty} (\partial_x^2 v_\lambda)\partial_x(\sigma_{rr}(t, u_0)\partial_x u_0 \partial_x u_\lambda) dx$$
$$\leq L_0 \|\partial_x^2 v_\lambda\|_{L^2} (\|\partial_x u_0\|_{L^\infty}^2 \|\partial_x u_\lambda\|_{L^2}$$
$$+ \|\partial_x^2 u_0\|_{L^2} \|\partial_x u_\lambda\|_{L^\infty} + \|\partial_x u_0\|_{L^\infty} \|\partial_x^2 u_\lambda\|_{L^2})$$
$$\leq L_0 \|v_\lambda\|_{H^2} (\|u_0\|_{H^2} \|\partial_x u_0\|_{H^1} \|\partial_x u_\lambda\|_{L^2}$$
$$+ \|\partial_x u_0\|_{H^1} \|\partial_x u_\lambda\|_{H^1} + \|\partial_x u_0\|_{H^1} \|\partial_x^2 u_\lambda\|_{L^2})$$
$$\leq L_0 \|v_\lambda\|_{H^2} (\|u_0\|_{H^2} + 2) \|\partial_x u_0\|_{H^1} \|\partial_x u_\lambda\|_{H^1}.$$

We estimate the first term on the right-hand side of (5.13) by (5.9), and combine the resulting inequality and the inequalities obtained above. This yields

$$\frac{1}{\lambda}(H^{(2)}(t+\lambda, u_\lambda, v_\lambda) - H^{(2)}(t, u_0, v_0))$$
$$\leq \frac{1}{2\lambda}\left(\int_t^{t+\lambda} h(s) ds\right) \|\partial_x^2 u_\lambda\|_{L^2}^2 + \frac{L_0}{2} \|\partial_x v_\lambda\|_{H^1} \|\partial_x^2 u_\lambda\|_{L^2}^2 - \gamma\delta_0 \|\partial_x^2 u_\lambda\|_{L^2}^2$$
$$+ L_0(\gamma \|u_0\|_{H^2} + \|v_\lambda\|_{H^2}(\|u_0\|_{H^2} + 2))(\|\partial_x u_0\|_{H^1} \vee \|\partial_x u_\lambda\|_{H^1})^2.$$

Combining this inequality with (5.6) and (5.10) we observe that the desired inequality (5.5) is satisfied for the function

$$g(r) = L_0 r \left\{ \left(\frac{L_0 r}{4\gamma}\right) \vee (3 + \gamma + r) \right\} \qquad \text{for } r \geq 0. \qquad \square$$

Let c_0 be the constant in (5.2), and define $\hat{H} : [0, \infty) \times H^2(\mathbf{R}) \times H^2(\mathbf{R}) \to \mathbf{R}$ by

$$\hat{H}(t, u, v) = \exp\left(-\frac{1}{c_0}\int_0^t h(s) ds\right) H(t, u, v)$$

for $(t,u,v) \in [0,\infty) \times H^2(\mathbf{R}) \times H^2(\mathbf{R})$. Then we have

$$\hat{H}(t,u,v) \le H(t,u,v) \le \exp\left(\frac{1}{c_0}\int_0^\infty h(s)ds\right)\hat{H}(t,u,v) \tag{5.14}$$

for $(t,u,v) \in [0,\infty) \times H^2(\mathbf{R}) \times H^2(\mathbf{R})$. Since g is continuous and $g(0) = 0$, we choose a number $R_0 > 0$ so small that

$$\text{if } r \ge 0 \text{ and } r^2 \le \frac{R_0}{c_0}\exp\left(\frac{1}{c_0}\int_0^\infty h(s)ds\right) \text{ then } g(r) < \gamma\delta_0, \tag{5.15}$$

and define a subset Ω of $[0,\infty) \times X$ by

$$\Omega = \{(t,(u,v)) \in [0,\infty) \times (H^2(\mathbf{R}) \times H^2(\mathbf{R})); \hat{H}(t,u,v) \le R_0\}.$$

Let $r_0 = \sqrt{R_0/C_0}$, where C_0 is the constant in (5.2). Then, by (5.2) we have

$$S_0 := \{(u,v) \in H^2(\mathbf{R}) \times H^2(\mathbf{R}); \|(u,v)\|_{H^2\times H^2} \le r_0\} \subset \Omega(t) \tag{5.16}$$

for any $t \in [0,\infty)$, and there exists a connected component C of Ω such that $[0,\infty) \times S_0 \subset C \subset \Omega$. Let R_0' be the positive number such that $(R_0')^2 = \frac{R_0}{c_0}\exp\left(\frac{1}{c_0}\int_0^\infty h(s)ds\right)$. Then, by (5.2) and (5.14) we have

$$\Omega(t) \subset S_0' := \{(u,v) \in H^2(\mathbf{R}) \times H^2(\mathbf{R}); \|(u,v)\|_{H^2\times H^2} \le R_0'\} \tag{5.17}$$

for any $t \in [0,\infty)$. Let V be the functional on $[0,\infty) \times X \times X$ defined by

$$V(t,(u,v),(\hat{u},\hat{v})) = \left(\int_{-\infty}^{\infty} (\hat{v}-v)^2 + \left(\int_u^{\hat{u}} \sqrt{\sigma_r(t,r)}dr\right)^2 dx\right)^{1/2}$$

for $(u,v),(\hat{u},\hat{v}) \in X$ and $t \in [0,\infty)$. It is easily seen that conditions $(V1)$–$(V4)$ are satisfied. In particular, we see that for each $t \in [0,\infty)$, $V(t,\cdot,\cdot)$ is a metric on X and

$$\begin{aligned}\min\{1,\sqrt{\delta_0}\}\|(u,v)-(\hat{u},\hat{v})\| &\le V(t,(u,v),(\hat{u},\hat{v})) \\ &\le (1 \vee \sqrt{L_0})\|(u,v)-(\hat{u},\hat{v})\|\end{aligned}$$

for $(u,v),(\hat{u},\hat{v}) \in X$. Consider the operator $A : \Omega \to X$ defined by

$$A(t,(u,v)) = (\partial_x v, \partial_x \sigma(t,u) - \gamma v)$$

for $(t,(u,v)) \in \Omega$. Then the nonlinear wave equation with dissipation (5.1) is converted into the initial value problem for A. We can prove that the initial value problem for A is globally well-posed, by Theorem 1 combined with the following theorem which will be proved by a sequence of propositions.

Theorem 5. *The operator A satisfies* ($\Omega 1$)–($\Omega 4$).

In view of (5.16) and (5.17), we are in a position to state the global solvability of the nonlinear wave equation with dissipation (5.1).

Corollary 1. *For any (u_0, v_0) such that $\|(u_0, v_0)\|_{H^2 \times H^2} \le r_0$, there exists a unique time global solution $(u(\cdot), v(\cdot))$ to* (5.1) *such that*

$$(u(\cdot), v(\cdot)) \in C^1([0, \infty); L^2(\mathbf{R}) \times L^2(\mathbf{R})) \cap L^\infty(0, \infty; H^2(\mathbf{R}) \times H^2(\mathbf{R})).$$

Remark 2. *Similar results are obtained in Yamada* [23] *and Matsumura* [14].

For the proof of Theorem 5 we follow the argument in [8]. We note here that

$$\|\partial_x w\|_{L^2}^2 \le \|w\|_{L^2} \|\partial_x^2 w\|_{L^2} \qquad \text{for } w \in H^2(\mathbf{R}). \tag{5.18}$$

Proposition 10. *The operator A is continuous on Ω.*

Proof. Let $(t, (u, v)), (\hat{t}, (\hat{u}, \hat{v})) \in \Omega$. Since $\sigma(t, 0) = 0$, we have

$$\sigma(t, u(x)) - \sigma(\hat{t}, u(x)) = u(x) \int_0^1 (\sigma_r(t, \hat{\theta} u(x)) - \sigma_r(\hat{t}, \hat{\theta} u(x))) d\hat{\theta}$$

and

$$\begin{aligned}
&\|\sigma(t, u) - \sigma(\hat{t}, u)\|_{L^2}^2 \\
&\quad = \int_{-\infty}^{\infty} \left((t - \hat{t}) u(x) \int_0^1 \int_0^1 \sigma_{tr}(\hat{t} + \theta(t - \hat{t}), \hat{\theta} u(x)) d\theta d\hat{\theta} \right)^2 dx \\
&\quad \le \int_{-\infty}^{\infty} \left(|t - \hat{t}| \cdot |u(x)| \int_0^1 h(\hat{t} + \theta(t - \hat{t})) d\theta \right)^2 dx \\
&\quad = \left(\int_{\hat{t}}^t h(s) ds \right)^2 \|u\|_{L^2}^2.
\end{aligned}$$

Since $\|u\|_{L^2} \le R_0'$ by (5.17) and $\|\sigma_r(\hat{t}, \cdot)\|_{L^\infty} \le L_0$, we get

$$\begin{aligned}
\|\sigma(t, u) - \sigma(\hat{t}, \hat{u})\|_{L^2} &\le \|\sigma(t, u) - \sigma(\hat{t}, u)\|_{L^2} + \|\sigma(\hat{t}, u) - \sigma(\hat{t}, \hat{u})\|_{L^2} \\
&\le \left| \int_{\hat{t}}^t h(s) ds \right| \|u\|_{L^2} + L_0 \|u - \hat{u}\|_{L^2} \\
&\le R_0' \left| \int_{\hat{t}}^t h(s) ds \right| + L_0 \|u - \hat{u}\|_{L^2}.
\end{aligned}$$

By (5.17) we have $\|\partial_x^2(v-\hat{v})\|_{L^2} \le \|\partial_x^2 v\|_{L^2} + \|\partial_x^2\hat{v}\|_{L^2} \le 2R_0'$. Since

$$\begin{aligned}\partial_x^2\sigma(t,u(x)) &= \partial_x(\sigma_r(t,u(x))\partial_x u(x))\\ &= \sigma_{rr}(t,u(x))(\partial_x u(x))^2 + \sigma_r(t,u(x))\partial_x^2 u(x),\end{aligned}$$

we get, by using the inequality $\|w\|_{L^\infty} \le \|w\|_{H^1}$ for $w \in H^1(\mathbf{R})$,

$$\begin{aligned}\|\partial_x^2(\sigma(t,u)-\sigma(\hat{t},\hat{u}))\|_{L^2} &\le \|\partial_x^2\sigma(t,u)\|_{L^2} + \|\partial_x^2\sigma(\hat{t},\hat{u})\|_{L^2}\\ &\le L_0(\|(\partial_x u)^2\|_{L^2} + \|(\partial_x\hat{u})^2\|_{L^2}) + L_0(\|\partial_x^2 u\|_{L^2} + \|\partial_x^2\hat{u}\|_{L^2})\\ &\le L_0(\|\partial_x u\|_{L^\infty}\|\partial_x u\|_{L^2} + \|\partial_x\hat{u}\|_{L^\infty}\|\partial_x\hat{u}\|_{L^2}) + 2L_0R_0'\\ &\le 2L_0(R_0')^2 + 2L_0R_0'.\end{aligned}$$

Thus, using (5.18), we have

$$\begin{aligned}&\|A(t,(u,v)) - A(\hat{t},(\hat{u},\hat{v}))\|^2\\ &\quad\le \|\partial_x(v-\hat{v})\|_{L^2}^2 + \|\partial_x(\sigma(t,u)-\sigma(\hat{t},\hat{u})) - \gamma(v-\hat{v})\|_{L^2}^2\\ &\quad\le \|\partial_x(v-\hat{v})\|_{L^2}^2 + 2\|\partial_x(\sigma(t,u)-\sigma(\hat{t},\hat{u}))\|_{L^2}^2 + 2\gamma^2\|v-\hat{v}\|_{L^2}^2\\ &\quad\le \|v-\hat{v}\|_{L^2}\|\partial_x^2(v-\hat{v})\|_{L^2} + 2\gamma^2\|v-\hat{v}\|_{L^2}^2\\ &\qquad + 2\|\sigma(t,u)-\sigma(\hat{t},\hat{u})\|_{L^2}\|\partial_x^2(\sigma(t,u)-\sigma(\hat{t},\hat{u}))\|_{L^2}\\ &\quad\le 2R_0'\|v-\hat{v}\|_{L^2} + 2\gamma^2\|v-\hat{v}\|_{L^2}^2\\ &\qquad + 4L_0R_0'(1+R_0')\left(R_0'\left|\int_{\hat{t}}^t h(s)\,ds\right| + L_0\|u-\hat{u}\|_{L^2}\right),\end{aligned}$$

which implies the continuity of A on Ω. □

PROPOSITION 11. *Condition* (Ω2) *is satisfied for the set* Ω.

PROOF. Let $t_n \in [0,\infty)$ with $t_n \uparrow t \in [0,\infty)$ as $n \to \infty$. Let $(u,v) \in X$ and let $\{(u_n,v_n)\}$ be a sequence in X such that $(u_n,v_n) \in \Omega(t_n)$ for $n \ge 1$ and $(u_n,v_n) \to (u,v)$ in X as $n \to \infty$. We have to show that $(u,v) \in \Omega(t)$. Since the sequence $\{(u_n,v_n)\}$ is bounded in $H^2(\mathbf{R}) \times H^2(\mathbf{R})$ it follows that $(u,v) \in H^2(\mathbf{R}) \times H^2(\mathbf{R})$ and the sequence $\{(u_n,v_n)\}$ converges weakly to (u,v) in $H^2(\mathbf{R}) \times H^2(\mathbf{R})$ as $n \to \infty$. By (5.18), we see that the sequence $\{(u_n,v_n)\}$ converges to (u,v) in $H^1(\mathbf{R}) \times H^1(\mathbf{R})$ as $n \to \infty$. Moreover, $\{(u_n,v_n)\}$ converges to (u,v) in $L^\infty(\mathbf{R}) \times L^\infty(\mathbf{R})$ as $n \to \infty$. Since $\hat{H}(t_n,u_n,v_n) \le R_0$ for $n \ge 1$, we have

$$
\begin{aligned}
&R_0 \exp\left(\frac{1}{c_0}\int_0^{t_n} h(s)ds\right)\\
&\quad \geq \int_{-\infty}^{\infty}\left(\int_0^{u_n}\sigma(t_n, r)dr + \frac{1}{2}v_n^2\right)dx\\
&\qquad + \frac{1}{2}\int_{-\infty}^{\infty}(\sigma_r(t_n, u_n)(\partial_x u_n)^2 + (\gamma u_n + \partial_x v_n)^2)dx\\
&\qquad + \frac{1}{2}\int_{-\infty}^{\infty}(\sigma_r(t_n, u_n)(\partial_x^2 u_n)^2 + (\gamma\partial_x u_n + \partial_x^2 v_n)^2)dx\\
&\quad = \int_{-\infty}^{\infty}\left(\int_0^{u_n}\sigma(t, r)dr + \frac{1}{2}v_n^2\right)dx\\
&\qquad + \frac{1}{2}\int_{-\infty}^{\infty}\{\sigma_r(t, u)((\partial_x u_n)^2 + (\partial_x^2 u_n)^2) + (\gamma u_n + \partial_x v_n)^2 + (\gamma\partial_x u_n + \partial_x^2 v_n)^2\}dx\\
&\qquad + \int_{-\infty}^{\infty}\left(\int_0^{u_n}(\sigma(t_n, r) - \sigma(t, r))dr\right)dx\\
&\qquad + \frac{1}{2}\int_{-\infty}^{\infty}\{(\sigma_r(t_n, u_n) - \sigma_r(t, u))((\partial_x u_n)^2 + (\partial_x^2 u_n)^2)\} \qquad \text{for } n \geq 1. \qquad (5.19)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left|\int_{-\infty}^{\infty}\left(\int_0^{u_n}(\sigma(t_n, r) - \sigma(t, r))dr\right)dx\right|\\
&\quad = \left|\int_{-\infty}^{\infty}(t_n - t)\left(\int_0^{u_n}\left(\int_0^1\int_0^1 \sigma_{tr}(t + \theta(t_n - t), \hat{\theta}r)d\theta d\hat{\theta}\right)r\, dr\right)dx\right|\\
&\quad \leq \left|\int_{-\infty}^{\infty}(t_n - t)\left(\int_0^{u_n}\left(\int_0^1 h(t + \theta(t_n - t))d\theta\right)r\, dr\right)dx\right|\\
&\quad = \frac{\|u_n\|_{L^2}^2}{2}\left|\int_t^{t_n} h(s)ds\right|
\end{aligned}
$$

and

$$
\begin{aligned}
|\sigma_r(t_n, u_n) - \sigma_r(t, u)| &\leq |\sigma_r(t_n, u_n) - \sigma_r(t_n, u)| + |\sigma_r(t_n, u) - \sigma_r(t, u)|\\
&\leq L_0\|u_n - u\|_{L^\infty} + \left|\int_t^{t_n} h(s)ds\right|
\end{aligned}
$$

for $n \geq 1$, we have $R_0 \geq \hat{H}(t, u, v)$ by taking the inferior limit in (5.19) as $n \to \infty$. □

Proposition 12. *There exists a real-valued continuous function ω defined on $[0,\infty)$ such that*

$$D_+V(t,(u,v),(\hat{u},\hat{v}))(A(t,(u,v)),A(t,(\hat{u},\hat{v})) \le \omega(t)V(t,(u,v),(\hat{u},\hat{v}))$$

for $(u,v),(\hat{u},\hat{v}) \in \Omega(t)$ *and* $t \in [0,\infty)$.

Proof. Let $(u,v),(\hat{u},\hat{v}) \in \Omega(t)$ for $t \in [0,\infty)$. Let $(\xi,\eta),(\hat{\xi},\hat{\eta}) \in X$. Then we get

$$\begin{aligned}
&2D_+V(t,(u,v),(\hat{u},\hat{v}))((\xi,\eta),(\hat{\xi},\hat{\eta}))V(t,(u,v),(\hat{u},\hat{v}))\\
&\quad = \liminf_{h\downarrow 0}\frac{1}{h}(V(t+h,(u,v)+h(\xi,\eta),(\hat{u},\hat{v})+h(\hat{\xi},\hat{\eta}))^2 - V(t,(u,v),(\hat{u},\hat{v}))^2)\\
&\quad = \liminf_{h\downarrow 0}\frac{1}{h}\left\{\int_{-\infty}^{\infty}((\hat{v}+h\hat{\eta}-(v+h\eta))^2-(\hat{v}-v)^2)dx\right.\\
&\qquad\qquad \left. + \int_{-\infty}^{\infty}\left(\left(\int_{u+h\xi}^{\hat{u}+h\hat{\xi}}\sqrt{\sigma_r(t+h,r)}dr\right)^2 - \left(\int_u^{\hat{u}}\sqrt{\sigma_r(t,r)}dr\right)^2\right)dx\right\}\\
&\quad = \int_{-\infty}^{\infty}\left(2(\hat{v}-v)(\hat{\eta}-\eta) + 2\int_u^{\hat{u}}\sqrt{\sigma_r(t,r)}dr\left\{(\hat{\xi}\sqrt{\sigma_r(t,\hat{u})}-\xi\sqrt{\sigma_r(t,u)})\right.\right.\\
&\qquad\qquad \left.\left. + \int_u^{\hat{u}}\frac{\sigma_{tr}(t,r)}{2\sqrt{\sigma_r(t,r)}}dr\right\}\right)dx. \qquad (5.20)
\end{aligned}$$

Substituting $(\xi,\eta) = A(t,(u,v))$ and $(\hat{\xi},\hat{\eta}) = A(t,(\hat{u},\hat{v}))$ into (5.20) yields

$$\begin{aligned}
&D_+V(t,(u,v),(\hat{u},\hat{v}))(A(t,(u,v)),A(t,(\hat{u},\hat{v}))V(t,(u,v),(\hat{u},\hat{v}))\\
&\quad = \int_{-\infty}^{\infty}\left((\hat{v}-v)(\partial_x(\sigma(t,\hat{u})-\sigma(t,u))-\gamma(\hat{v}-v))\right.\\
&\qquad\qquad + \int_u^{\hat{u}}\sqrt{\sigma_r(t,r)}dr\left((\partial_x\hat{v}\sqrt{\sigma_r(t,\hat{u})}-\partial_x v\sqrt{\sigma_r(t,u)})\right.\\
&\qquad\qquad\qquad \left.\left. + \int_u^{\hat{u}}\frac{\sigma_{tr}(t,r)}{2\sqrt{\sigma_r(t,r)}}dr\right)\right)dx\\
&\quad = -\gamma\int_{-\infty}^{\infty}(\hat{v}-v)^2dx - \int_{-\infty}^{\infty}\partial_x(\hat{v}-v)(\sigma(t,\hat{u})-\sigma(t,u))dx
\end{aligned}$$

$$+\int_{-\infty}^{\infty}\left(\int_{u}^{\hat{u}}\sqrt{\sigma_r(t,r)}dr((\partial_x\hat{v}\sqrt{\sigma_r(t,\hat{u})}-\partial_x v\sqrt{\sigma_r(t,u)}))\right)dx$$

$$+\int_{-\infty}^{\infty}\left(\int_{u}^{\hat{u}}\sqrt{\sigma_r(t,r)}dr\int_{u}^{\hat{u}}\frac{\sigma_{tr}(t,r)}{2\sqrt{\sigma_r(t,r)}}dr\right)dx$$

$$=-\gamma\int_{-\infty}^{\infty}(\hat{v}-v)^2dx+\int_{-\infty}^{\infty}\left(\int_{u}^{\hat{u}}\sqrt{\sigma_r(t,r)}dr\int_{u}^{\hat{u}}\frac{\sigma_{tr}(t,r)}{2\sqrt{\sigma_r(t,r)}}dr\right)dx$$

$$+\int_{-\infty}^{\infty}\partial_x\hat{v}\int_{u}^{\hat{u}}(\sqrt{\sigma_r(t,r)}\sqrt{\sigma_r(t,\hat{u})}-\sigma_r(t,r))drdx$$

$$+\int_{-\infty}^{\infty}\partial_x v\int_{\hat{u}}^{u}(\sqrt{\sigma_r(t,r)}\sqrt{\sigma_r(t,u)}-\sigma_r(t,r))drdx.$$

The second term on the right-hand side is estimated as follows:

$$\left|\int_{-\infty}^{\infty}\left(\int_{u}^{\hat{u}}\sqrt{\sigma_r(t,r)}dr\int_{u}^{\hat{u}}\frac{\sigma_{tr}(t,r)}{2\sqrt{\sigma_r(t,r)}}dr\right)dx\right|\leq\frac{\sqrt{L_0}h(t)}{2\sqrt{\delta_0}}\int_{-\infty}^{\infty}(\hat{u}-u)^2dx.$$

The third and fourth terms are estimated as follows:

$$\left|\int_{-\infty}^{\infty}\partial_x\hat{v}\int_{u}^{\hat{u}}(\sqrt{\sigma_r(t,r)}\sqrt{\sigma_r(t,\hat{u})}-\sigma_r(t,r))drdx\right|$$

$$\leq\|\partial_x\hat{v}\|_{L^\infty}\int_{-\infty}^{\infty}\left|\int_{u}^{\hat{u}}\frac{\sqrt{\sigma_r(t,r)}(\sigma_r(t,\hat{u})-\sigma_r(t,r))}{\sqrt{\sigma_r(t,\hat{u})}+\sqrt{\sigma_r(t,r)}}dr\right|dx$$

$$\leq L_0\|\hat{v}\|_{H^2}\int_{-\infty}^{\infty}\left|\int_{u}^{\hat{u}}|\hat{u}-r|dr\right|dx=L_0\|\hat{v}\|_{H^2}\|\hat{u}-u\|^2/2$$

and

$$\left|\int_{-\infty}^{\infty}\partial_x v\int_{\hat{u}}^{u}(\sqrt{\sigma_r(t,r)}\sqrt{\sigma_r(t,u)}-\sigma_r(t,r))drdx\right|\leq L_0\|v\|_{H^2}\|\hat{u}-u\|^2/2.$$

Setting $\omega(t)=C_0'(1+h(t))$ for a suitable positive number C_0', we conclude that

$$D_+V(t,(u,v),(\hat{u},\hat{v}))(A(t,(u,v)),A(t,(\hat{u},\hat{v})))\leq\omega(t)V(t,(u,v),(\hat{u},\hat{v}))$$

for $(u,v),(\hat{u},\hat{v})\in\Omega(t)$ and $t\in[0,\infty)$. □

PROPOSITION 13. *For any* $t\in[0,\infty)$ *and* $(u_0,v_0)\in\Omega(t)$,

$$\liminf_{\lambda\downarrow 0}\frac{1}{\lambda}d((u_0,v_0)+\lambda A(t,(u_0,v_0)),\Omega(t+\lambda))=0. \tag{5.21}$$

Proof. Let $t \in [0, \infty)$ and $(u_0, v_0) \in \Omega(t)$. By (5.15) and (5.17), we note that

$$-\gamma\delta_0 + g(\|(u_0, v_0)\|_{H^2 \times H^2}) < 0. \tag{5.22}$$

By Proposition 9, there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0]$, the problem

$$\begin{cases} (u_\lambda - u_0)/\lambda = \partial_x v_\lambda, \\ (v_\lambda - v_0)/\lambda = \sigma_r(t, u_0)\partial_x u_\lambda - \gamma v_\lambda \end{cases}$$

has a solution $(u_\lambda, v_\lambda) \in H^3(\mathbf{R}) \times H^3(\mathbf{R})$ satisfying the properties (i) and (ii) in Proposition 9. If it is proved that $(u_\lambda, v_\lambda) \in \Omega(t + \lambda)$ for sufficiently small $\lambda > 0$, then the subtangential condition (5.21) is shown to be satisfied by using the property (i) in Proposition 9.

We shall prove that $(u_\lambda, v_\lambda) \in \Omega(t + \lambda)$ for sufficiently small $\lambda > 0$. By (5.2) and (5.5), we have

$$\begin{aligned} &\frac{1}{\lambda}\left(\left(1 - \frac{1}{2c_0}\int_t^{t+\lambda} h(s)ds\right)H(t + \lambda, u_\lambda, v_\lambda) - H(t, u_0, v_0)\right) \\ &\quad \le (1 + \lambda^2)g(\|(u_0, v_0)\|_{H^2 \times H^2} \vee \|(u_\lambda, v_\lambda)\|_{H^2 \times H^2})(\|\partial_x u_0\|_{H^1} \vee \|\partial_x u_\lambda\|_{H^1})^2 \\ &\qquad - \gamma\delta_0\|\partial_x u_\lambda\|_{H^1}^2 \end{aligned} \tag{5.23}$$

for $\lambda \in (0, \lambda_0]$. Choose $\lambda_1 \in (0, \lambda_0]$ so that $\frac{1}{c_0}\int_t^{t+\lambda} h(s)ds \le 1$ for $\lambda \in (0, \lambda_1]$ and $t \in [0, \infty)$. Noting that $e^{-2r} \le 1 - r$ for $0 \le r \le 1/2$, we have

$$\exp\left(-\frac{1}{c_0}\int_t^{t+\lambda} h(s)ds\right) \le 1 - \frac{1}{2c_0}\int_t^{t+\lambda} h(s)ds$$

for $\lambda \in (0, \lambda_1]$. Hence

$$\begin{aligned} &\frac{1}{\lambda}(\hat{H}(t + \lambda, u_\lambda, v_\lambda) - \hat{H}(t, u_0, v_0)) \\ &\quad \le \exp\left(-\frac{1}{c_0}\int_0^t h(s)ds\right)(-\gamma\delta_0\|\partial_x u_\lambda\|_{H^1}^2 + (1 + \lambda^2)g(\|(u_0, v_0)\|_{H^2 \times H^2} \\ &\qquad \vee \|(u_\lambda, v_\lambda)\|_{H^2 \times H^2})(\|\partial_x u_0\|_{H^1} \vee \|\partial_x u_\lambda\|_{H^1})^2) \end{aligned} \tag{5.24}$$

for $\lambda \in (0, \lambda_1]$. Since $(u_\lambda, v_\lambda) \to (u_0, v_0)$ in $H^2(\mathbf{R}) \times H^2(\mathbf{R})$ as $\lambda \downarrow 0$, we have

$$\begin{aligned} &\limsup_{\lambda \downarrow 0} \frac{1}{\lambda}(\hat{H}(t + \lambda, u_\lambda, v_\lambda) - \hat{H}(t, u_0, v_0)) \\ &\quad \le \exp\left(-\frac{1}{c_0}\int_0^t h(s)ds\right)(-\gamma\delta_0 + g(\|(u_0, v_0)\|_{H^2 \times H^2}))\|\partial_x u_0\|_{H^1}^2. \end{aligned} \tag{5.25}$$

If $\|\partial_x u_0\|_{H^1} \neq 0$, then we have $(-\gamma\delta_0 + g(\|(u_0, v_0)\|_{H^2\times H^2}))\|\partial_x u_0\|_{H^2} < 0$ by (5.22). Hence (5.25) implies that $\hat{H}(t+\lambda, u_\lambda, v_\lambda) < \hat{H}(t, u_0, v_0) \leq R_0$ and $(u_\lambda, v_\lambda) \in \Omega(t+\lambda)$ for sufficiently small $\lambda > 0$. If $\|\partial_x u_0\|_{H^1} = 0$, then (5.24) implies that

$$\begin{aligned}&\frac{1}{\lambda}(\hat{H}(t+\lambda, u_\lambda, v_\lambda) - \hat{H}(t, u_0, v_0))\\ &\quad\leq \exp\left(-\frac{1}{c_0}\int_0^t h(s)ds\right)\\ &\qquad\times(-\gamma\delta_0 + (1+\lambda^2)g(\|(u_0, v_0)\|_{H^2\times H^2} \vee \|(u_\lambda, v_\lambda)\|_{H^2\times H^2}))\|\partial_x u_\lambda\|_{H^1}^2\end{aligned}$$

for $\lambda \in (0, \lambda_1]$. Since

$$\begin{aligned}&\lim_{\lambda\downarrow 0}(-\gamma\delta_0 + (1+\lambda^2)g(\|(u_0, v_0)\|_{H^2\times H^2} \vee \|(u_\lambda, v_\lambda)\|_{H^2\times H^2}))\\ &\quad= -\gamma\delta_0 + g(\|(u_0, v_0)\|_{H^2\times H^2}) < 0,\end{aligned}$$

the right-hand side of the above inequality is less than or equal to zero for sufficient small $\lambda > 0$; hence $\hat{H}(t+\lambda, u_\lambda, v_\lambda) \leq \hat{H}(t, u_0, v_0) \leq R_0$ and $(u_\lambda, v_\lambda) \in \Omega(t+\lambda)$ for sufficient small $\lambda > 0$. □

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