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Geometric log Hodge structures on the standard log point

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ABSTRACT. We construct natural polarized log Hodge structures associated to a projective log deformation over the standard log point.

Contents

1. Introduction

Since K. Kato and S. Usui introduced the notion of polarized log Hodge structure (see [15]), the theory rapidly grew. On the other hand, to apply the vast theory to geometry, evidently necessary are results of the type that a geometric family yields polarized log Hodge structures.

More precisely, let $f : P \to S$ be a projective log smooth morphism of fs log analytic spaces. Then, the question is whether f yields polarized log Hodge structures over S. The current situation concerning this question is summarized in Section 9 of [10] by T. Kajiwara, K. Kato, and the second author of this paper. Roughly speaking, the situation is as follows: Although

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there are satisfactory, affirmative answers in the case where the base S is log smooth, there are no nontrivial example when the base is not log smooth, even when the base is the standard log point.

The authors of [10] developed the theory of log Picard varieties and log Albanese varieties under the assumption that the answer to this question is affirmative (that f is *good* in their terminology), and stated that one can expect the goodness if f is exact. Note that the last condition that f is exact is always satisfied when S is the standard log point.

In this paper, we prove the following.

THEOREM 1.1 (= THEOREM 2.7). *A projective log deformation over the standard log point is good.*

That is, f is indeed good if S is the standard log point and if P is a projective log deformation which means that P is projective and is locally isomorphic to the special fiber of a semistable family with the natural log structure (this usage of "log deformation" was introduced by J. H. M. Steenbrink in [21]). Accordingly, the theory of [10] can be applied to such an f so that we now have for such an f the log Picard variety and the log Albanese variety, and so on.

For readers' convenience, we include here a vague but more understandable statement.

COROLLARY 1.2. *Let $f : P \to S$ be a log deformation over the standard log point. Then the triple*

$$(R^q f_*^{\log}\mathbf{Z}, (R^q f_* \omega^{\cdot}_{P/S}, F), \iota)$$

is a log Hodge structure of weight q for every integer q, where $f^{\log} : (P^{\log}, \mathcal{O}_P^{\log}) \to (S^{\log}, \mathcal{O}_S^{\log})$ is the associated morphism of ringed spaces, $\omega^{\cdot}_{P/S}$ denotes the relative log de Rham complex, F the decreasing filtration induced by the stupid filtration on $\omega^{\cdot}_{P/S}$, and $\iota : R^q f_*^{\log}\mathbf{Z} \to \mathcal{O}_S^{\log} \otimes R^q f_* \omega^{\cdot}_{P/S}$ the natural morphism induced by the canonical morphism $\mathbf{Z} \hookrightarrow \mathcal{O}_P^{\log} \to \omega^{\cdot,\log}_{P/S}$.

In Section 2, we review the definition of polarized log Hodge structure and the definition of the goodness and state our main theorem 2.7. In Section 3, we review the paper [5] which shows that a projective log deformation yields nilpotent orbits. Since a nilpotent orbit is an equivalent notion of a polarized log Hodge structure, this result in [5] already associates polarized log Hodge structures to a projective log deformation. However, the construction in [5] heavily depends on the various double complex constructions à la Steenbrink, and quite far from the definition of the goodness in [10] (which does not use any double complex). Thus, as is explained in Section 4, our task is to show that the two constructions in [5] and in [10] coincide. To see this, we introduce an intermediate one, and compare it with each of two one by one. In Section 5, we prove the coincidence of the construction of [5] including double complexes with the intermediate one. There, after recalling several definitions and results in [5], we prove the commutativity of the diagram as expected. In Section 6, we give several isomorphisms concerning log Riemann–Hilbert correspondences, which is essentially a part of the theory in [7]. In Section 7, we prove the coincidence of the construction of [10] with the intermediate one. A key is a comparison between the relative log de Rham complex and the absolute one. In Section 8, by using the results obtained so far, we complete the proof of the main theorem. Finally, in Section 9, we apply the geometric machinery of [10], and define the log Picard and log Albanese varieties for a projective log deformation and state some results.

We note that it is well-known that the answer to the original question is affirmative for the non-log case, that is, the case of usual analytic spaces. This is classical but we could not find an appropriate reference, so we include a sketch of the proof as an appendix.

Notation and Terminology. For the basic notions on log geometry, see [15]. Here we review some of them. A monoid means a commutative semigroup. For an analytic space S, we denote by $\mathcal{O}_S$ the sheaf of holomorphic functions over S. A *log structure* on S is a pair of a sheaf of monoids M and a homomorphism $\alpha : M \to \mathcal{O}_S$ of sheaves of monoids such that the induced homomorphism $\alpha^{-1}(\mathcal{O}_S^\times) \to \mathcal{O}_S^\times$ is an isomorphism ([15] 2.1.1). Here we regard $\mathcal{O}_S$ as a sheaf of monoids by multiplication. Let P be a monoid. Then P^{gp} is an abelian group that represents the functor $A \mapsto \mathrm{Hom}(P, A)$ (the set of the homomorphisms from P to an abelian group A as monoids). A monoid P is said to be *fs* (fine and saturated) if it is finitely generated, if the canonical homomorphism $P \to P^{\mathrm{gp}}$ is injective and if for any $a \in P^{\mathrm{gp}}$, a belongs to the image of P whenever there is an $n \geq 1$ such that a^n belongs to the image of P ([15] 2.1.4). A log structure M on an analytic space S is *fs* if, locally on S, there exist an fs monoid P and a homomorphism $\beta : P_S \to \mathcal{O}_S$ of sheaves of monoids such that the pushout of $P_S \leftarrow \beta^{-1}(\mathcal{O}_S^\times) \to \mathcal{O}_S^\times$ endowed with the induced homomorphism to $\mathcal{O}_S$ is isomorphic to M as a log structure ([15] 2.1.5). An *fs log analytic space* is a pair of an analytic space and an fs log structure on it ([15] 2.1.5). For an fs log analytic space S, we denote by M_S the log structure on it. A morphism of fs log analytic spaces is called proper (resp. separated) if its underlying continuous map of topological spaces is so (cf. [15] 0.7.5). See 2.1.11 of [15] for the definition of log smoothness of a morphism of fs log analytic spaces. There is a natural functor from the category of fs log analytic spaces to that of ringed spaces: $S \mapsto (S^{\log}, \mathcal{O}_S^{\log})$. See [15] 2.2.3–2.2.4.

Throughout the paper, we fix a standard log point

$$0 := (\mathrm{Spec}\,\mathbf{C}, \mathbf{N} \oplus \mathbf{C}^{\times}),$$

which is an fs log analytic space. Here the structural homomorphism $M_0 = \mathbf{N} \oplus \mathbf{C}^{\times} \to \mathcal{O}_0$ sends the generator 1 of $\mathbf{N}$ to zero. This 1 as a section of M_0 is denoted by t. The topological space $0^{\log}$ is canonically identified with the unit circle $\{\alpha \in \mathbf{C} \mid \alpha\bar{\alpha} = 1\}$ by definition. Let $\log t$ be a section of the sheaf of logarithms $\mathscr{L}_0 \subset \mathcal{O}_0^{\log}$ such that $\exp(\log t) = t \in M_0^{\mathrm{gp}}$, which is determined modulo $2\pi i\mathbf{Z}$. We often identify a sheaf of abelian groups on 0 with the module of the global sections of it.

For simplicity, we often denote the pullback of a sheaf on a space by the same symbol as that for the original sheaf.

Sometimes, we omit the dot in the symbol $A^{\cdot}$ for a complex. For example, $\omega_{Y/0}$ denotes the relative log de Rham complex $\omega_{Y/0}^{\cdot}$.

2. Main theorem

In this section, we state our main theorem 2.7. For this, we recall the definition of polarized log Hodge structure and the definition of the goodness only in the case where the base is the standard log point 0, which are enough for our purpose. See [9, 2.3], [15, 2.4], and [10] for the general theories.

2.1. Let q be an integer. A *log Hodge structure* of weight q over the standard log point 0 is a triple $(H_{\mathbf{Z}}, H_{\mathcal{O}}, \iota)$, where $H_{\mathbf{Z}}$ is a locally free sheaf of finitely generated $\mathbf{Z}$-modules over $0^{\log}$, $H_{\mathcal{O}}$ is a $\mathbf{C}$-vector space regarded as a sheaf of $\mathcal{O}_0$-modules endowed with a decreasing filtration $F^{\cdot}$, and ι is an isomorphism

$$\iota : \mathcal{O}_0^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}} \cong \mathcal{O}_0^{\log} \otimes_{\mathcal{O}_0} H_{\mathcal{O}}$$

of $\mathcal{O}_0^{\log}$-modules satisfying the following conditions (i)–(iii). Let z be the point of $0^{\log}$ corresponding to 1, where we identify the set $0^{\log}$ with the set $\{\alpha \in \mathbf{C} \mid \alpha\bar{\alpha} = 1\}$.

(i) For any sufficiently shifted specialization $a : \mathcal{O}_{0,z}^{\log} \to \mathbf{C}$ (which means that $\exp(a(\log t))$ is sufficiently near to $0 \in \mathbf{C}$), the specialization $\mathbf{C} \otimes_{\mathcal{O}_{0,z}^{\log}} \iota$ gives a Hodge structure of weight q in the usual sense.

(ii) Any specialization $\mathbf{C} \otimes_{\mathcal{O}_{0,z}^{\log}} \iota$ together with the monodromy filtration $W(N)[-q]$, where N is the monodromy, is a mixed Hodge structure in the usual sense.

(iii) Griffiths transversality is satisfied, that is,

$$\nabla(F^p) \subset \omega_0^1 \otimes_{\mathcal{O}_0} F^{p-1}$$

for all $p \in \mathbf{Z}$, where $\nabla : H_{\mathcal{O}} \to \omega_0^1 \otimes_{\mathcal{O}_0} H_{\mathcal{O}}$ is induced from $d : \mathcal{O}_0^{\log} \to \omega_0^1 \otimes_{\mathcal{O}_0} \mathcal{O}_0^{\log}$.

We explain that the general definition is equivalent to the above definition in this case. Since we consider only the pure log Hodge structure, the admissibility ([9, 2.3.5 (i)]) is an empty condition. As for the condition [9, 2.3.5 (ii)], the monodromy cone (denoted by "$C(s)$" there) is isomorphic to $\mathbf{R}_{\geq 0}$, and has two faces. The part of [9, 2.3.5 (ii)] where "σ" there is trivial is equivalent to the above condition (i). The part of [9, 2.3.5 (ii)] where "σ" is the whole monodromy cone is equivalent to the above condition (ii) because for any two specializations a, b, the data $\imath \otimes_{\mathcal{O}_{0,z}^{\log}, a} \mathbf{C}$ with $W(N)[-q]$ is a mixed Hodge structure if and only if the data $\imath \otimes_{\mathcal{O}_{0,z}^{\log}, b} \mathbf{C}$ with $W(N)[-q]$ is a mixed Hodge structure (hence, (ii) is even equivalent to the condition that "There is a specialization...''). Finally, the condition [9, 2.3.5 (iii)] is equivalent to the above condition (iii).

Remark 2.2. The authors learned from K. Kato that (i) in the above definition is in fact a consequence of the other conditions. (Though we do not use this fact, we note that it makes the last sentence of 8.2 in the proof of the main theorem unnecessary.) We sketch the proof here. It is enough to prove the following:

Let N be a nilpotent endomorphism on a finite dimensional $\mathbf{R}$-vector space H and let F be a decreasing filtration on $H_{\mathbf{C}}$. Let $(W(N)[-q], F)$ be the mixed Hodge structure. Then, $\exp(iyN)F$ is a Hodge structure of weight q for any sufficiently large $y > 0$.

To see this, let $\hat{F}$ be the $\mathbf{R}$-split mixed Hodge structure associated with $(W(N)[-q], F)$ of Cattani–Kaplan–Schmid [2] (cf. [15, 6.1.2]). Let ν be the splitting of $\hat{F}$. Then we have $\mathrm{Ad}(\nu(t))N = t^{-2}N$ for any $t \in \mathbf{R}_{>0}$ and $\nu(t)F \to \hat{F}$ $(t \to \infty)$ ([15, Lemma 6.1.11 (ii)]) so that $\nu(\sqrt{y})\exp(iyN)F \to \exp(iN)\hat{F}$ $(y \to \infty)$. Since $\exp(iN)\hat{F}$ is a Hodge structure by [2, Lemma 3.12], $\nu(\sqrt{y})\exp(iyN)F$ is a Hodge structure for any sufficiently large $y > 0$. This implies that $\exp(iyN)F$ is a Hodge structure for any sufficiently large $y > 0$.

Next we review the definition of the goodness based on [10, Section 7]. Here we adopt a simplified description. Later in 9.2 and 9.3, we discuss the equivalence of the definition here and that in [10, Section 7]. Let $f : Y \to 0$ be a proper, separated and log smooth fs log analytic space over 0. Assume that for any $y \in Y$, the cokernel of $\mathbf{Z} = M_0^{\mathrm{gp}}/\mathcal{O}_0^{\times} \to M_{Y,y}^{\mathrm{gp}}/\mathcal{O}_{Y,y}^{\times}$ is torsion-free.

Definition 2.3. Let $q \geq 0$. We say that $Y/0$ is *good for* $\mathscr{H}^q$ if the following conditions (i)–(iii) are satisfied. We say that $Y/0$ is *good* if it is good for $\mathscr{H}^q$ for any q.

(i) The three homomorphisms

$$\mathcal{O}_0^{\log} \otimes_{\mathbf{Z}} R^q f_*^{\log}\mathbf{Z} \leftarrow \mathcal{O}_0^{\log} \otimes_{\mathbf{C}} \tau_*(\mathcal{O}_0^{\log} \otimes_{\mathbf{Z}} R^q f_*^{\log}\mathbf{Z}) \\ \to \mathcal{O}_0^{\log} \otimes_{\mathbf{C}} \tau_* R^q f_*^{\log} \omega_{Y/0}^{\cdot,\log} \leftarrow \mathcal{O}_0^{\log} \otimes_{\mathbf{C}} R^q f_* \omega_{Y/0}^{\cdot},$$

where the first homomorphism is induced by the adjoint map, the second one is by the natural map $f^{\log-1}\mathcal{O}_0^{\log} \to \omega_{Y/0}^{\cdot,\log}$, and the third one is induced by the composite

$$\tau_* R^q f_*^{\log} \omega_{Y/0}^{\cdot,\log} \leftarrow R^q f_* R\tau_* \omega_{Y/0}^{\cdot,\log} \xleftarrow{\sim} R^q f_* \omega_{Y/0}^{\cdot},$$

are isomorphisms.

Here, $\omega_{Y/0}^{\cdot}$ is the relative analytic log de Rham complex.

(ii) For any integer r, the natural map $R^q f_* \omega_{Y/0}^{\cdot \geq r} \to R^q f_* \omega_{Y/0}^{\cdot}$ is injective, and the natural map $R^q f_* \omega_{Y/0}^{\cdot \geq r} \to R^q f_* \omega_{Y/0}^{r}$ is surjective.

(iii) The isomorphisms in (i) together with the filtration induced by the injections in (ii) give a log Hodge structure of weight q.

2.4. We discuss the relationship with [10, Section 7].

First, note that there is a mistake in [10, 7.1]: The first arrow in the last line of ibid. p. 164 should be converted. Hence, the condition (i) there should be the same form of the above condition 2.3 (i), i.e., "The three homomorphisms. . . ."

Assume that we made this modification on [10, 7.1]. Then, our definition of goodness 2.3 is precisely equivalent to the goodness of [10, 7.1], and also equivalent to another goodness of [10, 7.2] (note that our f is always exact because the log of the base 0 is just $\mathbf{N}$). We prove these equivalences in the last section.

More strongly, for $Y \to 0$ as above, the condition 2.3 (i) always holds, which we will prove in 6.1. Similarly, both the condition [10, 7.1 (i)] and the modified (i) in [10, 7.2] also always hold. Further, these two isomorphisms in the derived category coincide, as we will see in the last section (9.3).

2.5. We summarize the current situation of the problem of the goodness. See [10, Section 9] for details. Let $f : P \to S$ be a proper, separated, vertical and log smooth morphism of fs log analytic spaces. Assume that for any $p \in P$, the cokernel of $M_{S,f(p)}^{\mathrm{gp}}/\mathcal{O}_{S,f(p)}^{\times} \to M_{P,p}^{\mathrm{gp}}/\mathcal{O}_{P,p}^{\times}$ is torsion-free.

Then, we can expect that, under some Kähler type conditions, f is good either if S is log smooth or if f is exact. For the former case, there is a satisfactory result by Kato–Matsubara–Nakayama ([13]). On the other hand, there have not been known any nontrivial example if S is not log smooth. The main theorem 2.7 in this paper provides such examples for the first time.

2.6. A morphism $f : Y \to 0$ of fs log analytic spaces is called a *log deformation* ([21]) if Y is locally isomorphic to an open subspace of

$$\mathrm{Spec}(\mathbf{C}[t_1, \dots, t_{d+1}]/(t_1 \cdots t_{d+1})) \qquad (d \geq 0)$$

endowed with the log structure defined by $\mathbf{N}^{d+1}$ sending e_j to t_j and endowed with the structure morphism sending $1 \in \mathbf{N}$ to $t_1 \cdots t_{d+1}$. Here $(e_j)_{1 \leq j \leq d+1}$ is the canonical basis of $\mathbf{N}^{d+1}$. (Equivalently, f is a log deformation if and only if Y is locally isomorphic to the log special fiber of a semistable family over a log pointed disk endowed with the natural log structure.)

A log deformation is said to be projective if the underlying space of Y is projective.

Theorem 2.7. *A projective log deformation $f : Y \to 0$ is good.*

3. Review of [5]

In this section, we briefly review [5].

3.1. Let $f : Y \to 0$ be a projective log deformation. We describe the irreducible decomposition of Y by $Y = \bigcup_{\lambda \in \Lambda} Y_\lambda$. We set

$$Y_{\underline{\lambda}} = Y_{\lambda_0} \cap Y_{\lambda_1} \cap \cdots \cap Y_{\lambda_k}$$

for a subset $\underline{\lambda} = \{\lambda_0, \lambda_1, \dots, \lambda_k\} \subset \Lambda$ (for $\underline{\lambda} = \varnothing$, we set $Y_\varnothing = Y$) as in [5, 3.2]. The inclusion $Y_{\underline{\lambda}} \to Y$ is denoted by $a_{\underline{\lambda}}$. In what follows, we often omit the symbol $(a_{\underline{\lambda}})_*$ for short. A log structure on $Y_{\underline{\lambda}}$ is induced by $a_{\underline{\lambda}} : Y_{\underline{\lambda}} \to Y$. Thus an augmented cubical log analytic space $a : Y_\bullet \to Y$ is obtained. The relative log de Rham complex $\omega_{Y_{\underline{\lambda}}/0}$ for $\underline{\lambda} \subset \Lambda$ defines a co-cubical complex $\omega_{Y_\bullet/0}$ on Y. On $Y_{\underline{\lambda}}$, a morphism of complexes

$$d \log t \wedge : \omega_{Y_{\underline{\lambda}}} \to \omega_{Y_{\underline{\lambda}}}[1]$$

is given by

$$\omega^p_{Y_{\underline{\lambda}}} \ni x \mapsto d \log t \wedge x \in \omega^{p+1}_{Y_{\underline{\lambda}}}$$

for every p. Then a complex $\mathbf{C}[u] \otimes \omega_{Y_{\underline{\lambda}}}$, where u is an indeterminate, is obtained by the differential

$$\mathrm{id} \otimes d + (2\pi\sqrt{-1})^{-1} \frac{d}{du} \otimes d \log t \wedge : \mathbf{C}[u] \otimes \omega^p_{Y_{\underline{\lambda}}} \to \mathbf{C}[u] \otimes \omega^{p+1}_{Y_{\underline{\lambda}}}$$

for every p. We denote it by $\omega_{Y_{\underline{\lambda}}}[u]$ for short. In particular, we have a complex $\omega_Y[u]$ on Y as the case of $\underline{\lambda} = \varnothing$. Thus we obtain a co-cubical complex $\omega_{Y_\bullet}[u]$ on Y.

3.2. As in [5, 5.5], we set

$$K_{\mathbf{C}} = \mathscr{C}(\omega_{Y_\bullet}[u]),$$

where $\mathscr{C}$ denotes the Čech complex of the co-cubical complex (see e.g. [5, 2.5]). We define the complex $K_{\mathbf{Q}}$ similarly by replacing $\omega_{Y_\bullet}$ with the Koszul complex on $Y_\bullet$. The increasing filtrations W on $K_{\mathbf{Q}}$ and $K_{\mathbf{C}}$, and the decreasing filtration F on $K_{\mathbf{C}}$ are defined in [5, 5.5]. Moreover the morphism $K_{\mathbf{Q}} \to K_{\mathbf{C}}$ is also defined in [5, 5.5]. Theorem 5.9 of [5] states that these data yield a mixed Hodge structure

$$(H^q(Y, K_{\mathbf{Q}}), W[q], H^q(Y, K_{\mathbf{C}}), F)$$

for every q.

3.3. The morphism

$$\frac{d}{du} \otimes \mathrm{id} : \mathbf{C}[u] \otimes \omega^p_{Y_{\underline{\lambda}}} \to \mathbf{C}[u] \otimes \omega^p_{Y_{\underline{\lambda}}}$$

induces an endomorphism of the co-cubical complex $\omega_{Y_\bullet}[u]$. Then a morphism of complexes

$$\frac{d}{du} \otimes \mathrm{id} : K_{\mathbf{C}} \to K_{\mathbf{C}}$$

is induced. We obtain a morphism

$$H^q\left(Y, \frac{d}{du} \otimes \mathrm{id}\right) : H^q(Y, K_{\mathbf{C}}) \to H^q(Y, K_{\mathbf{C}})$$

for every q, denoted by N_K in [5, 5.16]. Similarly, we obtain a morphism of complexes $K_{\mathbf{Q}} \to K_{\mathbf{Q}}$ and a morphism $H^q(Y, K_{\mathbf{Q}}) \to H^q(Y, K_{\mathbf{Q}})$ for every q. The latter is also denoted by N_K in [5, 5.16]. These morphisms are compatible with the morphism $K_{\mathbf{Q}} \to K_{\mathbf{C}}$ above. Theorem 8.16, the main result of [5], claims the following.

THEOREM 3.4. *For every q, there exists a bilinear form S_q on $H^q(Y, K_{\mathbf{Q}})$ such that*

$$(H^q(Y, K_{\mathbf{Q}}), W[q], H^q(Y, K_{\mathbf{C}}), F, N_K, S_q)$$

is a polarized mixed Hodge structure in the sense of Cattani–Kaplan–Schmid [2, Definition (2.26)].

4. Q-structures

The main step of our proof of Theorem 2.7 is to show the compatibility of the two **Q**-structures: that in 2.3 and that by [5] reviewed in Section 3.

To prove this compatibility, we introduce an intermediate **Q**-structure in this section, and later, in Sections 5 and 7, we prove that this intermediate **Q**-structure coincides with each **Q**-structure of two, respectively.

4.1. Fix an integer q. Let $H_{\mathbf{Q}} = R^q f_*^{\log}\mathbf{Q}$. Let $H_{\mathcal{O}} = H^q(Y, \omega^{\cdot}_{Y/0})$. We will prove in 6.1 that we have the quasi-isomorphisms

$$\mathcal{O}_0^{\log} \otimes H_{\mathbf{Q}} \xleftarrow{\sim} \mathcal{O}_0^{\log} \otimes \tau_*(\mathcal{O}_0^{\log} \otimes H_{\mathbf{Q}}) \xrightarrow{\sim} \mathcal{O}_0^{\log} \otimes \tau_* R^q f_*^{\log} \omega^{\cdot,\log}_{Y/0} \xleftarrow{\sim} \mathcal{O}_0^{\log} \otimes H_{\mathcal{O}} \qquad (1)$$

as in 2.3 (i).

Let z be the point of $0^{\log}$ corresponding to 1. Taking the stalk at z of (1) and specializing it by the ring homomorphism $\mathcal{O}_{0,z}^{\log} = \mathbf{C}\left[\frac{\log t}{2\pi i}\right] \to \mathbf{C}$; $\frac{\log t}{2\pi i} \mapsto 0$, we have an isomorphism

$$\mathbf{C} \otimes H_{\mathbf{Q},z} \cong H_{\mathcal{O}}. \qquad (2)$$

Then, the main part of the proof of Theorem 2.7 is the following statement.

4.1.1. The **Q**-structure defined by (2) coincides with the **Q**-structure defined by the isomorphism

$$\mathbf{C} \otimes H^q(Y, K_{\mathbf{Q}}) \cong H^q(Y, K_{\mathbf{C}}) \cong H_{\mathcal{O}} \qquad (3)$$

constructed in [5, Theorem 5.9].

4.2. To prove 4.1.1, we introduce an intermediate **Q**-structure.

Consider the following commutative diagram of ringed spaces:

$$\begin{array}{ccccccc} Y_z^{\log} & \xrightarrow{i} & Y_\infty^{\log} & \xrightarrow{\pi} & Y^{\log} & \xrightarrow{\tau} & Y \\ \downarrow{\scriptstyle f^{\log}} & & \downarrow{\scriptstyle f_\infty} & & \downarrow{\scriptstyle f^{\log}} & & \downarrow{\scriptstyle f} \\ \{0\} & \xrightarrow{i} & \mathbf{R} & \xrightarrow{\pi} & 0^{\log} & \xrightarrow{\tau} & 0, \end{array}$$

where the bottom π is the map $\mathbf{R} \ni r \mapsto e^{2\pi i r} \in \{\alpha \mid \alpha\bar{\alpha} = 1\} = 0^{\log}$, and the left two squares are cartesian.

Consider the composite of quasi-isomorphisms

$$R(\tau\pi)_*\mathbf{C} \xrightarrow{\sim} R(\tau\pi)_*\pi^{-1}\omega_Y^{\cdot,\log} \xleftarrow{\sim} \omega^{\cdot}_Y[u] \xrightarrow{\sim} \omega^{\cdot}_{Y/0} \qquad (4)$$

on Y, where the first quasi-isomorphism is by log Poincaré lemma ([14, Theorem (3.8)]), the second is by [6, Lemma 3.3], and the last one is by [14, Remark (4.10)] and sends the indeterminate u to 0.

Consider its $R^q f_*$ on 0:

$$\mathbf{C} \otimes H^q(Y_\infty^{\log}, \mathbf{Q}) = H^q(Y_\infty^{\log}, \mathbf{C}) \cong H_{\mathcal{O}}. \qquad (5)$$

Then, the statement 4.1.1 reduces to the following two statements:

4.2.1. The **Q**-structure defined by (3) coincides with the **Q**-structure defined by (5).

4.2.2. The **Q**-structure defined by (2) coincides with the **Q**-structure defined by (5).

We will prove 4.2.1 in Section 5, and 4.2.2 in Section 7.

5. Double complexes

5.1. In this section, we prove 4.2.1. The strategy is as follows. Just as Steenbrink uses the commutativity of the diagram

$$\begin{array}{ccccc} \mathbf{Q}\text{-nearby cycle} & \longrightarrow & & \longrightarrow & A_{\mathbf{Q}} \\ \downarrow & & & & \downarrow \\ \mathbf{C}\text{-nearby cycle} & \longrightarrow & \Omega_{X/S}(\log Y)\otimes \mathcal{O}_Y & \longrightarrow & A_{\mathbf{C}} \end{array}$$

in the proof of [20, Theorem (4.19)] (notation be as there), we construct the commutative diagram

$$\begin{array}{ccccc} R(\tau\pi)_*\mathbf{Q} & \longrightarrow & & \longrightarrow & K_{\mathbf{Q}} \\ \downarrow & & & & \downarrow \\ R(\tau\pi)_*\mathbf{C} & \xrightarrow{(4)} & \omega_{Y/0} & \longrightarrow & K_{\mathbf{C}}. \end{array}$$

Then, by taking $R^q f_*$, we have 4.2.1.

To this end, we recall several definitions in [6] and correct small mistakes on the sign of the differentials.

Definition 5.2. Let (K_n, W) be a family of complexes equipped with an increasing filtration W for every non-negative integer n, and $\theta_n : K_n \to K_{n+1}[1]$, $n \geq 0$, a morphism of complexes, such that the conditions (1.4.1)–(1.4.5) in [6, Definition 1.4] are satisfied. Then we set

- $D^{p,q} = K_q^{p+q+1}/W_q$ for $p, q \in \mathbf{Z}_{\geq 0}$
- $d' : D^{p,q} \to D^{p+1,q}$ is the morphism induced by $-d_{K_q}$, where d_{K_q} denotes the differential of the complex K_q
- $d'' : D^{p,q} \to D^{p,q+1}$ is induced by the morphism $-\theta_q : K_q \to K_{q+1}[1]$

for every $p, q \in \mathbf{Z}_{\geq 0}$. Then we can easily see the equality $d'd'' + d''d' = 0$. Thus we obtain a double complex $D = D(K_n, W, \theta_n)$. The associated single complex is denoted by $sD(K_n, W, \theta_n)$ and called the Steenbrink complex of $\{(K_n, W), \theta_n\}$. The Steenbrink complex $sD(K_n, W, \theta_n)$ is equipped with the increasing filtration L defined in [6, Definition 1.4] and with the decreasing filtration F defined in [6, Remark 1.6].

For the case where there exist K and $\theta : K \to K[1]$ such that $(K_n, W) = (K, W)$ and $\theta_n = \theta$ for all n, $sD(K_n, W, \theta_n)$ is simply denoted by $sD(K, W, \theta)$ in this article.

Remark 5.3. Compared with Definition 1.4 in [6], we change the signs of d' and d''. By the definition above, we obtain the formula

$$\mathrm{gr}_m^L \, sD(K_n, W, \theta_n) = \bigoplus_{\substack{q \geq 0 \\ q \geq -m}} \mathrm{gr}_{m+2q+1}^W K_q[1]$$

for every m.

Remark 5.4. As described in [6, Remark 1.7], the construction of the Steenbrink complex satisfies the natural functoriality. In particular, we have the following functoriality:

Let $\{(K_n, W), \theta_n\}$ be as above, K' a complex, and $f : K' \to K_0[1]$ a morphism of complexes satisfying the condition $\theta_0 f = 0$. Then the morphism

$$(K')^p \to D(K_n, W, \theta_n)^{p,0} = K_0^{p+1}/W_0$$

induced by f defines a morphism of complexes $K' \to sD(K_n, W, \theta_n)$.

Definition 5.5. Let $\delta : K \to K(-1)$ be a morphism of complexes. A complex $\rho(K,\delta)$ is defined by $\rho(K,\delta)^p = K^p \oplus K^{p-1}(-1)$ with the differential $d : \rho(K,\delta)^p \to \rho(K,\delta)^{p+1}$ defined by $d(x, y) = (dx, -dy + \delta(x))$ for $(x, y) \in K^p \oplus K^{p-1}(-1) = \rho(K,\delta)^p$. The morphisms of complexes $\theta : \rho(K,\delta) \to \rho(K,\delta)(1)[1]$ and $\mu : K \to \rho(K,\delta)(1)[1]$ are defined by

$$\theta(x, y) = (0, x) \in K^{p+1}(1) \oplus K^p = \rho(K,\delta)^{p+1}(1)$$

for $(x, y) \in K^p \oplus K^{p-1}(-1) = \rho(K,\delta)^p$ and

$$\mu(x) = (0, x) \in K^{p+1}(1) \oplus K^p = \rho(K,\delta)^{p+1}(1)$$

for $x \in K^p$. It is easy to check that they actually define morphisms of complexes.

For $K_n = \rho(K,\delta)(n+1)$ with the canonical filtration $W = \tau$ and $\theta_n = \theta(n+1)$, the Steenbrink complex $sD(K_n, W, \theta_n)$ is denoted by $SZ(K,\delta)$ and called the Steenbrink–Zucker complex for (K,δ). Then we obtain a morphism of complexes

$$\mu : K \to SZ(K,\delta) \tag{6}$$

by Remark 5.4 because we have $\theta_0 \mu = \theta(1)\mu = 0$ by definition.

REMARK 5.6. The definition of the morphism μ is different from that in [6, Remark 1.13] because the sign of differential of the Steenbrink complex in Definition 5.2 is changed.

5.7. Let $f : Y \to 0$ be a log deformation. We fix an injective resolution $\mathbf{Q}_{Y^{\log}} \to I$ of $\mathbf{Q}_{Y^{\log}}$. Then we have the monodromy automorphism $T : (\tau\pi)_*\pi^{-1}I \to (\tau\pi)_*\pi^{-1}I$. A subcomplex $B(I)$ of $(\tau\pi)_*\pi^{-1}I$ is defined by

$$B(I) = \bigcup_{m \geq 0} (\mathrm{Ker}(T - \mathrm{id})^{m+1}) \subset (\tau\pi)_*\pi^{-1}I$$

in [6, (3.4.1)]. The inclusion $B(I) \to (\tau\pi)_*\pi^{-1}I$ is a quasi-isomorphism by [6, Lemma 3.5]. Moreover, $\log T$ is well-defined on $B(I)$ by definition.

5.8. We consider the commutative diagram

$$\begin{array}{ccc} L^0 & \xrightarrow{\varepsilon} & L^1 \\ {\scriptstyle v_0}\downarrow & & \downarrow{\scriptstyle v_1} \\ \mathcal{O}_Y & \xrightarrow[\mathbf{e}]{} & M_Y^{\mathrm{gp}} \end{array} \tag{7}$$

of abelian sheaves on Y which appeared in [6, (5.1.1)], where L^0 and L^1 are torsion-free abelian sheaves and the morphism $\mathbf{e} : \mathcal{O}_Y \to M_Y^{\mathrm{gp}}$ is defined by $\mathbf{e}(f) = \exp(2\pi\sqrt{-1}f)$ for $f \in \mathcal{O}_Y$. As in [6, (5.1.2)], we assume that there exists an element of $\Gamma(Y, L^1)$ whose image by v_1 is $t \in \Gamma(Y, M_Y)$. We denote this element by $t \in \Gamma(Y, L^1)$ by abuse of the notation.

We obtain the Koszul complexes $\mathrm{Kos}^n(\varepsilon)$ and $\mathrm{Kos}^n(\mathbf{e})$ by

$$\mathrm{Kos}^n(\varepsilon)^p = \Gamma_{n-p}(L^0) \otimes \bigwedge^p L^1$$

$$\mathrm{Kos}^n(\mathbf{e})^p = \Gamma_{n-p}(\mathcal{O}_Y) \otimes \bigwedge^p M_Y^{\mathrm{gp}}$$

as in [21, Section 1]. (For the details, see [21], [5], [6]). An increasing filtration W on $\mathrm{Kos}^n(\varepsilon)$ is defined in [21, (2.8)]. Similarly, an increasing filtration W is defined on $\mathrm{Kos}^n(\mathbf{e})$ (see also [5, 3.15]). Moreover, the morphism

$$\Gamma_{n-p}(\mathcal{O}_Y) \otimes \bigwedge^p M_Y^{\mathrm{gp}} \ni x \otimes y \mapsto 1^{[1]}x \otimes y \in \Gamma_{n+1-p}(\mathcal{O}_Y) \otimes \bigwedge^p M_Y^{\mathrm{gp}}$$

defines a morphism of complexes $\mathrm{Kos}^n(\mathbf{e}) \to \mathrm{Kos}^{n+1}(\mathbf{e})$ as [5, (3.11.2)]. Thus we obtain an inductive system

$$\cdots \to \mathrm{Kos}^n(\mathbf{e}) \to \mathrm{Kos}^{n+1}(\mathbf{e}) \to \cdots$$

and set

$$\mathrm{Kos}_Y(M_Y) = \varinjlim_n \mathrm{Kos}^n(\mathbf{e})$$

as in [5, 3.11]. The increasing filtration W is obtained by

$$W_m \operatorname{Kos}_Y(M_Y) = \varinjlim_n W_m \operatorname{Kos}^n(\mathbf{e})$$

for every m.

The morphism

$$\Gamma_{n-p}(L^0) \otimes \bigwedge^p L^1 \ni x \otimes y \mapsto x \otimes t \wedge y \in \Gamma_{n-p}(L^0) \otimes \bigwedge^{p+1} L^1$$

defines a morphism of complexes

$$\operatorname{Kos}^n(\varepsilon) \to \operatorname{Kos}^{n+1}(\varepsilon)[1]$$

denoted by $t\wedge$ for short. Similarly, we obtain a morphism of complexes

$$t\wedge : \operatorname{Kos}^n(\mathbf{e}) \to \operatorname{Kos}^{n+1}(\mathbf{e})[1].$$

Moreover, the morphism

$$t\wedge : \operatorname{Kos}_Y(M_Y) \to \operatorname{Kos}_Y(M_Y)$$

is similarly defined in [5, (3.13.1)].

From the commutative diagram (7), the morphism of complexes

$$\operatorname{Kos}^n(\varepsilon) \to \operatorname{Kos}^n(\mathbf{e})$$

is obtained for every n. On the other hand, we have the canonical morphism

$$\operatorname{Kos}^n(\mathbf{e}) \to \operatorname{Kos}_Y(M_Y) = \varinjlim_n \operatorname{Kos}^n(\mathbf{e})$$

for every n. These morphisms preserve the filtrations W by definition. The morphism of complexes $\psi : \operatorname{Kos}_Y(M_Y) \to \omega_Y$ is defined as in [5, (3.11.3)], which preserves the filtration W on the both sides. Then we have the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{Kos}^n(\varepsilon) & \xrightarrow{t\wedge} & \operatorname{Kos}^{n+1}(\varepsilon)[1] \\
\big\downarrow & & \big\downarrow \\
\operatorname{Kos}^n(\mathbf{e}) & \xrightarrow{t\wedge} & \operatorname{Kos}^{n+1}(\mathbf{e})[1] \\
\big\downarrow & & \big\downarrow \\
\operatorname{Kos}_Y(M_Y) & \xrightarrow{t\wedge} & \operatorname{Kos}_Y(M_Y)[1] \\
\psi\big\downarrow & & \big\downarrow\psi \\
\omega_Y & \xrightarrow[(2\pi\sqrt{-1})^{-1} d\log t\wedge]{} & \omega_Y[1],
\end{array}$$

where the commutativity of the bottom square is proved in [5, (3.13.2)]. The composite $\mathrm{Kos}^n(\varepsilon) \to \omega_Y$ of the vertical arrows coincides with the morphism ϕ defined in [6, (5.1.4)].

Definition 5.9. In the situation above, we set

$$A_{\mathbf{Q}} = sD(\mathrm{Kos}_Y(M_Y), W, t\wedge), \qquad A_{\mathbf{C}} = sD(\omega_Y, W, d\log t\wedge)$$

as in [5, (5.17)].

On the other hand, we set

$$K_n = \mathrm{Kos}^{n_0+n+1}(\varepsilon)(n+1), \qquad \theta_n = (2\pi\sqrt{-1})t\wedge : K_n \to K_{n+1}[1],$$

where $n_0 = \dim Y + 1$. Thus we obtain the Steenbrink complex $sD(K_n, W, \theta_n)$.

We have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Kos}^{n_0+n+1}(\varepsilon)(n+1) & \xrightarrow{(2\pi\sqrt{-1})t\wedge} & \mathrm{Kos}^{n_0+n+2}(\varepsilon)(n+2)[1] \\
{\scriptstyle (2\pi\sqrt{-1})^{-n-1}\,\mathrm{id}}\downarrow & & \downarrow{\scriptstyle (2\pi\sqrt{-1})^{-n-2}\,\mathrm{id}} \\
\mathrm{Kos}^{n_0+n+1}(\varepsilon) & \xrightarrow{t\wedge} & \mathrm{Kos}^{n_0+n+2}(\varepsilon)[1] \\
\downarrow & & \downarrow \\
\mathrm{Kos}_Y(M_Y) & \xrightarrow{t\wedge} & \mathrm{Kos}_Y(M_Y)[1] \\
{\scriptstyle (2\pi\sqrt{-1})^{n+1}\psi}\downarrow & & \downarrow{\scriptstyle (2\pi\sqrt{-1})^{n+2}\psi} \\
\omega_Y & \xrightarrow[d\log t\wedge]{} & \omega_Y[1],
\end{array}$$

where the both composites of the vertical arrows coincide with ϕ. Therefore we obtain the commutative diagram

$$\begin{array}{ccc}
sD(K_n, W, \theta_n) & \longrightarrow & A_{\mathbf{Q}} \\
\downarrow & & \downarrow \\
A_{\mathbf{C}} & = & A_{\mathbf{C}}
\end{array} \tag{8}$$

of the Steenbrink complexes. The vertical arrows $sD(K_n, W, \theta_n) \to A_{\mathbf{C}}$ and $A_{\mathbf{Q}} \to A_{\mathbf{C}}$ are denoted by $sD(\phi)$ and α respectively.

Remark 5.10. The triple $((A_{\mathbf{Q}}, L), (A_{\mathbf{C}}, L, F), \alpha)$ above coincides with (A, W, F) defined in [5, Definition 5.18]. On the other hand, the triple

$$((sD(K_n, W, \theta_n), L), (A_{\mathbf{C}}, L, F), sD(\phi))$$

coincides with (5.1.5) in [6] except the sign of the differentials.

5.11. As in [6, (3.4)] and the proof of [6, Lemma (3.5)], we have the commutative diagram

$$\begin{array}{ccccccc} R(\tau\pi)_*\mathbf{Q} & \xrightarrow{\sim} & (\tau\pi)_*\pi^{-1}I & \xleftarrow{\sim} & B(I) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ R(\tau\pi)_*\mathbf{C} & \xrightarrow{\sim} & (\tau\pi)_*\pi^{-1}I_{\mathbf{C}} & \xleftarrow{\sim} & B(I)_{\mathbf{C}} & \xleftarrow{\sim} & \omega_Y[u] \end{array}$$

whose horizontal arrows are quasi-isomorphisms. The diagram

$$\begin{array}{ccc} \omega_Y[u] & \xrightarrow{\frac{d}{du}} & \omega_Y[u] \\ \downarrow & & \downarrow \\ B(I)_{\mathbf{C}} & \xrightarrow[\log T]{} & B(I)_{\mathbf{C}} \end{array} \tag{9}$$

is commutative as stated in [6, (3.6)]. We set $\delta = -(2\pi\sqrt{-1})^{-1}\log T$ on $B(I)$ and $\delta = -(2\pi\sqrt{-1})^{-1}\frac{d}{du}$ on $\omega_Y[u]$ for the simplicity of notation.

The morphism of complexes

$$SZ(B(I),\delta) \to SZ(B(I)_{\mathbf{C}},\delta) = SZ(B(I),\delta)_{\mathbf{C}}$$

coincides with the natural morphism induced by the base extension $\mathbf{Q} \hookrightarrow \mathbf{C}$. Moreover, we obtain the morphism of complexes

$$SZ(\omega_Y[u],\delta) \to SZ(B(I)_{\mathbf{C}},\delta)$$

by the commutative diagram (9). It turns out that this is a (filtered) quasi-isomorphism by [6, Lemma 3.7].

On the other hand, the morphism

$$\varphi : \rho(\omega_Y[u],\delta)^p \to \omega_Y^p$$

is defined by

$$\varphi(x,y) = x_0 + d\log t \wedge y_0$$

for $(x,y) \in \omega_Y^p[u] \oplus \omega_Y^{p-1}[u]$, where $x = \sum_j x_j(u^j/j!)$ and $y = \sum_j y_j(u^j/j!)$. It is easy to check that this defines a morphism of complexes $\varphi : \rho(\omega_Y[u],\delta) \to \omega_Y$. (This morphism is denoted by ψ in [6, (3.9.1)].) Moreover, we can easily check that the diagram

$$\begin{array}{ccc} \rho(\omega_Y[u],\delta) & \xrightarrow{\theta} & \rho(\omega_Y[u],\delta)[1] \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi[1]} \\ \omega_Y & \xrightarrow[d\log t\wedge]{} & \omega_Y[1] \end{array}$$

is commutative. Thus the morphism of complexes

$$SZ(\omega_Y[u],\delta) = sD(\rho(\omega_Y[u],\delta),\tau,\theta) \to sD(\omega_Y, W, d\log t\wedge) = A_{\mathbf{C}}$$

is obtained by the functoriality of the Steenbrink complexes. The diagram

$$SZ(B(I)_{\mathbf{C}},\delta) \xleftarrow{\sim} SZ(\omega_Y[u],\delta) \to A_{\mathbf{C}} \tag{10}$$

defines a morphism

$$\alpha : SZ(B(I)_{\mathbf{C}},\delta) \to A_{\mathbf{C}}$$

in the (filtered) derived category.

Theorem 5.12. *The data*

$$((A_{\mathbf{Q}},L),(A_{\mathbf{C}},L,F),\alpha), \tag{11}$$

$$((SZ(B(I),\delta),L),(A_{\mathbf{C}},L,F),\alpha), \tag{12}$$

$$((sD(K_n,W,\theta_n),L),(A_{\mathbf{C}},L,F),sD(\phi)) \tag{13}$$

are cohomological mixed Hodge complexes on Y.

Proof. For $((A_{\mathbf{Q}},L),(A_{\mathbf{C}},L,F),\alpha)$, Theorem 5.21 of [5] implies the conclusion. For the other two, the same proofs as those for Theorem 3.12 in [6] and for (5.4) in [21] work.

5.13. In order to relate (12) and (13), the commutative diagram in the derived category

$$\begin{array}{ccccc} SZ(B(I),\delta) & \xleftarrow{\sim} & SZ(\mathrm{Kos}^{n_0}(\tilde{\varepsilon}),\delta) & \xrightarrow{\sim} & sD(K_n,W,\theta_n) \\ \downarrow & & \downarrow & & \downarrow \\ A_{\mathbf{C}} & = & A_{\mathbf{C}} & = & A_{\mathbf{C}} \end{array} \tag{14}$$

is constructed in the paragraph before Proposition 5.5 and in the proof of Proposition 5.7 in [6], where $((SZ(\mathrm{Kos}^{n_0}(\tilde{\varepsilon}),\delta),L),(A_{\mathbf{C}},L,F))$ together with the middle vertical morphism is a cohomological mixed Hodge complex in [6, Proposition 5.5].

5.14. The comparison morphisms

$$A_{\mathbf{Q}} \to K_{\mathbf{Q}}, \qquad A_{\mathbf{C}} \to K_{\mathbf{C}} \tag{15}$$

defined in Definition 5.26 and Definition 5.24 in [5] give us the commutative diagram

$$\begin{array}{ccc} A_{\mathbf{Q}} & \longrightarrow & K_{\mathbf{Q}} \\ \big\downarrow & & \big\downarrow \\ A_{\mathbf{C}} & \longrightarrow & K_{\mathbf{C}} \end{array} \tag{16}$$

by [5, Lemma 5.27].

Combining the commutative diagrams (8), (14) and (16), we obtain the commutative diagram

$$\begin{array}{ccccccccc} SZ(B(I),\delta) & \longleftarrow & SZ(\mathrm{Kos}^{n_0}(\tilde{\varepsilon}),\delta) & \longrightarrow & sD(K_n, W, \theta_n) & \longrightarrow & A_{\mathbf{Q}} & \longrightarrow & K_{\mathbf{Q}} \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ A_{\mathbf{C}} & = & A_{\mathbf{C}} & = & A_{\mathbf{C}} & = & A_{\mathbf{C}} & \longrightarrow & K_{\mathbf{C}} \end{array}$$

in the derived category.

5.15. Now it suffices to construct a commutative diagram

$$\begin{array}{ccccc} R(\tau\pi)_*\mathbf{Q} & & \longrightarrow & & SZ(B(I),\delta) \\ \big\downarrow & & & & \big\downarrow \alpha \\ R(\tau\pi)_*\mathbf{C} & \xrightarrow{(4)} & \omega_{Y/0} & \longrightarrow & A_{\mathbf{C}} \end{array}$$

in the derived category.

Because we have an isomorphism

$$B(I) \to R(\tau\pi)_*\mathbf{Q}$$

in the derived category, the diagram

$$R(\tau\pi)_*\mathbf{Q} \xleftarrow{\sim} B(I) \xrightarrow{\mu} SZ(B(I),\delta) \tag{17}$$

defines a morphism $R(\tau\pi)_*\mathbf{Q} \to SZ(B(I),\delta)$ in the derived category, where μ denotes the morphism defined in (6).

On the other hand, the morphism

$$d\log t\wedge : \omega_Y \to \omega_Y[1]$$

induces a morphism $\omega_Y \to A_{\mathbf{C}}$ as in Remark 5.4. Then we obtain a morphism

$$\omega_{Y/0} \to A_{\mathbf{C}} \tag{18}$$

because the morphism $\omega_Y \to A_{\mathbf{C}}$ clearly factors through the morphism $\omega_Y \to \omega_{Y/0}$.

What we have to prove is the following:

PROPOSITION 5.16. *The morphisms* (17) *and* (18) *make the diagram* (5.15) *commutative.*

PROOF. It is sufficient to prove that the diagram

$$\begin{array}{ccccc}
R(\tau\pi)_*\mathbf{Q} & \xleftarrow{\sim} & B(I) & \xrightarrow{\mu} & SZ(B(I),\delta) \\
\downarrow & & \downarrow & & \downarrow \\
R(\tau\pi)_*\mathbf{C} & \xleftarrow{\sim} & B(I)_{\mathbf{C}} & \xrightarrow{\mu} & SZ(B(I),\delta)_{\mathbf{C}} \\
 & & \uparrow\wr & & \uparrow\wr \\
 & & \omega_Y[u] & \xrightarrow{\mu} & SZ(\omega_Y[u],\delta) \\
 & & \downarrow & & \downarrow{\scriptstyle sD(\varphi)} \\
 & & \omega_{Y/0} & \longrightarrow & A_{\mathbf{C}}
\end{array} \tag{19}$$

is commutative because the morphism (4) is represented by

$$R(\tau\pi)_*\mathbf{C} \simeq (\tau\pi)_*\pi^{-1}I_{\mathbf{C}} \xleftarrow{\sim} B(I)_{\mathbf{C}} \xleftarrow{\sim} \omega_Y[u] \to \omega_{Y/0}$$

in the derived category and because the morphism $\alpha : SZ(B(I),\delta) \to A_{\mathbf{C}}$ is represented as the composite of the right vertical arrows in (19) in the derived category. The squares except the bottom are trivially commutative by definition.

The diagram

$$\begin{array}{ccc}
\omega_Y^p[u] & \xrightarrow{\mu} & \omega_Y^{p+1}[u] \oplus \omega_Y^p[u] \\
\downarrow & & \downarrow{\scriptstyle\varphi} \\
\omega_Y^p & \xrightarrow[d\log t\wedge]{} & \omega_Y^{p+1}
\end{array}$$

is commutative because we have

$$\varphi(\mu(x)) = \varphi(0,x) = d\log t \wedge x_0$$

for $x = \sum x_j(u^j/j!) \in \omega_Y^p[u]$. Then we can easily check the commutativity of the bottom square in the diagram (19).

5.17. Now we prove the coincidence of the morphisms N_K on $H^q(Y,K_{\mathbf{Q}})$ and the logarithm of the monodromy automorphism on $H_{\mathbf{Q},z}$ under the identification above.

It is known that the monodromy automorphism on the stalk $H_{\mathbf{Q},z}$ is induced from the monodromy automorphism on $\Gamma(\mathbf{R},\pi^{-1}H_{\mathbf{Q}}) \simeq (\tau\pi)_*\pi^{-1}H_{\mathbf{Q}}$

(see e.g. Iversen [8, IV Theorem 9.7]). Thus the monodromy automorphism is induced by

$$T : (\tau\pi)_*\pi^{-1}I \to (\tau\pi)_*\pi^{-1}I$$

because

$$(\tau\pi)_*\pi^{-1}H_{\mathbf{Q}} \simeq (\tau\pi)_*\pi^{-1}R^qf_*^{\log}\mathbf{Q} \simeq (\tau\pi)_*R^q(f_\infty)_*\mathbf{Q}$$
$$\simeq R^q(f\tau\pi)_*\mathbf{Q} \simeq R^qf_*(\tau\pi)_*\pi^{-1}I \simeq H^q(Y, B(I))$$

by the base change.

Therefore, it is sufficient to prove that $\log T$ on $H^q(Y, B(I)_{\mathbf{C}})$ coincides with N_K on $H^q(Y, K_{\mathbf{C}})$ under the identification induced by the morphism

$$B(I)_{\mathbf{C}} \xleftarrow{\sim} \omega_Y[u] \to \omega_{Y/0} \to A_{\mathbf{C}} \to K_{\mathbf{C}}$$

in the derived category. We already know that $\log T$ on $B(I)_{\mathbf{C}}$ is identified with $\frac{d}{du}$ on $\omega_Y[u]$ as in [6]. On the other hand, the canonical morphism $a : Y_\bullet \to Y$ induces the morphism

$$a^* : \omega_Y[u] \to K_{\mathbf{C}}$$

by definition. It is easy to see that the diagram

$$\begin{array}{ccc} \omega_Y[u] & \xrightarrow{\frac{d}{du}} & \omega_Y[u] \\ {\scriptstyle a^*}\downarrow & & \downarrow{\scriptstyle a^*} \\ K_{\mathbf{C}} & \xrightarrow[\frac{d}{du}\otimes \mathrm{id}]{} & K_{\mathbf{C}} \end{array}$$

is commutative. Therefore the following lemma implies the conclusion.

Lemma 5.18. *The morphism*

$$H^q(Y, a^*) : H^q(Y, \omega_Y[u]) \to H^q(Y, K_{\mathbf{C}})$$

coincides with the isomorphism induced by the morphisms

$$\omega_Y[u] \to \omega_{Y/0} \to A_{\mathbf{C}} \to K_{\mathbf{C}}$$

above for any q.

Proof. The canonical morphism $a : Y_\bullet \to Y$ induces the morphism

$$a^* : \omega_{Y/0} \to \mathscr{C}(\omega_{Y_\bullet/0})$$

as in [5, 5.7], where $\mathscr{C}$ denotes the Čech complex of the co-cubical complex $\omega_{Y_\bullet/0}$ as in 3.2 (see e.g. [5, 2.5]). Similarly to the morphism $\omega_Y[u] \to \omega_{Y/0}$

in (4), we have a morphism of complexes

$$\omega_{Y_{\underline{\lambda}}}[u] \to \omega_{Y_{\underline{\lambda}}/0}$$

for every $\underline{\lambda} \subset \Lambda$. They form a morphism of co-cubical complexes $\omega_{Y_\bullet}[u] \to \omega_{Y_\bullet/0}$. Thus the morphism of complexes

$$K_{\mathbf{C}} \to \mathscr{C}(\omega_{Y_\bullet/0})$$

is obtained in [5, 5.5]. As in the proof of Theorem 5.29 in [5], we can easily check that the diagram

$$\begin{array}{ccccc} \omega_Y[u] & \longrightarrow & \omega_{Y/0} & \longrightarrow & A_{\mathbf{C}} \\ {\scriptstyle a^*}\downarrow & & {\scriptstyle a^*}\downarrow & & \downarrow \\ K_{\mathbf{C}} & \longrightarrow & \mathscr{C}(\omega_{Y_\bullet/0}) & \longleftarrow & K_{\mathbf{C}} \end{array}$$

is commutative. Therefore we obtain the conclusion because the morphism $K_{\mathbf{C}} \to \mathscr{C}(\omega_{Y_\bullet/0})$ above induces the isomorphism

$$H^q(Y, K_{\mathbf{C}}) \xrightarrow{\sim} H^q(Y, \mathscr{C}(\omega_{Y_\bullet/0}))$$

for any q by Theorem 5.9 in [5].

6. Log Riemann–Hilbert correspondences

In this section, we prove that the condition of 2.3 (i) is in fact always satisfied. This is essentially a part of [7]. In the course of the proof, we also prove some facts which will be used to prove 4.2.2.

Proposition 6.1. *Let $f : Y \to 0$ be a proper, separated and log smooth fs log analytic space over* 0. *Assume that for any $y \in Y$, the cokernel of $\mathbf{Z} = M_0^{\mathrm{gp}}/\mathcal{O}_0^\times \to M_{Y,y}^{\mathrm{gp}}/\mathcal{O}_{Y,y}^\times$ is torsion-free. Then, $Y \to 0$ satisfies the condition* (i) *in* 2.3.

For a later use, we prove the following generalized statement.

Proposition 6.2. *Let $f : P \to S$ be a proper, separated, exact and log smooth morphism of fs log analytic spaces. Assume that for any $p \in P$, the cokernel of $M_{S,f(p)}^{\mathrm{gp}}/\mathcal{O}_{S,f(p)}^\times \to M_{P,p}^{\mathrm{gp}}/\mathcal{O}_{P,p}^\times$ is torsion-free. Let $q \geq 0$. Then the following hold.*

(i) *The three homomorphisms*

$$\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} R^q f_*^{\log} \mathbf{Z} \leftarrow \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} R^q f_*^{\log} \mathbf{Z})$$
$$\to \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \tau_* R^q f_*^{\log} \omega_{P/S}^{\cdot,\log} \leftarrow \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} R^q f_* \omega_{P/S}^{\cdot},$$

where the first homomorphism is induced by the adjoint map, the second one is by the natural map $f^{\log -1}\mathcal{O}_S^{\log} \to \omega_{P/S}^{\cdot,\log}$ *together with the projection formula, and the third one is induced by the composite*

$$\tau_* R^q f_*^{\log} \omega_{P/S}^{\cdot,\log} \leftarrow R^q f_* R\tau_* \omega_{P/S}^{\cdot,\log} \overset{\sim}{\leftarrow} R^q f_* \omega_{P/S}^{\cdot},$$

are isomorphisms.

(ii) *Two homomorphisms*

$$\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} R^q f_*^{\log} \mathbf{Z} \to R^q f_*^{\log} \omega_{P/S}^{\cdot,\log} \leftarrow \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} R^q f_* \omega_{P/S}^{\cdot},$$

where the first homomorphism is induced by the projection formula and the natural map $f^{\log -1}\mathcal{O}_S^{\log} \to \omega_{P/S}^{\cdot,\log}$ *and the second one is by the projection formula, are isomorphisms. Further the composite of these two coincides with the composite of three in* (i).

6.3. We prepare for the proof of the above proposition.

First we show the following vanishing results.

$$R^p \tau_* R^q f_*^{\log} \mathcal{O}_S^{\log} = 0 \qquad (p > 0). \tag{20}$$

$$R^p \tau_* R^q f_*^{\log} \omega_{P/S}^{\cdot,\log} = 0 \qquad (p > 0). \tag{21}$$

As for (20), we note that, by [7, Theorem (6.3)], $R^q f_*^{\log} \mathcal{O}_S^{\log}$ is isomorphic to $\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} V^q$ for some $\mathcal{O}_S$-module V^q. Then, [7, Proposition (3.7) (3)] implies the desired vanishing (20).

As for (21), we use the following proposition.

Recall that a morphism $f : P \to S$ is said to be log injective if for any $p \in P$, the homomorphism $M_{S,f(p)}/\mathcal{O}_{S,f(p)}^{\times} \to M_{P,p}/\mathcal{O}_{P,p}^{\times}$ is injective (cf. [18, Definition (5.5.1)]).

Proposition 6.4. *Let* $f : P \to S$ *be a proper, separated and log injective morphism of fs log analytic spaces. Assume that for any* $p \in P$*, the cokernel of* $M_{S,f(p)}^{\mathrm{gp}}/\mathcal{O}_{S,f(p)}^{\times} \to M_{P,p}^{\mathrm{gp}}/\mathcal{O}_{P,p}^{\times}$ *is torsion-free. Let* F *be an* $\mathcal{O}_P$*-module. Then the natural homomorphism*

$$\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} Rf_* F \to Rf_*^{\log}(\mathcal{O}_P^{\log} \otimes_{\tau^{-1}\mathcal{O}_P} \tau^{-1} F)$$

is a quasi-isomorphism.

We remark that the case where f is exact and $F = \mathcal{O}_P$ of this proposition is explained in [10, 8.13]. The proof below is essentially the same.

Proof. Factor f into $P \xrightarrow{v} P' \xrightarrow{f'} S$ such that the underlying morphism of v is the identity and that f' is strict. Note that both v and f' are proper, separated and log injective. Then, we have

$$\mathcal{O}_{P'}^{\log} \otimes_{\mathcal{O}_P} F \xrightarrow{\sim} Rv_*^{\log}(\mathcal{O}_P^{\log} \otimes_{\mathcal{O}_P} F), \tag{22}$$

i.e., 6.4 holds for v. This is a generalization of [16, Lemma 4.5], and proved at stalks as follows. Let $s \in P^{\log}$. Let $s' \in P'^{\log}$ and $p \in P$ be its images. Let $B := F_p$. Let $K := \mathrm{Hom}(C, \mathbf{Z})$, where C is the cokernel of $M_{S,f(p)}^{\mathrm{gp}}/\mathcal{O}_{S,f(p)}^{\times} \to M_{P,p}^{\mathrm{gp}}/\mathcal{O}_{P,p}^{\times}$. Then, K acts on the stalk $(\mathcal{O}_P^{\log} \otimes_{\mathcal{O}_P} F)_s = \mathcal{O}_{P,s}^{\log} \otimes_{\mathcal{O}_{P,p}} B$ and we can see that the cohomology $H^q(K, (\mathcal{O}_P^{\log} \otimes_{\mathcal{O}_P} F)_s)$ is zero for $q > 0$ and is naturally isomorphic to $\mathcal{O}_{P',s'}^{\log} \otimes_{\mathcal{O}_{P,p}} B = (\mathcal{O}_{P'}^{\log} \otimes_{\mathcal{O}_P} F)_{s'}$ for $q = 0$ (cf. the case where $n = 1$ of [7, Lemma (8.6.3.1) (1)]). Here we use the assumption of torsion-freeness and log injectiveness. Hence, we have (22) and

$$Rf_*'^{\log}(\mathcal{O}_{P'}^{\log} \otimes_{\mathcal{O}_P} F) \xrightarrow{\sim} Rf_*^{\log}(\mathcal{O}_P^{\log} \otimes_{\mathcal{O}_P} F).$$

The rest is to show that

$$\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} Rf_*' F \xrightarrow{\sim} Rf_*'^{\log}(\mathcal{O}_{P'}^{\log} \otimes_{\mathcal{O}_P} F),$$

i.e., that 6.4 holds also for f'. Since $\mathcal{O}_{P'}^{\log} = \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \mathcal{O}_P$ ([7, Lemma (3.6) (1)]), we have $\mathcal{O}_{P'}^{\log} \otimes_{\mathcal{O}_P} F = \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} F$. Further, since $P'^{\log} = S^{\log} \times_S P$, the proper base change theorem and the projection formula imply the desired result.

By 6.4, we have

$$\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} R^q f_* \omega_{P/S}^{\cdot} \xrightarrow{\sim} R^q f_*^{\log} \omega_{P/S}^{\cdot,\log}. \tag{23}$$

Hence, by [7, Proposition (3.7) (3)], we get (21).

6.5. We prove the proposition 6.2 (i). First, by [7, Theorem (6.3)] and its proof, we know that the composite

$$\begin{aligned}\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} Rf_*^{\log}\mathbf{Z} &\xleftarrow{\sim} \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} Rf_*^{\log}\mathbf{Z}) \to \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} R\tau_* Rf_*^{\log}\mathcal{O}_S^{\log} \\ &\to \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} R\tau_* Rf_*^{\log}\omega_{P/S}^{\cdot,\log} = \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} Rf_* R\tau_* \omega_{P/S}^{\cdot,\log} \\ &\xleftarrow{\sim} \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} Rf_* \omega_{P/S}^{\cdot}\end{aligned} \tag{24}$$

is an isomorphism in the derived category.

Here, by (20), the second map is a quasi-isomorphism. Hence, the third one is also a quasi-isomorphism. Taking the q-th cohomology $\mathcal{H}^q$, we have the commutative diagram

$$\begin{array}{ccccccc} \tau^*\mathcal{H}^q(\tau_*(Rf_*^{\log}\mathcal{O}_S^{\log})) & \xrightarrow{\sim} & \tau^* R^q\tau_* Rf_*^{\log}\mathcal{O}_S^{\log} & \xrightarrow{\sim} & \tau^* R^q f_* R\tau_*\omega_{P/S}^{\cdot,\log} & \xleftarrow{\sim} & \tau^* R^q f_*\omega_{P/S}^{\cdot} \\ \simeq\downarrow & & \downarrow & & \downarrow & & \\ \mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} R^q f_*^{\log}\mathbf{Z} & \xleftarrow{\mathrm{adj}} & \tau^*\tau_* R^q f_*^{\log}\mathcal{O}_S^{\log} & \longrightarrow & \tau^*\tau_* R^q f_*^{\log}\omega_{P/S}^{\cdot,\log}. & & \end{array}$$

Here, $\tau^*(-)$ means $\mathcal{O}_S^{\log} \otimes \tau^{-1}(-)$. By (20) and (21), the middle vertical arrow and the right vertical arrow are quasi-isomorphisms, respectively. Hence the bottom arrows are also quasi-isomorphisms. This completes the proof of 6.2 (i).

6.6. We prove 6.2 (ii). It is enough to show that the composite of quasi-isomorphisms in (24) coincides with the composite

$$\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} Rf_*^{\log}\mathbf{Z} \to Rf_*^{\log}\omega_{P/S}^{\cdot,\log} \xleftarrow{\sim} \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} Rf_*\omega_{P/S}^{\cdot}, \tag{25}$$

where the latter morphism is a quasi-isomorphism by (23).

First we have the commutative diagram

$$\begin{array}{ccccc}
\mathcal{O}_S^{\log} \otimes R\tau_* Rf_*^{\log}\omega_{P/S}^{\cdot,\log} & = & \mathcal{O}_S^{\log} \otimes Rf_* R\tau_*\omega_{P/S}^{\cdot,\log} & \xleftarrow{\sim} & \mathcal{O}_S^{\log} \otimes Rf_*\omega_{P/S}^{\cdot} \\
{\scriptstyle\text{adj}}\downarrow & & \downarrow & & \downarrow \\
Rf_*^{\log}\omega_{P/S}^{\cdot,\log} & \xleftarrow{\text{adj}} & Rf_*^{\log}(\mathcal{O}_S^{\log} \otimes R\tau_*\omega_{P/S}^{\cdot,\log}) & \xleftarrow{\sim} & Rf_*^{\log}(\mathcal{O}_S^{\log} \otimes \omega_{P/S}^{\cdot}),
\end{array} \tag{26}$$

where the top row is nothing but the second and the third lines in (24) and the composite of the right vertical arrow and the bottom horizontal arrows coincides with the latter morphism in (25). Since the latter morphism in (25) is a quasi-isomorphism, the left vertical arrow in (26) is a quasi-isomorphism.

Second we have another commutative diagram

$$\begin{array}{ccccc}
\mathcal{O}_S^{\log} \otimes \tau_*(\mathcal{O}_S^{\log} \otimes Rf_*^{\log}\mathbf{Z}) & \xrightarrow{\sim} & \mathcal{O}_S^{\log} \otimes R\tau_* Rf_*^{\log}\mathcal{O}_S^{\log} & \xrightarrow{\sim} & \mathcal{O}_S^{\log} \otimes R\tau_* Rf_*^{\log}\omega_{P/S}^{\cdot,\log} \\
\simeq\downarrow & & \downarrow{\scriptstyle\text{adj}} & & \downarrow\simeq \\
\mathcal{O}_S^{\log} \otimes Rf_*^{\log}\mathbf{Z} & = & \mathcal{O}_S^{\log} \otimes Rf_*^{\log}\mathbf{Z} & \longrightarrow & Rf_*^{\log}\omega_{P/S}^{\cdot,\log},
\end{array} \tag{27}$$

where the left vertical arrow and the top row are the ones which appeared in (24), the right vertical arrow is the same as the left vertical one in (26), and the bottom horizontal arrow is the former morphism in (25). This implies the desired compatibility and that the former morphism in (25) is a quasi-isomorphism, which completes the proof of 6.2.

7. Log de Rham complexes: absolute and relative

In this section, we prove 4.2.2. Let the notation be as in Section 4.

7.1. We define a homomorphism $s(Y) : f^{\log-1}\mathcal{O}_0^{\log} \to R\pi_*\mathbf{C}$ as the adjoint of the homomorphism of the constant sheaves $\pi^{-1}f^{\log-1}\mathcal{O}_0^{\log} = \mathbf{C}\left[\frac{\log t}{2\pi i}\right] \to \mathbf{C}$ on $Y_\infty^{\log}$ sending $\frac{\log t}{2\pi i}$ to 0. Note that $s(Y)$ is the pullback of $s(0)$ in the sense that

$s(Y)$ coincides with $f^{\log-1}\mathcal{O}_0^{\log} \xrightarrow{f^{\log-1}(s(0))} f^{\log-1}R\pi_*\mathbf{C} \xrightarrow{\sim} R\pi_*\mathbf{C}$, where the last quasi-isomorphism is by the facts that $Y^{\log}$ is a topological manifold and that $\pi : Y_\infty^{\log} \to Y^{\log}$ is locally trivial with discrete fibers. Note that we also have $R\pi_*\mathbf{C} = \pi_*\mathbf{C}$ on $Y^{\log}$ by the same reason.

We will reduce 4.2.2 to the following:

Proposition 7.2. *The composite*

$$R\tau_* f^{\log-1}\mathcal{O}_0^{\log} \to R\tau_*\omega_{Y/0}^{\cdot,\log} \xleftarrow{\sim} \omega_{Y/0}^{\cdot} \tag{28}$$

induced by the natural homomorphism $f^{\log-1}\mathcal{O}_0^{\log} \to \omega_{Y/0}^{\cdot,\log}$ *coincides with the composite*

$$\begin{aligned} R\tau_* f^{\log-1}\mathcal{O}_0^{\log} &\xrightarrow{R\tau_*(s(Y))} R(\tau\pi)_*\mathbf{C} \xrightarrow{\sim} R(\tau\pi)_*\pi^{-1}\omega_Y^{\cdot,\log} \\ &\xleftarrow{\sim} \omega_Y^{\cdot}[u] \xrightarrow{\sim} \omega_{Y/0}^{\cdot} \end{aligned} \tag{29}$$

of the morphism $R\tau_*(s(Y))$ *and the morphism* (4) *in* 4.2.

7.3. We prove 7.2 in several steps.

First, we introduce an extended log de Rham complex. Let u be an indeterminate. Consider the complex $\omega_Y^{\cdot,\log}[u]$ on $Y^{\log}$ whose component of degree p is $\omega_Y^{p,\log} \otimes_{\mathbf{C}} \mathbf{C}[u]$, where $\mathbf{C}[u]$ is the constant sheaf, and whose derivation is defined by $u \mapsto \frac{d\log t}{2\pi i} \in \omega_Y^{1,\log}$. Let $e : f^{\log-1}\mathcal{O}_0^{\log} \to \omega_Y^{\cdot,\log}[u]$ be the morphism sending $\frac{\log t}{2\pi i}$ to $\frac{\log t}{2\pi i} - u$. Then, the composite

$$f^{\log-1}\mathcal{O}_0^{\log} \xrightarrow{e} \omega_Y^{\cdot,\log}[u] \xrightarrow{u\mapsto 0} \omega_{Y/0}^{\cdot,\log}$$

is nothing but the natural map $f^{\log-1}\mathcal{O}_0^{\log} \to \omega_{Y/0}^{\cdot,\log}$.

7.4. Next, we define a morphism of complexes

$$w : \pi^{-1}\omega_Y^{\cdot,\log}[u] \to \pi^{-1}\omega_Y^{\cdot,\log}$$

on $Y_\infty^{\log}$ by sending u to $\frac{\log t}{2\pi i} \in \mathcal{O}_Y^{\log}$. The well-definedness is shown as follows. For any p, let $u^m\eta \in \pi^{-1}\omega_Y^{p,\log}[u]$, where $\eta \in \pi^{-1}\omega_Y^{p,\log}$. Then,

$$d(u^m\eta) = mu^{m-1}\frac{d\log t}{2\pi i} \wedge \eta + u^m\, d\eta.$$

On the other hand, $w(u^m\eta) = \left(\frac{\log t}{2\pi i}\right)^m \eta$, and

$$d\left(\left(\frac{\log t}{2\pi i}\right)^m \eta\right) = m\left(\frac{\log t}{2\pi i}\right)^{m-1}\frac{d\log t}{2\pi i} \wedge \eta + \left(\frac{\log t}{2\pi i}\right)^m d\eta.$$

Hence, $wd(u^m\eta) = dw(u^m\eta)$.

The diagram

$$\begin{array}{ccc} \pi^{-1}f^{\log-1}\mathcal{O}_0^{\log} & \xrightarrow{\pi^{-1}(e)} & \pi^{-1}\omega_Y^{\cdot,\log}[u] \\ \downarrow & & \downarrow{\scriptstyle w} \\ \mathbf{C} & \xrightarrow{\sim} & \pi^{-1}\omega_Y^{\cdot,\log}, \end{array}$$

where the left vertical arrow is the morphism defined by $\frac{\log t}{2\pi i} \mapsto 0$, and the bottom quasi-isomorphism is by the log Poincaré lemma ([14, Theorem (3.8)]), is commutative. This is seen by $w\left((\pi^{-1}(e))\left(\frac{\log t}{2\pi i}\right)\right) = w\left(\frac{\log t}{2\pi i} - u\right) = \frac{\log t}{2\pi i} - \frac{\log t}{2\pi i} = 0$.

By taking the adjoint, we have a commutative diagram

$$\begin{array}{ccc} f^{\log-1}\mathcal{O}_0^{\log} & \xrightarrow{e} & \omega_Y^{\cdot,\log}[u] \\ {\scriptstyle s(Y)}\downarrow & & \downarrow \\ R\pi_*\mathbf{C} & \xrightarrow{\sim} & R\pi_*\pi^{-1}\omega_Y^{\cdot,\log}. \end{array}$$

7.5. The last commutativity implies that the morphism (29) in 7.2 coincides with

$$R\tau_*f^{\log-1}\mathcal{O}_0^{\log} \xrightarrow{R\tau_*(e)} R\tau_*\omega^{\cdot,\log}[u] \longrightarrow R(\tau\pi)_*\pi^{-1}\omega_Y^{\cdot,\log} \xleftarrow{\sim} \omega_Y^{\cdot}[u] \xrightarrow{\sim} \omega_{Y/0}^{\cdot}.$$

Hence the composite of the morphism (29) and the natural homomorphism $\omega_{Y/0}^{\cdot} \xrightarrow{\sim} R\tau_*\omega_{Y/0}^{\cdot,\log}$ coincides with

$$R\tau_*f^{\log-1}\mathcal{O}_0^{\log} \xrightarrow{R\tau_*(e)} R\tau_*\omega_Y^{\cdot,\log}[u] \longrightarrow R(\tau\pi)_*\pi^{-1}\omega_Y^{\cdot,\log} \xleftarrow{\sim} \omega_Y^{\cdot}[u]$$
$$\longrightarrow R\tau_*\omega_Y^{\cdot,\log}[u] \xrightarrow{u\mapsto 0} R\tau_*\omega_{Y/0}^{\cdot,\log}.$$

On the other hand, 7.3 implies that the first morphism

$$R\tau_*f^{\log-1}\mathcal{O}_0^{\log} \to R\tau_*\omega_{Y/0}^{\cdot,\log}$$

in 7.2 coincides with the composite

$$R\tau_*f^{\log-1}\mathcal{O}_0^{\log} \xrightarrow{R\tau_*(e)} R\tau_*\omega_Y^{\cdot,\log}[u] \xrightarrow{u\mapsto 0} R\tau_*\omega_{Y/0}^{\cdot,\log}.$$

Therefore, 7.2 reduces to the statement that the composite

$$R\tau_*\omega_Y^{\cdot,\log}[u] \to R(\tau\pi)_*\pi^{-1}\omega_Y^{\cdot,\log} \xleftarrow{\sim} \omega_Y^{\cdot}[u] \to R\tau_*\omega_Y^{\cdot,\log}[u]$$

is the identity.

7.6. We prove the last statement. By [7, Proposition (3.7) (3)], $\omega_Y^{\cdot}[u] \to R\tau_*\omega_Y^{\cdot,\log}[u]$ is a quasi-isomorphism. Hence, this statement is equivalent to that the composite

$$\omega_Y^{\cdot}[u] \to R\tau_*\omega_Y^{\cdot,\log}[u] \to R(\tau\pi)_*\pi^{-1}\omega_Y^{\cdot,\log} \xleftarrow{\sim} \omega_Y^{\cdot}[u]$$

is the identity. To see this, we consider the morphisms of complexes and the statement is deduced from the following two properties of the quasi-isomorphism $\omega_Y^{\cdot}[u] \xrightarrow{\sim} R(\tau\pi)_*\pi^{-1}\omega_Y^{\cdot,\log}$ by [6, Lemma 3.3].

(i) It sends the indeterminate u to $\frac{\log t}{2\pi i} \in \mathcal{O}_Y^{\log}$.

(ii) The composite $\omega_Y^{\cdot} \to \omega_Y^{\cdot}[u] \to R(\tau\pi)_*\pi^{-1}\omega_Y^{\cdot,\log}$ is the one induced by the adjoint map.

This completes the proof of 7.2.

PROPOSITION 7.7. *We have the following.*

(i) *The homomorphism*

$$Rf_*R\tau_*f^{\log-1}\mathcal{O}_0^{\log} \to Rf_*R\tau_*\omega_{Y/0}^{\cdot,\log}$$

is a quasi-isomorphism.

(ii) *The homomorphism*

$$Rf_*R\tau_*(s(Y)) : Rf_*R\tau_*f^{\log-1}\mathcal{O}_0^{\log} \to Rf_*R(\tau\pi)_*\mathbf{C}$$

is a quasi-isomorphism.

PROOF. (i) By 6.2 (ii), we have

$$Rf_*^{\log}f^{\log-1}\mathcal{O}_0^{\log} \xrightarrow{\sim} Rf_*^{\log}\omega_{Y/0}^{\cdot,\log}.$$

By sending this by $R\tau_*$, we obtain (i).

(ii) By (i), the image by Rf_* of the morphism (28) in 7.2 is a quasi-isomorphism. Hence, by 7.2, the image by Rf_* of the morphism (29) in 7.2 is also a quasi-isomorphism. This implies (ii).

7.8. We prove 4.2.2 by the above proposition 7.2.

First, we prove that the homomorphism induced from

$$\begin{aligned}\mathcal{O}_0^{\log} \otimes Rf_*^{\log}\mathbf{C} &\xrightarrow{\sim} Rf_*^{\log}f^{\log-1}\mathcal{O}_0^{\log} \xleftarrow{\text{adj}} \tau^{-1}R\tau_*Rf_*^{\log}f^{\log-1}\mathcal{O}_0^{\log} \\ &= \tau^{-1}Rf_*R\tau_*f^{\log-1}\mathcal{O}_0^{\log} \xrightarrow{\sim} \tau^{-1}Rf_*R\tau_*\omega_{Y/0}^{\cdot,\log} \xleftarrow{\sim} \tau^{-1}Rf_*\omega_{Y/0}^{\cdot}\end{aligned} \tag{30}$$

by taking $\mathcal{H}^q$, taking the stalk at z and specializing by $\frac{\log t}{2\pi i} \mapsto 0$ coincides with (2). Here the first arrow in the second line of (30) is a quasi-isomorphism by 7.7 (i).

By (20) and (21), the composite of the three quasi-isomorphisms in 2.3 (i) coincides with the $\mathcal{H}^q$ of the homomorphism

$$\begin{aligned}\mathcal{O}_0^{\log} \otimes Rf_*^{\log}\mathbf{C} \xleftarrow{\sim} \mathcal{O}_0^{\log} \otimes R\tau_*(\mathcal{O}_0^{\log} \otimes Rf_*^{\log}\mathbf{C}) \xrightarrow{\sim} \mathcal{O}_0^{\log} \otimes R\tau_* Rf_*^{\log}\omega_{Y/0}^{\cdot,\log} \\ = \mathcal{O}_0^{\log} \otimes Rf_* R\tau_*\omega_{Y/0}^{\cdot,\log} \xleftarrow{\sim} \mathcal{O}_0^{\log} \otimes Rf_*\omega_{Y/0}^{\cdot}.\end{aligned} \tag{31}$$

Hence, (2) coincides with the homomorphism induced from

$$\begin{aligned}&\text{the stalk of } (\mathcal{O}_0^{\log} \otimes Rf_*^{\log}\mathbf{C}) \leftarrow R\tau_*(\mathcal{O}_0^{\log} \otimes Rf_*^{\log}\mathbf{C}) \\ &\xrightarrow{\sim} R\tau_* Rf_*^{\log} f^{\log-1}\mathcal{O}_0^{\log} \xrightarrow{\sim} R\tau_* Rf_*^{\log}\omega_{Y/0}^{\cdot,\log} = Rf_* R\tau_*\omega_{Y/0}^{\cdot,\log} \xleftarrow{\sim} Rf_*\omega_{Y/0}^{\cdot}\end{aligned} \tag{32}$$

by taking $\mathcal{H}^q$ and specializing by $\frac{\log t}{2\pi i} \mapsto 0$. Since it is clear that (30) and (32) are compatible, we see that the homomorphism induced from (30) and (2) coincide, as desired.

Since the stalk of Rf_* of the composite $R(\tau\pi)_*\mathbf{C} \xrightarrow{\sim} R(\tau\pi)_*\pi^{-1}\omega_Y^{\cdot,\log} \xleftarrow{\sim} \omega_Y^{\cdot}[u] \xrightarrow{\sim} \omega_{Y/0}^{\cdot}$ in (29) in 7.2 coincides with (5), 7.2 reduces 4.2.2 to the following claim.

Claim. The quasi-isomorphism

$$Rf_* R\tau_*(s(Y)) : Rf_* R(\tau\pi)_*\mathbf{C} \xleftarrow{\sim} Rf_* R\tau_* f^{\log-1}\mathcal{O}_0^{\log}$$

in 7.7 (ii) is compatible with the stalk of the homomorphism $\mathcal{O}_0^{\log} \otimes Rf_*^{\log}\mathbf{C} \xrightarrow{\sim} Rf_*^{\log} f^{\log-1}\mathcal{O}_0^{\log} \xleftarrow{\text{adj}} \tau^{-1}R\tau_* Rf_*^{\log} f^{\log-1}\mathcal{O}_0^{\log} = \tau^{-1}Rf_* R\tau_* f^{\log-1}\mathcal{O}_0^{\log}$.

To prove this claim, we use the following lemma.

Lemma 7.9. *The diagram*

$$\begin{array}{ccc} f^{\log-1}\mathcal{O}_0^{\log} & \longrightarrow & \pi_*\mathbf{C} \\ \downarrow & & \downarrow \\ (\pi i)_*(\pi i)^{-1} f^{\log-1}\mathcal{O}_0^{\log} & \longrightarrow & (\pi i)_*\mathbf{C} \end{array}$$

is commutative, where the upper arrow is induced by the homomorphism $s(Y)$ in 7.1*, the vertical arrows are the adjoint morphisms, and the bottom arrow is $(\pi i)_*$ of the morphism of constant sheaves $(\pi i)^{-1} f^{\log-1}\mathcal{O}_0^{\log} \to (\pi i)_*\mathbf{C}$ on Y_∞ defined by sending $\frac{\log t}{2\pi i}$ to* 0.

Proof. By taking the adjoint, the commutativity is reduced to that of

$$\begin{array}{ccc} \pi^{-1} f^{\log-1}\mathcal{O}_0^{\log} & \xrightarrow{\frac{\log t}{2\pi i}\mapsto 0} & \mathbf{C} \\ {\scriptstyle\text{adj}}\downarrow & & \downarrow \\ i_* i^{-1}\pi^{-1} f^{\log-1}\mathcal{O}_0^{\log} & \xrightarrow[\frac{\log t}{2\pi i}\mapsto 0]{} & i_*\mathbf{C}. \end{array}$$

7.10. To prove the above claim, we consider the following commutative diagram:

$$\begin{array}{ccccc} Rf_*^{\log} f^{\log-1}\mathcal{O}_0^{\log} & \xleftarrow{\text{adj}} & \tau^{-1}R\tau_* Rf_*^{\log} f^{\log-1}\mathcal{O}_0^{\log} & \xrightarrow{\sim} & \tau^{-1}R\tau_* Rf_*^{\log}\pi_*\mathbf{C} \\ \downarrow & & \downarrow & & \downarrow \\ Rf_*^{\log}(\pi i)_*(\pi i)^{-1}f^{\log-1}\mathcal{O}_0^{\log} & \xleftarrow{\text{adj}} & \tau^{-1}R\tau_* Rf_*^{\log}(\pi i)_*(\pi i)^{-1}f^{\log-1}\mathcal{O}_0^{\log} & \longrightarrow & \tau^{-1}R\tau_* Rf_*^{\log}(\pi i)_*\mathbf{C}, \end{array}$$

where the right square is induced from the commutative diagram in 7.9. By taking the stalk at z of this diagram, we get the following commutative diagram:

$$\begin{array}{ccccc} R\Gamma\left(Y_z^{\log}, \mathbf{C}\left[\frac{\log t}{2\pi i}\right]\right) & \xleftarrow{\text{adj}} & R\Gamma\left(Y^{\log}, \mathbf{C}\left[\frac{\log t}{2\pi i}\right]\right) & \xrightarrow{\sim} & R\Gamma(Y_\infty^{\log}, \mathbf{C}) \\ \simeq\downarrow & & \downarrow & & \downarrow\simeq \\ R\Gamma\left(Y_{\infty,z}^{\log}, \mathbf{C}\left[\frac{\log t}{2\pi i}\right]\right) & \xleftarrow{\text{id}} & R\Gamma\left(Y_{\infty,z}^{\log}, \mathbf{C}\left[\frac{\log t}{2\pi i}\right]\right) & \xrightarrow[\frac{\log t}{2\pi i}\mapsto 0]{} & R\Gamma(Y_{\infty,z}^{\log}, \mathbf{C}). \end{array}$$

This implies the claim and completes the proof of 4.2.2.

Remark 7.11. It is plausible that the natural homomorphism

$$f^{\log-1}\mathcal{O}_0^{\log} \to \omega_{Y/0}^{\cdot,\log}$$

in 7.2 is a quasi-isomorphism, or more generally, that

$$f^{\log-1}\mathcal{O}_S^{\log} \to \omega_{P/S}^{\cdot,\log}$$

in 6.2 is a quasi-isomorphism. If this is the case, it makes proofs in Sections 6–7 considerably simpler.

8. Proof of main theorem

In this section, we prove the main theorem 2.7.

8.1. Let $f: Y \to 0$ be as in 2.7. We claim that the data

$$(H_{\mathbf{Q}}, W(N), H_{\mathcal{O}}, F, N), \tag{33}$$

where F is the Hodge filtration and N is the monodromy logarithm on $H_{\mathbf{Q}}$, are isomorphic to

$$(H^q(Y, K_{\mathbf{Q}}), W[q], H^q(Y, K_{\mathbf{C}}), F, N_K) \tag{34}$$

in Theorem 3.4. In fact, we have a filtered quasi-isomorphism (18)

$$(\omega_{Y/0}, F) \xrightarrow{\sim} (A_{\mathbf{C}}, F),$$

where F on the source is defined by $\omega_{Y/0}^{\cdot \geq r}$, and the morphism $A_{\mathbf{C}} \to K_{\mathbf{C}}$ in (15) such that Theorem 5.21 and Theorem 5.29 in [5] enable us to identify

$$(H_{\mathcal{O}}, F) \simeq (H^q(Y, A_{\mathbf{C}}), F) \simeq (H^q(Y, K_{\mathbf{C}}), F) \tag{35}$$

as filtered $\mathbf{C}$-vector spaces. The $\mathbf{Q}$-structures coincide by 4.1.1, which is now proved by the results in Sections 5 and 7. The coincidence of N on $H_{\mathbf{Q}}$ and N_K on $H^q(Y, K_{\mathbf{Q}})$ is proved in 5.17. Then the weight filtrations coincide because they are the monodromy filtrations of the compatible operators N and N_K.

8.2. Now we prove the goodness as follows. By 6.1, the first condition (i) of the goodness is always satisfied. The second condition is by the coincidence of the Hodge filtration (35) and the fact that $(A_{\mathbf{C}}, F)$ is a part of a CMHC ([5, Theorem 5.21]). The third condition of the goodness, that is, that the data form an LH, is proved as follows. The Griffiths transversality (2.1 (iii)) holds because the construction in [5] satisfies the Griffiths transversality (a part of the definition of polarized MHS in [2, Definition (2.26)]). Condition 2.1 (ii) also holds because the construction in [5] is a MHS. Then, [2, Corollary (3.13)] implies Condition 2.1 (i).

9. Log Picard varieties and log Albanese varieties

In this section, we apply the main result 2.7 to the theory of log Picard and log Albanese varieties.

9.1. First we recall the definition of log Picard variety and log Albanese variety briefly. See [10] for the details.

Let S be an fs log analytic space. Let $\mathscr{A}_S$ be the category of log complex tori over S and let $\mathscr{H}_S$ be the category of log Hodge structures H on S of weight -1 satisfying $F^{-1}H_{\mathcal{O}} = H_{\mathcal{O}}$ and $F^1 H_{\mathcal{O}} = 0$. Then we have a category equivalence

$$\mathscr{H}_S \simeq \mathscr{A}_S.$$

Let $P \to S$ be as in 6.2. Assume that it is good for $\mathscr{H}^1$ in the sense of [10, 7.1]. Then the log Picard variety $A^*_{P/S}$ is the log complex torus corresponding to $\mathscr{H}^1(P)(1)$ and the log Albanese variety $A_{P/S}$ is the log complex torus corresponding to the dual log Hodge structure of $\mathscr{H}^1(P)$. Here $\mathscr{H}^1(P)$ is the log Hodge structure underlain by $R^1 f^{\log}_* \mathbf{Z}$ which is given by the definition of the goodness.

9.2. We prove that a projective log deformation $Y \to 0$ is good even in the sense of [10, 7.1] so that there are well-defined log complex tori $A^*_{Y/0}$ and $A_{Y/0}$.

For this, we prove that our definition of goodness coincides with that in [10]. As in 2.3, let $Y \to 0$ be a proper, separated and log smooth morphism of fs log analytic spaces such that for any $y \in Y$, the cokernel of $M_0^{\mathrm{gp}}/\mathcal{O}_0^{\times} \to M_{Y,y}^{\mathrm{gp}}/\mathcal{O}_{Y,y}^{\times}$ is torsion-free.

We do not recall the definition of the goodness in [10] here, but recall that its main difference from the goodness in Section 2 of this paper lies in that, in [10], the sheaves are all considered on the big sites.

Let $f : P \to S$ be as in 6.2. Then, any (fs) base change of f also satisfies the assumption of 6.2 so that the conclusion of 6.2 is satisfied also for any base changed morphism. This means that f satisfies the condition (i) in [10, 7.1]. (Actually, the modified one as explained in 2.4.)

Further, [11, Proposition 5.1] (the proper base change theorem in log Betti cohomology) implies that the construction of $R^q f^{\log}\mathbf{Z}$ is compatible with any base change. This means that the former part of the condition (iii) in [10, 7.1] is satisfied.

Thus, in particular, $Y \to 0$ as above always satisfies (i) and the former part of (iii) in [10, 7.1].

Next, in order to see that the condition (ii) for $Y \to 0$ in [10, 7.1] is equivalent to our (ii) in 2.3, it is enough to show that the constructions of the two maps in 2.3 (ii) are compatible with any base change. (Note that the injection there is a split injection.) This is by the fact that the construction of the relative differential sheaf is compatible with any (fs) base change (cf. [12, 1.7 and 3.12]) and by the usual flat base change theorem.

Finally, the latter part of the condition (iii) in [10, 7.1] is the same as our 2.3 (ii). This completes the proof of the equivalence of our definition of goodness and that in [10, 7.1]. In particular, our main theorem 2.7 implies that a projective log deformation $Y \to 0$ is good even in the sense of [10, 7.1].

9.3. There is another definition of the goodness in [10, 7.2]. For an f as in 6.2 (especially, for a projective log deformation), both goodness in [10, 7.1] and in [10, 7.2] coincide and the quasi-isomorphisms given by the first conditions in both definitions also coincide. This is a consequence of 6.2 (ii).

9.4. To discuss the applications, we further introduce notation.

Let (fs/S) be the category of fs log analytic spaces over S endowed with the usual topology. Recall that a log complex torus over S is a sheaf of abelian groups on (fs/S).

Let $(\mathrm{fs}/S)^{\log}$ be the site of pairs (U, T), where $T \in (\mathrm{fs}/S)$ and U is an open set of $T^{\log}$, defined in [9, 3.1.6] (cf. [10, 3.2]). For a sheaf F on $(\mathrm{fs}/S)^{\log}$, we denote by $\tau_* F$ the sheaf on (fs/S) defined by $(\tau_* F)(T) = F((T^{\log}, T))$ for $T \in (\mathrm{fs}/S)$.

Let $P \in (\mathrm{fs}/S)$ and $q \in \mathbf{Z}$.

For a sheaf F of abelian groups on (fs/P), let $\mathscr{H}^q(P,F)$ be the sheaf of abelian groups on (fs/S) whose restriction on any $T \in (\mathrm{fs}/S)$ is $R^q(P \times_S T \to T)_*(F_{P\times_S T})$. Here $F_{P\times_S T}$ is the restriction of F on the small site of $P \times_S T$.

For a sheaf F of abelian groups on $(\mathrm{fs}/P)^{\log}$, let $\mathscr{H}^q(P^{\log},F)$ (denoted by $\mathscr{H}^q(\tau^{-1}(P),F)$ in [10]) be the sheaf of abelian groups on $(\mathrm{fs}/S)^{\log}$ whose restriction on the small site of $(T^{\log},T)$ for any $T \in (\mathrm{fs}/S)$ is $R^q(P \times_S T \to T)_*^{\log}(F_{(P\times_S T)^{\log}})$. Here $F_{(P\times_S T)^{\log}}$ is the restriction of F on the small site of $(P \times_S T)^{\log}$.

Let $\mathbf{G}_m$ (resp. $\mathbf{G}_{m,\log}$) be the sheaf of abelian groups on (fs/P) defined by $\mathbf{G}_m(T) = \Gamma(T,\mathcal{O}_T^\times)$ (resp. $\mathbf{G}_{m,\log}(T) = \Gamma(T,M_T^{\mathrm{gp}})$) for $T \in (\mathrm{fs}/P)$.

Now we apply Section 8 of [10] on log Picard varieties. Note that the assumption 8.1 of [10] is satisfied (cf. the remark after it). Then, ibid. 8.2 and 8.3 imply the following.

Theorem 9.5. *Let $Y \to 0$ be a projective log deformation.*

(i) *Let*

$$\mathscr{H}^1(Y,\mathbf{G}_m)_0 = \mathrm{Ker}(\mathscr{H}^1(Y,\mathbf{G}_m) \to \mathscr{H}^2(Y,\mathbf{Z})),$$
$$\mathscr{H}^1(Y,\mathbf{G}_{m,\log})_0 = \mathrm{Ker}(\mathscr{H}^1(Y,\mathbf{G}_{m,\log}) \to \tau_*\mathscr{H}^2(Y^{\log},\mathbf{Z})).$$

Then we have canonical embeddings

$$\mathscr{H}^1(Y,\mathbf{G}_m)_0 \subset A_{Y/0}^* \subset \mathscr{H}^1(Y,\mathbf{G}_{m,\log})_0.$$

(ii) *We have a canonical isomorphism*

$$\mathscr{H}^1(Y,\mathbf{G}_{m,\log})/\mathscr{H}^1(Y,\mathbf{G}_{m,\log})_0 \simeq \mathscr{H}om(\mathbf{Z},\mathscr{H}^2(Y)(1)).$$

Here $\mathscr{H}^2(Y)$ is the log Hodge structure underlain by $R^2(Y^{\log} \to 0^{\log})_*\mathbf{Z}$ which is given by the definition of the goodness, and $\mathscr{H}om$ is the $\mathscr{H}om$ for log Hodge structures.

We remark that in (i), more precisely, [10, Theorem 8.2] describes the groups $\mathscr{H}^1(Y,\mathbf{G}_m)_0$, $\mathscr{H}^1(Y,\mathbf{G}_{m,\log})_0$, and the quotients $A_{Y/0}^*/\mathscr{H}^1(Y,\mathbf{G}_m)_0$ and $\mathscr{H}^1(Y,\mathbf{G}_{m,\log})_0/A_{Y/0}^*$ in terms of the log Hodge structure $\mathscr{H}^1(Y)(1)$. We omit here the details.

Corollary 9.6. *We have an exact sequence*

$$0 \to \mathscr{E}xt^1(A_{Y/0},\mathbf{G}_{m,\log}) \to \mathscr{H}^1(Y,\mathbf{G}_{m,\log}) \to \tau_*\mathscr{H}^2(Y^{\log},\mathbf{Z}) \to \mathscr{H}^2(Y,\mathcal{O}_Y).$$

Next, we apply Section 10 of [10] on log Albanese varieties.

PROPOSITION 9.7. *Let $Y \to 0$ be a connected projective log deformation. Then, for a log complex torus B over* 0, *the following sequence is exact.*

$$0 \to B(0) \to B(Y) \to \mathrm{Hom}(A_{Y/0}, B).$$

9.8. Finally we discuss some problems.

PROBLEM 1. Prove that the last arrow $B(Y) \to \mathrm{Hom}(A_{Y/0}, B)$ is surjective so that each section e of $Y/0$ induces the *log Albanese map* $\psi_e : Y \to A_{Y/0}$.

To prove this, it is enough to show that the extension "$\psi_e(a)$" of log Hodge structures in the page 177 of [10] is admissible.

PROBLEM 2 (cf. [10, Problem 10.6]). Prove the following: Let $Y/0$ be a projective log deformation. Then we have the inclusion

$$\mathscr{H}^1(P, \mathbf{G}_{m,\log})_0 \cap \mathrm{Image}(\mathscr{H}^1(P, \mathbf{G}_m) \to \mathscr{H}^1(P, \mathbf{G}_{m,\log})) \subset A^*_{Y/0}.$$

When this statement is valid, the inclusion map should be called the *log Abel–Jacobi map*.

More generally, we can ask the following.

PROBLEM 3 (cf. [10, 10.7]). Define a *log Abel–Jacobi map of degree r* whose target is the *log intermediate Jacobian* $\mathscr{E}xt^1(\mathbf{Z}, H^{2r-1}(Y)(r))$ of degree r, where $\mathscr{E}xt^1$ is taken in the category of log mixed Hodge structures.

Problem 1 should be the case of Problem 3 where $r = \dim Y$, and Problem 2 is the case $r = 1$.

Appendix

Because we could not find an appropriate reference, we give a sketch of the proof of the following basic statement, though it is not used in the text.

THEOREM A.1. *Let $f : X \to S$ be a proper, separated, smooth and Kähler morphism of complex analytic spaces. Then the triple*

$$(R^q f_* \mathbf{Z}, (R^q f_* \Omega_{X/S}, F), \iota)$$

is a variation of Hodge structures of weight q for every integer q, where $\Omega_{X/S}$ denotes the relative de Rham complex, F the decreasing filtration induced by the stupid filtration on $\Omega_{X/S}$, and $\iota : R^q f_* \mathbf{Z} \to R^q f_* \Omega_{X/S}$ the morphism induced *by the canonical morphism* $\mathbf{Z} \hookrightarrow \mathcal{O}_X \to \Omega_{X/S}$. *Moreover, the Hodge to de Rham spectral sequence degenerates at* E_1*-terms.*

PROOF. By the result of Siebenmann [19, Corollary 6.14] (cf. [14, Proposition 3.5, Remark 3.5.1]) $R^q f_* \mathbf{Z}$ is a locally constant abelian sheaf of finite rank on S. By the relative Poincaré lemma [4, Théorème 2.23 (ii)] and by the projection formula, the morphism ι induces an isomorphism $\mathcal{O}_S \otimes R^q f_* \mathbf{Z} \xrightarrow{\sim} R^q f_* \Omega_{X/S}$. Thus $R^q f_* \Omega_{X/S}$ is a locally free $\mathcal{O}_S$-module of finite rank and compatible with any base change.

Now we take a point $s \in S$ and set $A = \mathcal{O}_{S,s}$ for a while. We denote by $\mathfrak{m}$ the maximal ideal of A. By [1, Théorème 4.1] (cf. [17, Section 5]), there exists a complex K of $\mathcal{O}_{S,s}$-modules satisfying the following:

- K is bounded from the above and the below.
- K^n is a free A-module of finite rank for every n.
- For a coherent $\mathcal{O}_S$-module $\mathscr{F}$, we have a functorial isomorphism

$$Rf_*(\Omega^p_{X/S} \otimes f^*\mathscr{F})_s \simeq K \otimes \mathscr{F}_s$$

  in the derived category.

Therefore we have

$$\begin{aligned}
&l_{A/\mathfrak{m}^{n+1}}(R^q f_*(\Omega^p_{X/S} \otimes A/\mathfrak{m}^{n+1})_s) \\
&\quad = l_{A/\mathfrak{m}^{n+1}}(H^q(K \otimes A/\mathfrak{m}^{n+1})) \\
&\quad \leq l_{A/\mathfrak{m}^{n+1}}(A/\mathfrak{m}^{n+1}) \dim H^q(K \otimes A/\mathfrak{m}) \\
&\quad = l_{A/\mathfrak{m}^{n+1}}(A/\mathfrak{m}^{n+1}) \dim R^q f_*(\Omega^p_{X/S} \otimes A/\mathfrak{m})_s
\end{aligned} \tag{36}$$

for every $n \geq 0$ by the case $k = 0$ of [3, Corollaire (3.4)], where $l_{A/\mathfrak{m}^{n+1}}(-)$ denotes the lengths of the finite modules over $A/\mathfrak{m}^{n+1}$.

On the other hand, we have the isomorphism

$$A/\mathfrak{m}^{n+1} \otimes (R^q f_* \mathbf{Z})_s \simeq R^q f_*(\Omega_{X/S} \otimes A/\mathfrak{m}^{n+1})_s$$

for every n, q by the relative Poincaré lemma and by the projection formula. Thus we obtain

$$l_{A/\mathfrak{m}^{n+1}}(R^q f_*(\Omega_{X/S} \otimes A/\mathfrak{m}^{n+1})_s) = l_{A/\mathfrak{m}^{n+1}}(A/\mathfrak{m}^{n+1}) \dim R^q f_*(\Omega_{X/S} \otimes A/\mathfrak{m})_s$$

for every n, q. By considering the Hodge to de Rham spectral sequences for $A/\mathfrak{m}^{n+1}$ and $A/\mathfrak{m}$, we conclude that the equality holds in (36) for every n, p, q and that the Hodge to de Rham spectral sequences for $A/\mathfrak{m}^{n+1}$ and $A/\mathfrak{m}$

degenerate at E_1-terms as in [3, Section 5]. By [3, Corollaire (3.4), Remarque (3.3.3)] again, $H^q(K)$ is a free A-module and the canonical morphism $H^q(K) \otimes M \to H^q(K \otimes M)$ is an isomorphism for every A-module M and for every q. Therefore $R^q f_* \Omega^p_{X/S}$ is a locally free $\mathcal{O}_S$-module of finite rank and compatible with any base change for every p, q. Moreover the Hodge to de Rham spectral sequence degenerates at E_1-terms by the local freeness and the base change property of $R^q f_* \Omega^p_{X/S}$ and $R^q f_* \Omega_{X/S}$, and by the E_1-degeneration of the Hodge to de Rham spectral sequence for $A/\mathfrak{m}$. Once we know the base change property above, it is easy to see that the pair

$$((R^q f_* \mathbf{Z})_s, (R^q f_* \Omega_{X/S}, F) \otimes \mathbf{C}(s))$$

equipped with the isomorphism induced by ι is a Hodge structure of weight q.

What remains to check is the Griffiths transversality. For this purpose, we consider Ω_X and Ω_S. We denote by $G^r \Omega^p_X$ the image of the morphism $\Omega^r_S \otimes \Omega^{p-r}_X \to \Omega^p_X$ induced by the wedge product. Then G defines a finite decreasing filtration on Ω_X. On the other hand, the stupid filtration on Ω_S is denoted by G for a while. By the smoothness of f, we have

$$\mathrm{gr}^r_G \Omega_X \simeq f^* \Omega^r_S \otimes \Omega_{X/S}[-r] = f^{-1} \Omega^r_S \otimes \Omega_{X/S}[-r]$$

for every r. Therefore the canonical morphism $f^{-1}\Omega_S \to \Omega_X$ is a filtered quasi-isomorphism by the relative Poincaré lemma and by the flatness of $\Omega^p_{X/S}$ over $f^{-1}\mathcal{O}_S$. Then we have

$$\begin{aligned} \Omega^p_S \otimes R^q f_* \mathbf{C} &\simeq R^q f_* f^{-1} \Omega^p_S \simeq E_1^{p,q}(Rf_* f^{-1}\Omega_S, G) \\ &\simeq E_1^{p,q}(Rf_* \Omega_X, G) \simeq \Omega^p_S \otimes R^q f_* \Omega_{X/S} \end{aligned}$$

by the projection formula and by the base change property of $R^q f_* \Omega_{X/S}$. Moreover, the morphism of E_1-terms

$$\Omega^p_S \otimes R^q f_* \Omega_{X/S} \to \Omega^{p+1}_S \otimes R^q f_* \Omega_{X/S}$$

is identified with the morphism

$$d \otimes \mathrm{id} : \Omega^p_S \otimes R^q f_* \mathbf{C} \to \Omega^{p+1}_S \otimes R^q f_* \mathbf{C}$$

under this identification. Thus we can check the Griffiths transversality in the usual way.

REMARK A.2. Note that S is not necessarily reduced in the theorem above.

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