HIROSHIMA MATH. J.
45 (2015), 9–34

Operator theory and the Oka extension theorem

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(Received December 21, 2012)
(Revised August 26, 2013)

ABSTRACT. For δ an m-tuple of analytic functions, we define an algebra $H^{\infty}_{\delta,\,\mathrm{gen}}$, contained in the bounded analytic functions on the analytic polyhedron $\{\|\delta^{l}(z)\| < 1, 1 \le l \le m\}$, and prove a representation formula for it. We give conditions whereby every function that is analytic on a neighborhood of $\{\|\delta^{l}(z)\| \le 1, 1 \le l \le m\}$ is actually in $H^{\infty}_{\delta,\,\mathrm{gen}}$. We use this to give a proof of the Oka extension theorem with bounds. We define an $H^{\infty}_{\delta,\,\mathrm{gen}}$ functional calculus for operators.

1. Introduction

In [4] a new proof of the Oka Extension Theorem for p-polyhedra was given using operator-theoretic methods. While the proof used both the functional calculus for commuting operator-tuples and the Oka-Weil approximation theorem for analytic functions defined on a neighborhood of p-polyhedra, it had the novel feature that it revealed norms for which extensions were obtained with precise bounds.

In this paper we shall improve upon the results from [4] in a number of ways. First, we shall eliminate our reliance on the functional calculus for general operator-tuples developed by J. Taylor, and in its place, use a simple functional calculus for commuting diagonalizable matrix-tuples together with a technical axiom described in Section 3 of the paper. Secondly, we shall not require the Oka-Weil approximation theorem. Rather, using our technical axiom we shall give an original proof of the stronger Oka Extension Theorem [10] that is entirely elementary in nature. Finally, rather than working on p-polyhedra, we shall work with analytic polyhedra defined in general domains of holomorphy.

Once the program described in the above paragraph is carried out, the door is opened for generalization of Oka and Cartan like results in several complex variables that involve the existence of extensions of holomorphic

Partially supported by National Science Foundation Grant DMS 1068830.
Partially supported by National Science Foundation Grants DMS 0966845 and DMS 1300280.
2010 *Mathematics Subject Classification*. Primary: 32E20, 46H30. Secondary: 47C99.
Key words and phrases. Oka extension, operator theory.

functions. We illustrate this point by giving a proof of a special case of the Cartan Extension Theorem for holomorphic functions defined on analytic varieties in domains of holomorphy.

Here is a prototypical example.

EXAMPLE 1. Let $\mathscr{A} = H^\infty(\mathbf{D}^2)$, the algebra of all bounded analytic functions on $\mathbf{D}^2$. It was shown in [1] that a function ϕ is in $H^\infty(\mathbf{D}^2)$ and has norm less than or equal to 1 if and only if there is a graded Hilbert space $\mathscr{L} = \mathscr{L}^1 \oplus \mathscr{L}^2$ and a unitary operator $V : \mathbf{C} \oplus \mathscr{L} \to \mathbf{C} \oplus \mathscr{L}$, which can be written in block form as

$$V = \begin{array}{c} \\ \mathbf{C} \\ \mathscr{L} \end{array} \begin{array}{c} \begin{array}{cc} \mathbf{C} & \mathscr{L} \end{array} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array},$$

so that, writing P^1 for the projection from $\mathscr{L}$ onto $\mathscr{L}^1$ and P^2 for the projection from $\mathscr{L}$ onto $\mathscr{L}^2$, we have

$$\phi(z) = A + B(z^1P^1 + z^2P^2)[I_{\mathscr{L}} - (z^1P^1 + z^2P^2)D]^{-1}C. \tag{1.1}$$

If now $T = (T^1, T^2)$ is a pair of commuting operators on a Hilbert space $\mathscr{H}$, one can modify (1.1) to

$$\begin{aligned} \phi(T) = I_{\mathscr{H}} \otimes A + I_{\mathscr{H}} \otimes B(T^1 \otimes P^1 + T^2 \otimes P^2) \\ [I_{\mathscr{H}\otimes\mathscr{L}} - (T^1 \otimes P^1 + T^2 \otimes P^2)I_{\mathscr{H}} \otimes D]^{-1} I_{\mathscr{H}} \otimes C. \end{aligned} \tag{1.2}$$

Formula (1.2) makes sense as long as the spectrum of the operator

$$(T^1 \otimes P^1 + T^2 \otimes P^2)I_{\mathscr{H}} \otimes D$$

does not contain the point 1. This will be guaranteed if T^1 and T^2 both have spectrum in the open disk $\mathbf{D}$. If, in addition, one assumes that they are both strict contractions (i.e. have norm less than one), then it can be shown that the formula (1.2) gives a contractive $H^\infty(\mathbf{D}^2)$ functional calculus for T. Indeed, the contractivity follows from calculating $I - \phi(T)^*\phi(T)$ and observing that it is positve; the fact that the map is an algebra homomorphism follows from observing that in both (1.1) and (1.2), if one expands the inverse in a Neumann series, one gets an absolutely convergent series of polynomials, provided $\|z\|$ (respectively, $\|T\|$) is less than one.

In [4], EXAMPLE 1 was generalized to polynomial polyhedra as follows. Let $\delta = (\delta^1, \ldots, \delta^m)$ be a polynomial mapping, which we think of as a map into $\mathbf{C}^m$ with the ℓ^∞-norm. Let

$$G_\delta = \{z \in \mathbf{C}^d : \|\delta(z)\| < 1\},$$

$$K_\delta = \{z \in \mathbf{C}^d : \|\delta(z)\| \leq 1\},$$

and for any domain $\Omega \subset \mathbf{C}^d$ let $O(\Omega)$ denote the algebra of holomorphic functions on Ω. Let us use $C\mathscr{L}(\mathscr{H})^d$ to denote the set of all commuting d-tuples of bounded linear operators on the Hilbert space $\mathscr{H}$. Define the family of operators $\mathscr{F}_\delta$ by

$$\mathscr{F}_\delta = \{T \in C\mathscr{L}(\mathscr{H})^d : \|\delta(T)\| \leq 1\}. \tag{1.3}$$

Then define an algebra H^∞_δ of holomorphic functions on G_δ by

$$H^\infty_\delta = \{\phi \in O(G_\delta) : \sup\{\|\phi(T)\| : T \in \mathscr{F}_\delta \text{ and } \sigma(T) \subset G_\delta\} < \infty\}. \tag{1.4}$$

There is a representation formula for functions in H^∞_δ very similar to (1.1); this generalizes to operators just as in (1.2). It was shown in [4] that this leads not only to a new proof of the Oka extension theorem [10], but that it gives sharp bounds for the norms of the extensions, which the function theory approaches do not yield.

1.1. Main results. The definition of H^∞_δ, and also of $\mathscr{F}_\delta$ if δ is allowed to be a holomorphic mapping on some domain, has the drawback that it requires knowledge of the Taylor spectra of the tuples T. In Section 2 we define versions of these objects in a more direct and elementary way that avoids Taylor's theory. This goal is accomplished through restricting our use of operators in defining norms to the diagonalizable, finite dimensional case. For commuting diagonalizable matrices there is no mystery to the functional calculus: one simply chooses a basis of eigenvectors, and applies the function to the joint eigenvalue.

In Section 2, we describe a set of matrices $\mathscr{F}_{\delta,\mathrm{gen}} \subsetneq \mathscr{F}_\delta$, and algebras $H^\infty_{\delta,\mathrm{gen}}$, and a localized version $H^\infty_{\delta,\mathrm{gen}}(E)$, that are substitutes for $H^\infty_{\delta,\mathrm{gen}}$. In Section 3 we prove a representation theorem for $H^\infty_{\delta,\mathrm{gen}}$ and $H^\infty_{\delta,\mathrm{gen}}(E)$.

If U is a domain of holomorphy, and $\delta \in O(U)^m$ is no longer assumed to be polynomial, the formal calculations work just as before, leading to an algebra $H^\infty_{\delta,\mathrm{gen}}$ of functions on $G_\delta = \{z \in U : \|\delta(z)\| \leq 1\}$. To be useful, we would like to know what functions are in $H^\infty_{\delta,\mathrm{gen}}$. In particular, we define δ to be utile (DEFINITION 4 below) if every function holomorphic on a neighborhood of K_δ is in $H^\infty_{\delta,\mathrm{gen}}$.

In Section 4, we prove two theorems that give conditions for δ to be utile in terms of operator theoretic properties of the set $\mathscr{F}_{\delta,\mathrm{gen}}$. These conditions are not always satisfied. However, we show in Section 5 that one can add more functions to the set δ to get a new family γ so that $K_\gamma = K_\delta$, and hence

the function theory is unchanged, but so that γ is utile, and therefore one can attack function theory problems using these operator theory tools.

To illustrate the use of these ideas, in Section 6 we prove refinements of the Oka extension theorem and the Oka-Weil theorem. In Section 7 we prove a special case of the Cartan extension theorem.

1.2. Functional calculus. If $\mathscr{A}$ is a topological algebra that contains the polynomials $\mathbf{C}[z^1, \dots, z^d]$, and if $T = (T^1, \dots, T^d)$ is a d-tuple of commuting operators on a Banach space $\mathscr{X}$, an *$\mathscr{A}$-functional calculus for T* is a continuous homomorphism π from $\mathscr{A}$ to $B(\mathscr{X})$, the bounded linear operators on $\mathscr{X}$, such that $\pi(z^l) = T^l$ for each $1 \le l \le d$.

In the ground-breaking papers [15, 16], J. L. Taylor developed a functional calculus for the algebras $O(U)$, where $O(U)$ denotes the algebra of all holomorphic functions on the open set $U \subseteq \mathbf{C}^d$. This was a multi-variable version of the Riesz-Dunford calculus in one variable. He also defined a spectrum for commuting d-tuples, now called the Taylor spectrum, and showed that for any commuting d-tuple T, there is an $O(U)$ functional calculus for T if and only if the Taylor spectrum of T lies in U.

A drawback to Taylor's approach is that it is often difficult to determine what the spectrum of T is, and the formulas defining $\phi(T)$ can be too complicated to handle. (We shall routinely write $\phi(T)$ for $\pi(\phi)$.)

In Section 8 we propose a different approach. The key idea is that we choose some basic functions δ^l, and then define an $H^\infty_{\delta,\text{gen}}$ functional calculus by writing down a relatively simple formula for $\phi(T)$ in terms of $\delta(T)$, that gives a well-defined operator whenever $\|\delta(T)\| < 1$.

The formula (a modification of (1.2)) can be thought of as a multi-variable version of the Sz.-Nagy-Foiaş functional calculus, which is an $H^\infty(\mathbf{D})$ functional calculus for (single) Hilbert space contractions that have spectrum in the open unit disk $\mathbf{D}$, or, more generally, have no unitary summand [14]. Our approach generalizes ideas of C.-G. Ambrozie and D. Timotin [5], of J. A. Ball and V. Bolotnikov [6], and of the authors and N. J. Young [4].

Our approach leads in Section 8 to a functional calculus for any p-polyhedron (a set of the form K_δ), or, more generally, any rationally convex set. The rational Oka-Weil theorem asserts that any holomorphic function on a neighborhood of a rationally convex set is a limit of rational functions, so at first blush the existence of the functional calculus is unremarkable. However, our functional calculus comes with a *norm control*, and this is its key property. In THEOREM 10, we show there is no ambiguity: when $\phi(T)$ is defined both by our definition and by Taylor's, the two definitions coincide.

2. $H^\infty_{\delta,\mathrm{gen}}(E)$

Let U be a domain of holomorphy in $\mathbf{C}^d$, fixed throughout the paper. We will take $\delta = (\delta_1, \ldots, \delta_m) \in O(U)^m$ to be an m-tuple of non-constant holomorphic functions on U and think of it as a map into $\mathbf{C}^m$ with the ℓ^∞-norm. (More generally, one can consider δ as a map into the r-by-s matrices. This is natural if one wishes to define a functional calculus not for d-tuples T of commuting contractions, but defined by other inequalities, for example if T is a row-contraction, defined by $T^1T^{1*} + \cdots + T^dT^{d*} \leq I$. For convenience we will not do our calculations in this generality.

We define two analytic polyhedra in U by $G_\delta = \delta^{-1}(\mathbf{D}^m)$ and $K_\delta = \delta^{-1}(\overline{\mathbf{D}^m})$. *We shall always assume that K_δ is compact.* If T is a d-tuple of pairwise commuting operators acting on an N-dimensional Hilbert space, we say that T is *generic* if there exist N linearly independent joint eigenvectors whose corresponding joint eigenvalues are distinct (equivalently, T is diagonalizable and $\sigma(T)$ has cardinality N). If $\Lambda = \{\lambda_1, \ldots, \lambda_N\}$ is a set consisting of exactly N distinct points in $\mathbf{C}^d$, we let $\mathscr{G}_\Lambda$ denote the set of generic d-tuples of pairwise commuting operators T acting on $\mathbf{C}^N$ with $\sigma(T) = \Lambda$. We shall let superscripts identify components of d-tuples and subscripts identify components of N-tuples. Thus, if $T = (T^1, \ldots, T^d) \in \mathscr{G}_\Lambda$, then there exist N vectors $k_1, \ldots, k_N \in \mathbf{C}^N$ such that

$$T^r k_j = \lambda_j^r k_j$$

for $1 \leq r \leq d$ and $1 \leq j \leq N$. If $T \in \mathscr{G}_\Lambda$ and f is holomorphic on a neighborhood of $\sigma(T)$, then we define $f(T)$ to be the unique operator on $\mathbf{C}^N$ that satisfies

$$f(T)k_j = f(\lambda_j)k_j, \qquad 1 \leq j \leq N. \tag{2.1}$$

We now define generic and local versions of the sets $\mathscr{F}_\delta$, H^∞_δ and the norm $\|\cdot\|_\delta$ from [4]. Fix a set $E \subseteq G_\delta$ and let $E^{\mathbf{C}}$ denote the space of complex valued functions on E. For finite $\Lambda \subset G_\delta$, let $\mathscr{F}_{\delta,\Lambda}$ be the set in $\mathscr{G}_\Lambda$ defined by

$$\mathscr{F}_{\delta,\Lambda} = \{T \in \mathscr{G}_\Lambda \mid \|\delta_l(T)\| \leq 1, l = 1, \ldots, m\}. \tag{2.2}$$

We then define a collection of generic operators, $\mathscr{F}_{\delta,\mathrm{gen}}$, by setting

$$\mathscr{F}_{\delta,\mathrm{gen}} = \bigcup_{\substack{\Lambda \text{ finite} \\ \Lambda \subset G_\delta}} \mathscr{F}_{\delta,\Lambda}. \tag{2.3}$$

For $f \in E^{\mathbf{C}}$, define $\|f\|_{\delta,\mathrm{gen}}$ by the formula,

$$\|f\|_{\delta,\mathrm{gen}} = \sup_{\substack{T \in \mathscr{F}_{\delta,\mathrm{gen}} \\ \sigma(T) \subseteq E}} \|f(T)\|. \tag{2.4}$$

Note that the notation, $\|f\|_{\delta,\text{gen}}$ sees E due to the fact that $E = \text{dom } f$. Finally, we define $H^\infty_{\delta,\text{gen}}(E)$ to consist of all $f \in E^{\mathbf{C}}$ such that $\|f\|_{\delta,\text{gen}}$ is finite.

Before continuing, we remark that the space $H^\infty_{\delta,\text{gen}}(E)$, as just defined, refines the space H_δ considered in [4] in two ways. First, by taking supremum over generic operators rather than operators with spectrum in G_δ, use of the Taylor functional calculus is avoided (though *a priori* it may be the case that $\|\cdot\|_{\delta,\text{gen}} \neq \|\cdot\|_\delta$). Secondly, we have localized the definition of H^∞_δ through the introduction of the set E. This represents a considerably more general scheme than considered in [4]. In particular, our work here will make it clear that many of the proofs in [4] do not require that the elements of H^∞_δ be holomorphic. However, when $E = G_\delta$ and $f = \phi$ is holomorphic on G_δ it turns out that $\|f\|_{\delta,\text{gen}} = \|\phi\|_\delta$—see THEOREM 1.

The following proposition is a straightforward analog of [4, Proposition 2.4]. For E a set, we let $\ell^\infty(E)$ denote the Banach algebra of bounded complex valued functions f on E equipped with the norm,

$$\|f\|_\infty = \sup_{e\in E} |f(e)|.$$

PROPOSITION 1. *$H^\infty_{\delta,\text{gen}}(E)$ equipped with $\|\cdot\|_{\delta,\text{gen}}$ is a Banach algebra. Furthermore, $H^\infty_{\delta,\text{gen}}(E) \subseteq \ell^\infty(E)$ and $\|f\|_\infty \leq \|f\|_{\delta,\text{gen}}$ for all $f \in H^\infty_{\delta,\text{gen}}(E)$.*

PROOF. The only assertion that is not immediately obvious is that $H^\infty_{\delta,\text{gen}}(E)$ is complete. Let f_n be a Cauchy sequence in $H^\infty_{\delta,\text{gen}}(E)$. It converges in $\ell^\infty(E)$ to a function f. For any T in $\mathscr{F}_{\delta,\text{gen}}$ with $\sigma(T) \subseteq E$, we have $f_n(T) \to f(T)$. Therefore

$$\|f(T)\| \leq \sup_n \|f_n(T)\| \leq \sup_n \|f_n\|_{\delta,\text{gen}},$$

so f is in $H^\infty_{\delta,\text{gen}}(E)$ as required. □

Further useful regularity of the elements of $H^\infty_{\delta,\text{gen}}(E)$ is revealed in the following proposition. For $\alpha_1, \alpha_2 \in \mathbf{D}$, we recall that the *Carathéodory pseudometric*, d, is defined on $\mathbf{D}$ by the formula,

$$d(\alpha_1,\alpha_2) = \left|\frac{\alpha_2 - \alpha_1}{1 - \overline{\alpha_1}\alpha_2}\right|, \qquad \alpha_1, \alpha_2 \in \mathbf{D}.$$

For any Banach space $\mathscr{X}$, we shall let ball $\mathscr{X}$ denote the closed unit ball of $\mathscr{X}$.

PROPOSITION 2. *If $f \in \text{ball } H^\infty_{\delta,\text{gen}}$ and $\lambda_1, \lambda_2 \in E$, then*

$$d(f(\lambda_1), \mathscr{F}(\lambda_2)) \leq \max_{1\leq l\leq m} d(\delta_l(\lambda_1), \delta_l(\lambda_2)). \tag{2.5}$$

PROOF. Fix $\lambda_1, \lambda_2 \in E$ with $\lambda_1 \neq \lambda_2$ and set $\Lambda = \{\lambda_1, \lambda_2\}$. For $T \in \mathscr{G}_\Lambda$, let θ_T denote the angle between the eigenspaces of T. By the remark following (2.7) in [2], elements of $\mathscr{G}_\Lambda$ are determined uniquely by this angle between their eigenspaces. Furthermore, by the calculations immediately following (2.13) in [2], if g is a function defined on Λ and $T \in \mathscr{G}_\Lambda$, then

$$\|g(T)\| \leq 1 \Leftrightarrow \sin \theta_T \geq d(g(\lambda_1), g(\lambda_2)). \tag{2.6}$$

Recalling (2.2), we see that (2.8) implies that if $T \in \mathscr{G}_\Lambda$, then

$$T \in \mathscr{F}_{\delta,\Lambda} \Leftrightarrow \sin \theta_T \geq \max_{1 \leq l \leq m} d(\delta_l(\lambda_1), \delta_l(\lambda_2)).$$

In particular, there exists $T \in \mathscr{F}_{\delta,\Lambda}$ such that

$$\sin \theta_T = \max_{1 \leq l \leq m} d(\delta_l(\lambda_1), \delta_l(\lambda_2)).$$

As $\|f(T)\| \leq 1$ for this T, we deduce from (2.6) that (2.5) holds. □

COROLLARY 1. *If $f \in H^\infty_{\delta,\mathrm{gen}}(E)$, then f is continuous on E.*

PROOF. If $f \neq 0$ in $H^\infty_{\delta,\mathrm{gen}}(E)$, then Proposition 1 implies that $\|f\|^{-1}_{\delta,\mathrm{gen}} f$ is continuous. □

The equivalence of conditions (a), (b), and (c) in the next proposition is an exact analog of from [4, Proposition 2.7].

PROPOSITION 3. *Let $E \subset G_\delta$ be finite. The following are equivalent.*

(a) $H^\infty_{\delta,\mathrm{gen}}(E) = \ell^\infty(E)$.

(b) $\lambda^r|_E \in H^\infty_{\delta,\mathrm{gen}}(E)$ *for* $r = 1, \ldots, d$.

(c) $\mathscr{F}_{\delta,E}$ *is bounded.*

(d) $\delta|_E$ *is one-to-one.*

PROOF. It is obvious that (a) implies (b) and that (b) implies (c).

(c) ⇒ (d): Let $E = \{e_1, \ldots, e_N\}$, and assume that $\delta(e_1) = \delta(e_2)$. Choose vectors $k_1, \ldots, k_N$ so that $\{k_2, \ldots, k_N\}$ are orthonormal, and k_1 is orthogonal to $\{k_3, \ldots, k_N\}$, but at angle θ to k_2. Then T defined by $T^r k_j = e_j^r k_j$ is in $\mathscr{F}_{\delta,E}$, but the norm of T will tend to infinity as θ tends to 0.

(d) ⇒ (a): It is sufficient to prove that for each $1 \leq j \leq N$, the function f_j defined by

$$f_j(e_i) = 0, \quad i \neq j, \qquad f_j(e_j) = 1$$

is in $H^\infty_{\delta,\mathrm{gen}}(E)$. Since δ is one-to-one, there exists some function $g_j : \delta(E) \to \mathbf{C}$ such that $g_j \circ \delta = f_j$. To prove that f_j is in $H^\infty_{\delta,\mathrm{gen}}(E)$, it suffices to prove that g_j is bounded on m-tuples of commuting generic contractions. This latter assertion follows from LEMMA 1, below. □

LEMMA 1. *Let $\Lambda = \{\lambda_1, \dots, \lambda_N\}$ be a set of N distinct points in $\mathbf{D}^m$. The function g_j, defined on Λ by $g_j(\lambda_i) = \delta_{ij}$, is bounded on m-tuples of commuting contractions.*

PROOF. For each $i \neq j$, choose r_i so that $\lambda_j^{r_i} \neq \lambda_i^{r_i}$. Define h by

$$h(\lambda) = \prod_{i \neq j} \frac{\lambda^{r_i} - \lambda_i^{r_i}}{1 - \overline{\lambda_i^{r_i}} \lambda^{r_i}}.$$

By von Neumann's inequality, each factor in $h(T)$ has norm at most one if T is an m-tuple of commuting contractions. Let $g_j = \dfrac{1}{h(\lambda_j)} h$. $\square$

3. Representing elements of $H^\infty_{\delta,\text{gen}}(E)$

In this section we shall derive three conditions that are necessary and sufficient for a given $f \in E^{\mathbf{C}}$ to be an element of ball $H^\infty_{\delta,\text{gen}}(E)$. These conditions, which are summarized in Theorem 1 below, represent the appropriate extension of [4, Theorem 4.5] to the $H^\infty_{\delta,\text{gen}}(E)$ setting. To state the theorem will require a number of definitions.

DEFINITION 1. Let f be a function on E. We say a 4-tuple (a, β, γ, D) is a (δ, E)-realization for f if $a \in \mathbf{C}$ and there exists a decomposed Hilbert space, $\mathscr{M} = \bigoplus_{l=1}^m \mathscr{M}_l$, such that

(i) the 2×2 matrix

$$V = \begin{bmatrix} a & 1 \otimes \beta \\ \gamma \otimes 1 & D \end{bmatrix},$$

acts isometrically on $\mathbf{C} \oplus \mathscr{M}$,

(ii) for each $\lambda \in E$, $\delta(\lambda)$ acts on $\mathscr{M}$ via the formula,

$$\delta(\lambda)(\textstyle\bigoplus_{l=1}^m x_l) = \bigoplus_{l=1}^m \delta_l(\lambda) x_l, \tag{3.1}$$

(iii) for each $\lambda \in E$,

$$f(\lambda) = a + \langle \delta(\lambda)(1 - D\delta(\lambda))^{-1} \gamma, \beta \rangle. \tag{3.2}$$

DEFINITION 2. For $E \subseteq G_\delta$, we let $\mathscr{C}_\delta(E)$ denote the collection of functions h defined on $E \times E$ that have the form

$$h(\lambda, \mu) = \sum_{l=1}^m (1 - \overline{\delta_l(\mu)} \delta_l(\lambda)) a_l(\lambda, \mu), \qquad \lambda, \mu \in E,$$

where for $l = 1, \dots, m$, $a_l(\lambda, \mu)$ is positive definite on E.

LEMMA 2. *If $E \subset G_\delta$ is finite and $f \in \text{ball } H^\infty_{\delta,\text{gen}}(E)$, then $1 - \overline{f(\mu)}f(\lambda) \in \mathscr{C}_\delta(E)$.*

PROOF. This follows from a Hahn-Banach separation argument, as in [3, Section 11.1]. In brief, let R be the set of functions on $E \times E$. One needs to show that every real linear functional L on R that is non-negative on the cone $\mathscr{C}_\delta(E)$ is non-negative on the function $1 - \overline{f(\mu)}f(\lambda)$. Let $\mathscr{V}$ denote the functions on E, and use L to define a sesqui-linear form on functions on E by

$$\langle \phi, \psi \rangle_L = \frac{1}{2} L(\overline{\psi(\mu)}\phi(\lambda) + \overline{\phi(\mu)}\psi(\lambda)) + \frac{i}{2} L(i(\overline{\phi(\mu)}\psi(\lambda) - \overline{\psi(\mu)}\phi(\lambda))).$$

By a normal families argument, one can show that $\mathscr{C}_\delta(E)$ is closed, so

$$\overline{\phi(\mu)}\phi(\lambda) = \sum_{k=0}^{\infty} \overline{\phi(\mu)}\overline{\delta_1(\mu)^k}\delta_1(\lambda)^k \phi(\lambda)[1 - \overline{\delta_1(\mu)}\delta_1(\lambda)] \in \mathscr{C}_\delta(E).$$

Therefore the form $\langle \cdot, \cdot \rangle_L$ is actually positive semi-definite, so after modding out by a null-space if necessary, one gets an inner product on $\mathscr{V}$. The fact that L is non-negative on $\mathscr{C}_\delta(E)$ means that multiplication by each δ_l is a contraction on $\mathscr{V}$. Since f is in ball $H^\infty_{\delta,\text{gen}}(E)$, multiplication by f is also a contraction, and applying it to the function that is constantly 1, one gets that

$$0 \leq \langle 1, 1 \rangle_L - \langle f, f \rangle_L = L(1 - \overline{f(\mu)}f(\lambda)),$$

as desired. □

DEFINITION 3. Let m be a positive integer and $\mathscr{H}$ a separable infinite dimensional Hilbert space. We let $\mathscr{F}_m$ denote the collection of pairwise commuting m-tuples of contractions acting on $\mathscr{H}$. If $F \in O(\mathbf{D}^m)$, we define $\|F\|_m$ by

$$\|F\|_m = \sup_{\substack{T \in \mathscr{F}_m \\ \sigma(T) \subset \mathbf{D}^m}} \|F(T)\|. \tag{3.3}$$

For now, by $F(T)$ we shall mean the Taylor functional calculus as defined in [15]; though we shall see in Section 8 that we can use a much simpler definition. We let H^∞_m denote the set of all $F \in O(\mathbf{D}^m)$ such that $\|F\|_m < \infty$.

PROPOSITION 4. *H^∞_m equipped with $\| \cdot \|_m$ is a Banach algebra. Furthermore, $H^\infty_m \subseteq H^\infty(\mathbf{D}^m)$ and if $F \in H^\infty_m$, then $\sup_{z \in \mathbf{D}^m} |F(z)| \leq \|F\|_m$.*

PROOF. This is proved for example in [4, Proposition 2.4]. □

LEMMA 3. *If $\{F_n\}$ is a sequence in ball H^∞_m, $F \in O(\mathbf{D}^m)$ and $F_n \to F$ in $O(\mathbf{D}^m)$, then $F \in \text{ball } H^\infty_m$.*

PROOF. If $F_n \to F$ and $\sigma(T) \subset \mathbf{D}^m$, then $F_n(T) \to F(T)$ in operator norm, by continuity of the Taylor functional calculus [15]. Hence, if $T \in \mathscr{F}_m$ and $\sigma(T) \subset \mathbf{D}^m$,

$$\|F(T)\| = \lim_{n\to\infty} \|F_n(T)\| \leq 1.$$

Therefore $\|F\|_m \leq 1$. □

LEMMA 4. *If $F \in \text{ball } H^\infty_m$, then there exist m positive definite functions on $\mathbf{D}^m$, $A_1, \ldots, A_m$, such that*

$$1 - \overline{F(w)}F(z) = \sum_{l=1}^{m} (1 - \overline{w_l} z_l) A_l(z, w)$$

for all $z, w \in \mathbf{D}^m$.

PROOF. This was proved in [1]. □

THEOREM 1. *Let $E \subseteq G_\delta$ and let $f \in E^{\mathbf{C}}$. The following conditions are equivalent.*

(a) $f \in \text{ball } H^\infty_{\delta,\text{gen}}(E)$.
(b) $[1 - \overline{f(\mu)} f(\lambda)] \in \mathscr{C}_\delta(E)$.
(c) *f has a (δ, E)-realization.*
(d) $\exists F \in \text{ball } H^\infty_m$; $\forall \lambda \in E \;\; f(\lambda) = F(\delta(\lambda))$.

PROOF. That (b) implies (c) follows from a Lurking Isometry argument. Indeed, suppose

$$[1 - \overline{f(\mu)} f(\lambda)] = \sum_{l=1}^{m} (1 - \overline{\delta_l(\mu)} \delta_l(\lambda)) a_l(\lambda, \mu), \tag{3.4}$$

where for $l = 1, \ldots, m$, $a_l(\lambda, \mu)$ is positive definite on E. For each $1 \leq l \leq m$, find Hilbert space valued functions u_l such that

$$a_l(\lambda, \mu) = \langle u_l(\lambda), u_l(\mu) \rangle.$$

Then (3.4) can be interpreted as saying that the map

$$V : \begin{bmatrix} 1 \\ \delta(\lambda)(\bigoplus u_l(\lambda)) \end{bmatrix} \mapsto \begin{bmatrix} f(\lambda) \\ (\bigoplus u_l(\lambda)) \end{bmatrix}$$

is an isometry, and this gives the (δ, E)-realization.

To see that (c) implies (d), let (a, β, γ, D) be a realization for f as in Definition 1. If we define F by the formula,

$$F(Z) = a + \langle Z(1 - DZ)^{-1} \gamma, \beta \rangle, \tag{3.5}$$

then $F \in \text{ball } H_m^\infty$ by [5], and $f = F \circ \delta | E$. To see that (d) implies (a), fix $F \in \text{ball } H_m^\infty$ and assume that $f = F \circ \delta | E$. If Λ is a finite subset of E and $T \in \mathscr{F}_{\delta,\Lambda}$, then $\delta(T) \in \mathscr{F}_m$. Hence,

$$\|f(T)\| = \|F(\delta(T))\| \leq 1.$$

We now turn to the task of proving that (a) implies (b). Assume that $f \in \text{ball } H^\infty_{\delta,\text{gen}}(E)$. Let $\{\lambda_k\}$ be a dense sequence in E, and for $n \geq 1$, set $\Lambda_n = \{\lambda_1, \ldots, \lambda_n\}$. Fix n. As $f \in \text{ball } H^\infty_{\delta,\text{gen}}(E)$, $f \in \text{ball } H^\infty_{\delta,\text{gen}}(\Lambda_n)$. Hence, by Lemma 2, $[1 - \overline{f(\mu)} f(\lambda)] \in \mathscr{C}_\delta(\Lambda_n)$. As (b) implies (c) and (c) implies (d), there exists $F_n \in \text{ball } H_m^\infty$ such that $f|\Lambda_n = (F_n \circ \delta)|\Lambda_n$.

By PROPOSITION 4, $\{F_n\}$ is uniformly bounded on $\mathbf{D}^m$. Hence, there exist $F \in O(\mathbf{D}^m)$ and a subsequence $\{F_{n_i}\}$ such that $\{F_{n_i}\} \to F$ as $i \to \infty$ in $O(\mathbf{D}^m)$. By Lemma 3, $F \in \text{ball } H_m^\infty$. Furthermore, as $f|\Lambda_n = (F_n \circ \delta)|\Lambda_n$ for each n, $f(\lambda_k) = F(\delta(\lambda_k))$ for all k. But $\{\lambda_k\}$ is dense in E, so that, by Corollary 1, $f(\lambda) = F(\delta(\lambda))$ for all $\lambda \in E$. Consequently, if $A_1, \ldots, A_m$ are as in Lemma 4,

$$1 - \overline{f(\mu)} f(\lambda) = 1 - \overline{F(\delta(\mu))} F(\delta(\lambda)) = \sum_{l=1}^{m} (1 - \overline{\delta_l(\mu)} \delta_l(\lambda)) A_l(\delta(\lambda), \delta(\mu)).$$

As the functions, $a_l(\lambda, \mu) = A_l(\delta(\lambda), \delta(\mu))$, are positive definite on E, this proves via Definition 2 that $1 - \overline{f(\mu)} f(\lambda) \in \mathscr{C}_\delta(E)$. □

4. Utility

A fundamental fact, which allows for the purely operator theoretic construction in Section 2 to have significance for several complex variables, is that the elements of $H^\infty_{\delta,\text{gen}}(G_\delta)$ are holomorphic functions on G_δ. We adopt the abbreviated notation, $H^\infty_{\delta,\text{gen}}$, for $H^\infty_{\delta,\text{gen}}(G_\delta)$.

PROPOSITION 5. *$H^\infty_{\delta,\text{gen}}$ equipped with $\|\cdot\|_{\delta,\text{gen}}$ is a Banach algebra. Furthermore, if $H^\infty(G_\delta)$ denotes the space of bounded holomorphic functions on G_δ equipped with the sup norm, $\|\cdot\|_\infty$, then $H^\infty_{\delta,\text{gen}} \subseteq H^\infty(G_\delta)$ and $\|\phi\|_\infty \leq \|\phi\|_{\delta,\text{gen}}$ for all $\phi \in H^\infty_{\delta,\text{gen}}$.*

PROOF. That $H^\infty_{\delta,\text{gen}}$ is a Banach algebra, $H^\infty_{\delta,\text{gen}} \subseteq L^\infty(G_\delta)$, and $\|\phi\|_\infty \leq \|\phi\|_{\delta,\text{gen}}$ whenever $\phi \in H^\infty_{\delta,\text{gen}}$ all follow from Proposition 1. That $H^\infty_{\delta,\text{gen}} \subseteq O(\mathbf{D}^m)$, follows from (a) implies (d) in THEOREM 1. □

We now come to one of the central points of the paper. Our goal is to use the operator theoretic construction of Section 2 and the representation result THEOREM 1 to study holomorphic functions on domains of holomorphy.

If this program is to be carried out, then there must be an ample supply of holomorphic functions in $H^\infty_{\delta,\text{gen}}$. Thus, the following definition is germane.

DEFINITION 4. Let $\delta \in O(U)^m$. We say that δ is *utile* if $f \in H^\infty_{\delta,\text{gen}}$ whenever f is holomorphic on a neighborhood of K_δ.

4.1. An operator-theoretic characterization of utility. THEOREM 2 below gives a characterization for when a tuple δ is utile in terms of the following natural operator theoretic notions.

DEFINITION 5. Let us agree to say that δ is *bounding* if $\mathscr{F}_{\delta,\text{gen}}$ is bounded, i.e., for each $r = 1, \ldots, d$,

$$\sup_{T \in \mathscr{F}_{\delta,\text{gen}}} \|T^r\| < \infty.$$

Note that if δ is bounding and $\{T_n\}$ is a sequence in $\mathscr{F}_{\delta,\text{gen}}$, then $\bigoplus_{n=1}^\infty T_n$ is a well defined bounded operator.

DEFINITION 6. We say that δ is *spectrally determining* if δ is bounding and if $\sigma(\bigoplus_{n=1}^\infty T_n) \subset K_\delta$ for every sequence $\{T_n\}$ in $\mathscr{F}_{\delta,\text{gen}}$.

As we have adopted the strategy of studying holomorphic functions defined on a neighborhood of K_δ, the following technical modification of the notions of bounding and spectrally determining will prove useful.

DEFINITION 7. We say that δ is *strictly bounding* (resp. *strictly spectrally determining*) if there exists $t < 1$ such that $t\delta$ is bounding (resp. spectrally determining).

Note that if $t < 1$, then $\mathscr{F}_{\delta,\text{gen}} \subset \mathscr{F}_{t\delta,\text{gen}}$ so that if δ is strictly bounding, then δ is bounding. Likewise, if δ is strictly spectrally determining, then δ is spectrally determining. To prove this, assume $t\delta$ is spectrally determining and let $\{T_n\}$ be a sequence in $\mathscr{F}_{\delta,\text{gen}}$. As $\mathscr{F}_{\delta,\text{gen}} \subset \mathscr{F}_{t\delta,\text{gen}}$, $T = \bigoplus_n T_n$ is a well defined operator with $\sigma(T) \subseteq K_{t\delta} \subset U$. Hence, the operators, $\delta_l(T)$, $l = 1, \ldots, m$ make sense by the Taylor functional calculus. Furthermore, as $T_n \in \mathscr{F}_{\delta,\text{gen}}$, for each $l = 1, \ldots, m$,

$$\|\delta_l(T)\| = \|\delta_l(\textstyle\bigoplus_n T_n)\| = \|\bigoplus_n \delta_l(T_n)\| \leq 1.$$

Hence, by the spectral mapping theorem, $\sigma(T) \subseteq K_\delta$, as was to be shown.

We used the fact that $\delta_l(\bigoplus_n T_n) = \bigoplus_n \delta_l(T_n)$. This follows from Lemma 5.

LEMMA 5. *Let $\mathscr{H}_n$ be a sequence of Hilbert spaces, and let T_n be in $C\mathscr{L}(\mathscr{H}_n)^d$. Assume that $T := \bigoplus_n T_n$ is bounded, and its spectrum lies inside U. Let $g \in O(U)$. Then $g(\bigoplus_n T_n) = \bigoplus_n g(T_n)$.*

PROOF. This result follows from Vasilescu's construction of the Taylor functional calculus as an integral of an operator-valued Martinelli kernel (see [17, Section III.11] or [8]. The integrand for T is the direct sum of the integrands for each T_n. □

The authors have been unable to resolve fully the following issue.

Question: Does bounding imply spectrally determining?

However, we show in THEOREM 2 that utile and spectrally determining are equivalent, and in THEOREM 5 that strictly bounding implies utile.

In some cases **Question** can be answered.

PROPOSITION 6. *Let K_δ be a p-polyhedron, i.e. $U = \mathbf{C}^d$, and $\delta_1, \ldots, \delta_m$ are polynomials. If δ is bounding, then δ is spectrally determining.*

PROOF. Fix a sequence $\{T_n\}$ in $\mathscr{F}_{\delta,\mathrm{gen}}$. As δ is bounding, $T = \bigoplus_{n=1}^{\infty} T_n$ is a well defined bounded operator. Furthermore, if p is a polynomial, $p(T) = \bigoplus_{n=1}^{\infty} p(T_n)$. Thus, if $1 \leq l \leq m$,

$$\|\delta_l(T)\| = \|\bigoplus_{n=1}^{\infty} \delta_l(T_n)\| \leq 1,$$

so that $\sigma(\delta(T)) \subseteq (\mathbf{D}^-)^m$. Hence, by the spectral mapping theorem for polynomials, $\delta(\sigma(T)) = \sigma(\delta(T)) \subseteq (\mathbf{D}^-)^m$. Thus, $\sigma(T) \subseteq \delta^{-1}((\mathbf{D}^-)^m) = K_\delta$. □

We now can state our characterization of utility in operator terms.

THEOREM 2. *Let $\delta \in O(U)^m$. Then δ is utile if and only if δ is spectrally determining.*

The proof of THEOREM 2 will be presented in Subsection 4.3. The proof of sufficiency will be a straightforward operator-theoretic proof that relies on the Taylor functional calculus while the proof of necessity will rely on some nontrivial function theory which we discuss in the next subsection. We record the following simple corollary of PROPOSITION 6 and THEOREM 2 for future reference.

PROPOSITION 7. *Let K_δ be a p-polyhedron. If δ is bounding, then δ is utile.*

4.2. Some function theory on domains of holomorphy. In this section we record some results from function theory on domains of holomorphy that will be used in the sequel. Recall that $O(U)$ is naturally a Fréchet space when endowed with the topology of uniform convergence on compact subsets of U. A basic fact which we shall require is the following "baby" Corona Theorem.

The result is well-known, though we cannot find a reference where it is explicitly stated. It can be proved by the observation that if α is not in U^-, then the functions $R^r_\alpha(\lambda)$ can be chosen to be continuous, because the maximal ideal space of $A(U^-)$ is U^- (see e.g. [12]). If $\alpha \in \partial U$, take a sequence of points α_n tending to α from outside U^- and use a normal families argument.

THEOREM 3. *If $\alpha \in \mathbf{C}^d \setminus U$, then there exist $R^1_\alpha, \dots, R^d_\alpha \in O(U)$ such that*

$$1 = \sum_{r=1}^{d} R^r_\alpha(\lambda)(\lambda^r - \alpha^r) \tag{4.1}$$

for all $\lambda \in U$.

Recall that if $\mathscr{A}$ is an algebra, then a *complex homomorphism* of $\mathscr{A}$ is a homomorphism defined on $\mathscr{A}$ whose target algebra is $\mathbf{C}$. The following result is also well-known.

THEOREM 4. *χ is a continuous complex homomorphism of $O(U)$ if and only if there exists $\alpha \in U$ such that $\chi(f) = f(\alpha)$ for all $f \in O(U)$.*

PROOF. Sufficiency is obvious. To prove necessity, let λ^r be the r-th coordinate function, and define $\alpha^r = \chi(\lambda^r)$. If $\alpha \notin U$, then applying χ to both sides of (4.1) would yield a contradiction.

If $\alpha \in U$, then by [11, Thm. VII.4.1], for every f in $O(U)$, there are, for $1 \le r \le d$, functions Q^r in $O(U)$ satisfying

$$f(\lambda) - f(\alpha) = \sum_{r=1}^{d} (\lambda^r - \alpha^r) Q^r(\lambda).$$

Applying χ to both sides, we get that $\chi(f) = f(\alpha)$. $\square$

4.3. The proof of the utility theorem. This subsection is devoted to a proof of Theorem 2. First assume that δ is spectrally determining. Fix f, holomorphic on a neighborhood of K_δ. We need to show that $f \in H^\infty_{\delta,\mathrm{gen}}$. To that end, fix a sequence, $\{T_n\}$ in $\mathscr{F}_{\delta,\mathrm{gen}}$. As $\{T_n\}$ in $\mathscr{F}_{\delta,\mathrm{gen}}$ and δ is spectrally determining, if we set $T = \bigoplus_n T_n$, then T is a well defined bounded operator with $\sigma(T) \subseteq K_\delta$. Hence, as f is holomorphic on a neighborhood of K_δ, $f(T)$, as defined by the Taylor functional calculus, is a well defined bounded operator. But $f(T) = \bigoplus_n f(T_n)$ by LEMMA 5, so $\|f(T_n)\| \le \|f(T)\|$ for all n. Summarizing, we have shown that if $\{T_n\}$ is a sequence in $\mathscr{F}_{\delta,\mathrm{gen}}$, then $\sup_n \|f(T_n)\| < \infty$. Thus, $\|f\|_{\delta,\mathrm{gen}} < \infty$ and $f \in H^\infty_{\delta,\mathrm{gen}}$.

Now assume that δ is utile. We first show that δ is bounding. As $f(\lambda) = \lambda^r$ is holomorphic on a neighborhood of K_δ and δ is utile, $f \in H^\infty_{\delta,\text{gen}}$. Hence,

$$\sup_{T \in \mathscr{T}_{\delta,\text{gen}}} \|T^r\| = \sup_{T \in \mathscr{T}_{\delta,\text{gen}}} \|f(T)\| = \|f\|_{\delta,\text{gen}} < \infty.$$

This proves that δ is bounding.

To complete the proof that δ is spectrally determining, fix a sequence, $\{T_n\}$ in $\mathscr{T}_{\delta,\text{gen}}$. As δ is bounding, $T = \bigoplus_n T_n$ is a well defined bounded operator. For definiteness, we assume T is in $C\mathscr{L}(\mathscr{H})^d$. We need to show that $\sigma(T) \subseteq K_\delta$.

Since δ is utile, $O(U) \subset H^\infty_{\delta,\text{gen}}$. It follows that the formula

$$\tau(f) = \bigoplus_n f(T_n), \qquad f \in O(U)$$

defines an algebra homomorphism, $\tau : O(U) \to \mathscr{L}(\mathscr{H})$, with the property that $\tau(1) = 1$, and which the closed graph theorem implies is continuous. Let $\mathscr{A}$ be the operator norm closed algebra generated by ran τ. Evidently, $\mathscr{A}$ is a commutative Banach algebra with unit, containing the components of T. We let $\sigma_{\mathscr{A}}(T)$ denote the algebraic spectrum of T in this algebra $\mathscr{A}$. It is well known that $\sigma(T) \subseteq \sigma_{\mathscr{A}}(T)$ (see, e.g., [8]). Also, by Gelfand Theory, if X denotes the space of complex homomorphisms of $\mathscr{A}$, then $\sigma_{\mathscr{A}}(T) = \{\chi(T) \,|\, \chi \in X\}$. Thus, the proof of THEOREM 2 will be complete if we can prove the following claim.

Claim 1. If $\chi \in X$, then $\chi(T) \in K_\delta$.

To prove this claim we shall require the following simple lemma.

LEMMA 6. *If δ is utile and Ω is a neighborhood of K_δ, then there exists a positive constant, c, such that*

$$\|f\|_{\delta,\text{gen}} \leq c \sup_{\lambda \in \Omega} |f(\lambda)|$$

for all $f \in H^\infty(\Omega)$.

PROOF. The utility of δ guarantees that the formula $L(f) = f|G_\delta$ defines a linear transformation $L : H^\infty(\Omega) \to H^\infty_{\delta,\text{gen}}$. The closed graph theorem implies that L is bounded. □

We now turn to the proof of **Claim 1**.

Let $\chi \in X$ and set $\alpha = \chi(T)$. As $\chi \circ \tau$ is a continuous complex homomorphism of $O(U)$, it follows from THEOREM 4 that

$$\alpha = \chi(T) = \chi(\tau(\lambda)) = (\chi \circ \tau)(\lambda) \in U.$$

To see that α is in K_δ, we argue by contradiction. If $\alpha \notin K_\delta$, then there exists $t < 1$ such that $K_\delta \subset G_{t\delta}$ and $\alpha \notin K_{t\delta}$. As $K_{t\delta}$ is $O(U)$-convex and $\alpha \in U \backslash K_{t\delta}$, there exists a sequence $\{f_k\}$ in $O(U)$ such that

$$\sup_{\lambda \in K_{t\delta}} |f_k(\lambda)| \leq 1 \tag{4.2}$$

and

$$|f_k(\alpha)| \to \infty. \tag{4.3}$$

Evidently, (4.2) implies via an application of LEMMA 6 that there exists a constant c such that $\|f_k\|_{\delta,\mathrm{gen}} \leq c$. But then,

$$|f_k(\alpha)| = |\chi(\tau(f_k))| \leq \|\tau(f_k)\| \leq \|f_k\|_{\delta,\mathrm{gen}} \leq c$$

for all k, contradicting (4.3). This completes the proof of **Claim 1**. □

4.4. Strictly bounding implies utile.

THEOREM 5. *If δ is strictly bounding and f is holomorphic on a neighborhood of K_δ, then there exists $F \in H^\infty_{\delta,\mathrm{gen}}$ such that $F = f$ on G_δ.*

PROOF. Suppose that δ is strictly bounding and f is holomorphic on a neighborhood of K_δ. Choose $t < 1$ so that $\mathscr{F}_{t\delta,\mathrm{gen}}$ is bounded, but sufficiently large so that in addition, f is holomorphic on $G_{t\delta}$.

We first construct a mapping from $t^{-1}\mathbf{D}^m$ into $\mathbf{C}^d$ that serves as a left inverse for δ. Fix r. As $\|\lambda^r\|_{t\delta,\mathrm{gen}} < \infty$, an application of the equivalence of (a) and (d) in THEOREM 1 (with $E = G_{t\delta}$ and $z = \lambda^r$) yields $\Phi^r \in H^\infty_m$ with

$$\lambda^r = \Phi^r(t\delta(\lambda)), \qquad \lambda \in G_{t\delta}. \tag{4.4}$$

Define $\Psi^r \in O(t^{-1}\mathbf{D}^m)$ by setting $\Psi^r(z) = \Phi^r(tz)$ for $z \in t^{-1}\mathbf{D}^m$. Finally, amalgamate the functions Ψ^r, $r = 1, \ldots, d$ into a mapping, $\Psi : t^{-1}\mathbf{D}^m \to \mathbf{C}^d$.

Now, $\delta : G_{t\delta} \to t^{-1}\mathbf{D}^m$ and $\Psi : t^{-1}\mathbf{D}^m \to \mathbf{C}^d$. Furthermore, (4.4) implies that $\Psi \circ \delta = \mathrm{id}_{G_{t\delta}}$. Hence, δ is one-to-one, proper, and unramified. As a consequence, not only does δ embed $G_{t\delta}$ as an analytic submanifold of $t^{-1}\mathbf{D}^m$, but $V = \delta(G_{t\delta})$ is an analytic variety and $\omega : V \to \mathbf{C}$ is holomorphic on V if and only if $\omega \circ \delta$ is holomorphic on $G_{t\delta}$. As f is holomorphic on $G_{t\delta}$, it follows that,

$$\omega(\delta(\lambda)) = f(\lambda), \qquad \lambda \in G_{t\delta}, \tag{4.5}$$

defines a holomorphic function, ω, on V. By the Cartan extension theorem, there exists $H \in O(t^{-1}\mathbf{D}^m)$ such that

$$H(z) = \omega(z), \qquad z \in V. \tag{4.6}$$

Comparing (4.5) and (4.6) we see that $f(\lambda) = H(\delta(\lambda))$ whenever $\lambda \in G_{t\delta}$. As $H \in O(t^{-1}\mathbf{D}^m)$ implies that $H \in H_m^\infty$ (this follows from PROPOSITION 6 and THEOREM 2), it follows that Condition (d) in THEOREM 1 holds. Hence, applying the equivalence of (a) and (d) in that theorem a second time, we deduce that $f \in H^\infty_{\delta,\mathrm{gen}}$, as was to be shown. □

5. Subordination and utility

If $\delta \in O(U)^m$ and $\gamma \in O(U)^n$ are both tuples of non-constant functions, we say that γ is an *extension of* δ if $n \geq m$ and $\gamma_l = \delta_l$ for $1 \leq l \leq m$. If γ is an extension of δ and in addition, $K_\gamma = K_\delta$, we say that γ *is subordinate to* δ. Note that if γ is subordinate to δ, then $G_\gamma = G_\delta$, $\mathscr{F}_{\gamma,\mathrm{gen}} \subseteq \mathscr{F}_{\delta,\mathrm{gen}}$, $\|\cdot\|_{\gamma,\mathrm{gen}} \leq \|\cdot\|_{\delta,\mathrm{gen}}$ and $H^\infty_{\delta,\mathrm{gen}} \subseteq H^\infty_{\gamma,\mathrm{gen}}$.

We now describe a procedure from [4] that is very effective for using $H^\infty_{\delta,\mathrm{gen}}$ to do function theory. Suppose one is given a p-polyhedron, K_δ and a function, f, holomorphic on a neighborhood of K_δ. As K_δ is a p-polyhedron, PROPOSITION 7 will imply that $f \in H^\infty_{\delta,\mathrm{gen}}$ provided δ is bounding. While δ may not be bounding, there nevertheless exists a bounding subordinate extension γ of δ. As $K_\gamma = K_\delta$, the classical function theory has not changed, but now, as $f \in H^\infty_{\gamma,\mathrm{gen}}$, the structure theory from Sections 2 and 3 of the paper can be applied to f. A particularly simple way to construct a bounding subordinate extension γ of δ is to set

$$\gamma = (\delta_1, \ldots, \delta_m, \varepsilon\lambda^1, \ldots, \varepsilon\lambda^d) \tag{5.1}$$

where ε is chosen sufficiently small that for each $r = 1, \ldots, d$, $|\varepsilon\lambda^r| \leq 1$ for all $\lambda \in K_\delta$ (recall that δ is such that K_δ is always compact). Note that if one assumes that $K_\delta \subset \mathbf{D}^m$ (as Oka did), then one can choose $\varepsilon = 1$.

In [4] the idea in the previous paragraph was used to give a new proof of Oka's Extension Theorem. We give another proof in Section 6 below. One novel feature of this new argument is that it is valid not only when γ is defined by (5.1), but rather, it merely requires that γ is bounding. Another interesting feature of the argument is that it provides bounds for the extension.

When one passes from p-polyhedra to the more general case of analytic polyhedra defined in domains of holomorphy, as in light of **Question**, the situation is more complicated. To imitate the argument for p-polyhedra one now needs to know that there are extensions of δ that are spectrally determining.

THEOREM 6. *If $\delta \in O(U)^m$ is non-constant, then there exist $n \geq m$ and $\gamma \in O(U)^n$ such that γ is subordinate to δ and γ is spectrally determining.*

PROOF. Without loss of generality, we may assume that δ is bounding and $\|\lambda^r\|_{\delta,\,\mathrm{gen}} \leq 1$ for each r. Note that this implies that if $T \in \mathscr{F}_{\delta,\,\mathrm{gen}}$, then $\|T^r\| \leq 1$ for $r = 1, \ldots, d$. Consequently,

$$\text{if } \{T_n\} \text{ is a sequence in } \mathscr{F}_{\delta,\,\mathrm{gen}}, \text{ then } \sigma\Big(\bigoplus_{n=1}^{\infty} T_n\Big) \subseteq (\mathbf{D}^-)^d. \tag{5.2}$$

We now define γ. Let $E = (\mathbf{D}^-)^d \backslash U$. (The set E may be empty, in which case we let $\gamma = \delta$, and skip to the last paragraph of the proof). By THEOREM 3, for each $\alpha \in E$ there exist functions $R_\alpha^1, \ldots, R_\alpha^d \in O(U)$ such that

$$\sum_{r=1}^{d} R_\alpha^r(\lambda)(\lambda^r - \alpha^r) = 1 \tag{5.3}$$

for all $\lambda \in U$. We set

$$M_\alpha^r = \max_{\lambda \in K_\delta} |R_\alpha^r(\lambda)| \tag{5.4}$$

and let $M_\alpha = (M_\alpha^1, \ldots, M_\alpha^d)$ so that $\|M_\alpha\|$, the Euclidean norm of M_α, is given by

$$\|M_\alpha\| = \left(\sum_{r=1}^{d} |M_\alpha^r|^2\right)^{1/2}. \tag{5.5}$$

We let $B_\alpha(\varepsilon)$ denote the Euclidean ball in $\mathbf{C}^d$ centered at α and fix positive $s < 1$. As E is compact and $\left\{ B_\alpha\left(\dfrac{s}{\|M_\alpha\|}\right) \,\middle|\, \alpha \in E \right\}$ is an open cover of E, there exist $\alpha_1, \ldots, \alpha_n \in E$ such that

$$E \subset \bigcup_{i=1}^{n} B_{\alpha_i}\left(\frac{s}{\|M_{\alpha_i}\|}\right). \tag{5.6}$$

For $r = 1, \ldots, d$ and $i = 1, \ldots, n$ we define functions $\rho_i^r \in O(U)$ by the formulas,

$$\rho_i^r = \frac{1}{M_{\alpha_i}^r} R_{\alpha_i}^r. \tag{5.7}$$

Finally, define an extension γ of δ by setting

$$\gamma = (\delta_1, \delta_2, \ldots, \delta_m, \rho_1^1, \rho_1^2, \ldots, \rho_1^d, \ldots\ldots, \rho_n^1, \rho_n^2, \ldots, \rho_n^d). \tag{5.8}$$

Note that (5.4) and (5.7) imply that for each r and i, $\max_{K_\delta} |\rho_i^r| \leq 1$ so that γ is subordinate to δ.

We now show that γ, as constructed above, is spectrally determining. Accordingly, fix a sequence $\{T_n\}$ in $\mathscr{F}_{\gamma,\,\mathrm{gen}}$ and set $T = \bigoplus T_n$. We need to show that $\sigma(T) \subseteq K_\delta$.

Claim 2. $\sigma(T) \subset U$.

To prove this claim, observe that (5.2) implies that it suffices to prove that

$$\beta \in E \Rightarrow \beta \notin \sigma(T). \tag{5.9}$$

To prove (5.9), fix $\beta \in E$. By (5.6) there exists i such that

$$\|\beta - \alpha_i\| \leq \frac{s}{\|M_{\alpha_i}\|}. \tag{5.10}$$

If for $S \in \mathscr{F}_{\gamma,\,\mathrm{gen}}$, we define an operator $X(S)$ by the formula,

$$X(S) = \sum_{r=1}^{d} R^r_{\alpha_i}(S)(\beta^r - \alpha_i^r),$$

then (5.3) implies that

$$\begin{aligned}\sum_{r=1}^{d} R^r_{\alpha_i}(S)(S^r - \beta^r) &= \sum_{r=1}^{d} R^r_{\alpha_i}(S)(S^r - \alpha_i^r) - \sum_{r=1}^{d} R^r_{\alpha_i}(S)(\beta^r - \alpha_i^r) \\ &= 1 - X(S). \end{aligned}\tag{5.11}$$

Also, since by the construction of γ and (5.7) we have that

$$\|R^r_{\alpha_i}(S)\| \leq M^r_{\alpha_i}, \tag{5.12}$$

we see via (5.5), (5.10) and (5.12) that

$$\begin{aligned}\|X(S)\| &= \left\| \sum_{r=1}^{d} R^r_{\alpha_i}(S)(\beta^r - \alpha_i^r) \right\| \\ &\leq \sum_{r=1}^{d} \|R^r_{\alpha_i}(S)\|\, |\beta^r - \alpha_i^r| \\ &\leq \sum_{r=1}^{d} M^r_{\alpha_i} |\beta^r - \alpha_i^r| \\ &\leq \left(\sum_{r=1}^{d} |M^r_{\alpha_i}|^2 \right)^{1/2} \left(\sum_{r=1}^{d} |\beta^r - \alpha_i^r|^2 \right)^{1/2} \\ &= \|M_{\alpha_i}\|\, \|\beta - \alpha_i\| \\ &\leq \|M_{\alpha_i}\| \frac{s}{\|M_{\alpha_i}\|} \\ &= s. \end{aligned}\tag{5.13}$$

We now let $S = T_n$ and direct sum. If we set

$$Y^r = \bigoplus_{n=1}^{\infty} R^r_{\alpha_i}(T_n),$$

then (5.12) implies that Y^r is a well defined bounded operator that commutes with T. Likewise, if we set

$$X = \bigoplus_{n=1}^{\infty} X(T_n),$$

then (5.13) implies that X is a well defined bounded operator that commutes with T with the added property that

$$\|X\| \leq s < 1. \tag{5.14}$$

Finally, observe that by (5.11),

$$\sum_{r=1}^{d} Y^r(T^r - \beta^r) = 1 - X. \tag{5.15}$$

As (5.14) and (5.15) imply that $\sum_{r=1}^{d} Y^r(T^r - \beta^r)$ is invertible, it follows that $\beta \notin \sigma(T)$. This completes the proof of (5.9) which establishes **Claim 2**.

To complete the proof of THEOREM 6 we need to show that $\sigma(T) \subseteq K_\delta$. As $\sigma(T) \subset U$ and the functions δ_l are in $O(U)$, it follows from the Taylor functional calculus that the operators $\delta_l(T)$ are well defined. Furthermore, by the spectral mapping theorem for analytic functions, $\delta(\sigma(T)) = \sigma(\delta(T))$. But $\|\delta_l(T)\| \leq 1$ for each l, so that $\sigma(\delta(T)) \subseteq (\mathbf{D}^-)^m$. Hence, $\sigma(T) \subseteq \delta^{-1}((\mathbf{D}^-)^m) = K_\delta$. □

6. The Oka extension theorem on analytic polyhedrons

As discussed in [4], the following result, THEOREM 7 below, represents a refinement of a classical theorem of Oka. However, unlike in [4], we will not use the Oka-Weil Theorem. We define

$$\|\Phi\|_{H^\infty_{\mathfrak{m}}} = \sup\{\|\Phi(T)\| : tT \in \mathscr{F}_m, \sigma(tT) \subset \mathbf{D}^m\}.$$

THEOREM 7. *Let U be a domain of holomorphy and assume that $\delta_1, \ldots, \delta_m \in O(U)$ with δ strictly bounding. If ϕ is holomorphic on a neighborhood of K_δ, then there exists Φ holomorphic on a neighborhood of $(\mathbf{D}^m)^-$ such that $\phi(\lambda) = \Phi \circ \delta(\lambda)$ for all $\lambda \in G_\delta$. Moreover, Φ can be chosen so that $\forall t < 1$ and sufficiently close to 1, we have*

$$\|\Phi\|_{H^\infty_{\mathfrak{m}}} = \|\phi\|_{H^\infty_{\delta,\,\mathrm{gen}}}.$$

PROOF. Let ϕ be holomorphic on a neighborhood of K_δ. Choose $t' < t < 1$ such that $\mathscr{F}_{t'\delta,\,\mathrm{gen}}$ is bounded and ϕ is holomorphic on a neighbor-

hood of $K_{t'\delta}$. By THEOREM 5, $\phi \in H^\infty_{t\delta,\,\mathrm{gen}}$. Hence, by THEOREM 1, there exists $\Psi \in H^\infty_m$ such that $\phi(\lambda) = \Psi(t\delta(\lambda))$ for all $\lambda \in G_{t\delta}$, and with $\|\Psi\|_{H^\infty_m} = \|\phi\|_{H^\infty_{t\delta,\,\mathrm{gen}}}$. If we define Φ by $\Phi(z) = \Psi(tz)$, then Φ is holomorphic on a neighborhood of $(\mathbf{D}^m)^-$ and $\phi(\lambda) = \Phi \circ \delta(\lambda)$ for all $\lambda \in G_\delta$. Moreover, $\|\Phi\|_{H^\infty_{\mathbf{m}}} = \|\Psi\|_{H^\infty_m} = \|\phi\|_{H^\infty_{t\delta,\,\mathrm{gen}}}$. □

We also get a generalization of the Oka-Weil theorem.

THEOREM 8. *Let U be a domain of holomorphy and assume that $\delta_1, \ldots, \delta_m \in O(U)$ with K_δ compact in $\mathbf{C}^d$. If ϕ is holomorphic on a neighborhood of K_δ, then ϕ can be uniformly approximated on K_δ by holomorphic functions on U.*

PROOF. By THEOREM 6, we can choose $\gamma \in O(U)^n$ such that $\gamma_l = \delta_l$ for $1 \le l \le m$, $K_\gamma = K_\delta$, and γ is spectrally determining. By THEOREM 2, γ is utile, so $\phi \in H^\infty_{\gamma,\,\mathrm{gen}}$. By THEOREM 1, there is a function Φ in $H^\infty_{\mathbf{n}}$ such that $\phi = \Phi \circ \gamma$. The partial sums of the Taylor series of Φ, composed with γ, give a sequence of polynomials in γ that converge uniformly to ϕ on K_δ.

7. A Cartan extension theorem

In this section U is a domain of holomorphy in $\mathbf{C}^d$. We assume that V is an analytic set in U with the special presentation: there exists $\eta = (\eta_1, \ldots, \eta_n) \in O(U)^n$ such that

$$V = \{\lambda \in U : \eta(\lambda) = 0\}. \tag{7.1}$$

Any domain of holomorphy U can be exhausted by a sequence $G_{\delta^k} \subset K_{\delta^k} \subset G_{\delta^{k+1}} \subset U$ of analytic polyhedra, by [11, Cor. II.3.11]. Choose such a sequence. We make a second assumption about V:

$$\begin{aligned} &\forall \varepsilon > 0, \qquad \forall k \ge 1, \qquad [F \in H^\infty_{\delta^{k+1},\,\mathrm{gen}}] \wedge [F|_{V \cap G_{\delta^{k+1}}} = 0] \\ &\Rightarrow \exists \phi_r \in H^\infty_{\delta^k,\,\mathrm{gen}}; \qquad \left\| F - \sum_{r=1}^n \eta_r \phi_r \right\|_{H^\infty_{\delta^k,\,\mathrm{gen}}} \le \varepsilon. \end{aligned} \tag{7.2}$$

Inequality (7.2) will hold, for example, (once we augment each δ^k to make it utile as we can by THEOREM 6) if there is some compact set $L \subset U$ with the property that whenever Ω is an open set with $L \subset \Omega \subset U$ and $F \in O(\Omega)$ vanishes on $V \cap \Omega$, then F is in the ideal in $O(\Omega)$ generated by η.

In this section we wish to prove a special case of the Cartan extension theorem, first proved in [7]; see [9, Thm. I.5] for a more recent treatment. It can be argued that the logic is circular, as we use THEOREM 5, whose proof used the Cartan extension theorem. However, whenever the answer to **Question** is

known to be yes, such as for a p-polyhedron by PROPOSITION 6, one does not need THEOREM 5.

THEOREM 9. *Let $V \subseteq U$ satisfy* (7.1) *and* (7.2). *Suppose f is holomorphic on a neighborhood Ω of V. Then f has a holomorphic extension to U.*

PROOF. By THEOREM 6, one can assume that each $\delta^k = (\delta_1^k, \dots, \delta_{n_k}^k)$ is spectrally determining, and therefore bounding. Moreover, by multiplying δ^k by a number slightly greater than one (and so slightly shrinking G_{δ^k}), one can assume that each δ^k is strictly bounding.

For each k, there exists a positive number M^k such that

$$\beta^k := (\delta_1^k, \dots, \delta_{n_k}^k, M^k\eta_1, \dots, M^k\eta_n)$$

will satisfy $K_{\beta^k} \subset\subset \Omega \cap G_{\delta^{k+1}}$. As β^k is an extension of δ^k, it is *a fortiori* strictly bounding, so by THEOREM 5 it is utile. Therefore by THEOREM 1, we can find a function Ψ^k in $H^\infty_{\mathbf{n_k}+\mathbf{n}}$ such that $\Psi^k \circ \beta^k$ is a holomorphic function on G_{β^k} that equals the extension of f to $\Omega \cap G_{\delta^k}$.

Now let

$$\Phi^k(\lambda^1, \dots, \lambda^{n_k}) = \Psi^k(\lambda^1, \dots, \lambda^{n_k}, 0, \dots, 0).$$

Then $\Phi^k \circ \delta^k$ is holomorphic on G_{δ^k}, and agrees with f on $V \cap G_{\delta^k}$. Define

$$h^k(\lambda^1, \dots, \lambda^d) = \Phi^k(\delta^k(\lambda)).$$

Each h^k is in $H^\infty_{\delta^k,\text{gen}}$ and provides an extension of f to G_{δ^k}; we would like to get an extension on all of U at once.

Let us define new functions inductively. To start, we let $g^1 = h^1$, $g^2 = h^2$ and $g^3 = h^3$.

Having defined g^k in $H^\infty_{\delta^k,\text{gen}}$ with

$$g^k|_{V \cap G_{\delta^k}} = f|_{V \cap G_{\delta^k}},$$

observe that $h^{k+1} - g^k$ is in $H^\infty_{\delta^k,\text{gen}}$ and vanishes on $V \cap G_{\delta^k}$. So by assumption (7.2), $h^{k+1} - g^k$ can be approximated in $H^\infty_{\delta^{k-1},\text{gen}}$ by functions of the form $\sum_{r=1}^n \eta_r\phi_r$. By THEOREM 1, each ϕ_r in turn can be uniformly approximated on $K_{\delta^{k-2}}$ by polynomials in δ^{k-1}, so in particular by functions holomorphic on all of U. Therefore there is a function q^{k+1} in $O(U)$ such that $q^{k+1}|_V = 0$ and

$$\|h^{k+1} - q^{k+1} - g^k\|_{H^\infty(G_{\delta^{k-2}})} \le 2^{-k}. \tag{7.3}$$

Define $g^{k+1} = h^{k+1} - q^{k+1}$. By (7.3), the functions $\{g^k\}$ will be a Cauchy sequence on every compact subset of U, so, by passing to a subsequence, one can extract a limit that is holomorphic on U. Call this limit function F. As

each g^k agrees with f on $V \cap G_{\delta^k}$, we conclude that F is our desired extension of f to U. $\square$

8. Functional calculus

We shall use formula (3.2) to define a functional calculus for $H^\infty_{\delta,\mathrm{gen}}$. For $\mathscr{M} = \bigoplus_{l=1}^m \mathscr{M}_l$ a graded Hilbert space, let P_l be the orthogonal projection onto $\mathscr{M}_l$. If $S = (S^1, \ldots, S^m)$ is an m-tuple of operators on a Hilbert space $\mathscr{H}$, we shall let S_P denote the operator on $\mathscr{H} \otimes \mathscr{M}$ given by

$$S_P = \sum_{l=1}^m S^l \otimes P^l.$$

In this notation, $\delta(\lambda)$ from (3.1) becomes $\delta(\lambda)_P$. For any vector $\xi \in \mathscr{M}$, define

$$R_\xi : \mathscr{H} \to \mathscr{H} \otimes \mathscr{M}, \qquad v \mapsto v \otimes \xi.$$

DEFINITION 8. As in DEFINTION 1, let f be a function on G_δ with a (δ, G_δ)-realization (a, β, γ, D). Let $T \in \mathscr{F}_\delta$ be a d-tuple of commuting operators on a Hilbert space $\mathscr{H}$, and assume that $\sigma(T) \subset G_\delta$. Then we define $f(T)$ by

$$f(T) := aI_{\mathscr{H}} + R_\beta^* \delta(T)_P [I_{\mathscr{H}} \otimes I_{\mathscr{M}} - (I_{\mathscr{H}} \otimes D)\delta(T)_P]^{-1} R_\gamma. \tag{8.1}$$

Some remarks are in order.

1. Let $S = (\delta_1(T), \ldots, \delta_m(T))$. The assertion $\sigma(T) \subset G_\delta$ seems to require the calculation of the Taylor spectrum of T. But by the spectral mapping theorem, this becomes the much more innocuous assertion that
$$\sigma(S^l) \subset \mathbf{D}, \qquad 1 \le \forall l \le m.$$
2. To use (8.1), we need to be able to define $\delta_l(T)$ for each $1 \le l \le m$. If δ_l is a polynomial or rational function, this is immediate; or if δ itself comes from a (δ', G_δ)-realization, we can define $\delta(T)$ by the analogous formula. For general δ_l, it is still possible to define $\delta(T)$ in some circumstances—for example if T is a multiplication operator on a space of holomorphic functions. It is, however, essential to the definition that we know how to define $\delta(T)$.
3. In order for the right-hand side of (8.1) to make sense, we need to know that $[I_{\mathscr{H}} \otimes I_{\mathscr{M}} - (I_{\mathscr{H}} \otimes D)S_P]$ is invertible. This follows from PROPOSITION 8 below.
4. To speak unambiguously of $f(T)$, we need to know that the operator defined by (8.1) and the one defined by the Taylor functional calculus agree. This follows from THEOREM 10 below.

5. For our notion of functional calculus to be useful, we would like to know that $H^\infty_{\delta,\mathrm{gen}}$ has a rich supply of functions. In Section 4 we address the issue of when every function holomorphic on a neighborhood of K_δ is in $H^\infty_{\delta,\mathrm{gen}}$.

PROPOSITION 8. *The operator $I_{\mathscr{H}} \otimes I_{\mathscr{M}} - (I_{\mathscr{H}} \otimes D)\delta(T)_P$ is invertible whenever the Taylor spectrum of T is in G_δ.*

PROOF. It is sufficient to prove that if $\sigma(S) \subset \mathbf{D}^m$, then $\sigma[(I_{\mathscr{H}} \otimes D)S_P] \subset \mathbf{D}$. By LEMMA 7, below, there is an invertible operator A so that if $R = A^{-1}SA$, then each R^l is a strict contraction. Then

$$(A^{-1} \otimes I_{\mathscr{M}})\left[(I_{\mathscr{H}} \otimes D)\left(\sum_{l=1}^m S^l \otimes P^l\right)\right](A \otimes I_{\mathscr{M}})$$

$$= (I_{\mathscr{H}} \otimes D)\left(\sum_{l=1}^m R^l \otimes P^l\right).$$

The latter expression is a product of a strict contraction with a contraction, so its spectrum is in $\mathbf{D}$. So as $(I_{\mathscr{H}} \otimes D)S_P$ is similar to it, the spectrum is the same. □

The following lemma is multivariable version of a well-known result of G.-C. Rota [13].

LEMMA 7. *If $\sigma(S) \subset \mathbf{D}^m$, then there is an invertible operator A so that $\|A^{-1}S^lA\| < 1,\ 1 \le l \le m$.*

PROOF. As each S^l has spectral radius less than 1, choose $N > 0$ such that $\|(S^l)^N\| \le C < 1,\ 1 \le l \le m$. Define a new norm on $\mathscr{H}$ by

$$|||v||| = \frac{1}{N^m}\sqrt{\sum_p \|p(S)v\|^2},$$

where the sum is over all monomials p of degree at most $N-1$ in each variable. The norm $|||\cdot|||$ satisfies the parallelogram law, so is a Hilbert space norm. Moreover, it is similar to the norm $\|\cdot\|$, and

$$|||S^lv|||^2 - |||v|||^2 = \frac{1}{N^{2m}}\sum_q \|(S^l)^N q(S)v\|^2 - \|q(S)v\|^2$$

$$\le \frac{1}{N^{2m}}\sum_q (C^2-1)\|q(S)v\|^2$$

$$\le \frac{1}{N^{2m}}(C^2-1)\|v\|^2$$

where q ranges over monomials that have no positive powers of z^l. So in the $|||\cdot|||$ norm, each S^l is a strict contraction. Let A be the similarity between the Hilbert space with the $|||\cdot|||$ norm and the original $\|\cdot\|$ norm. □

THEOREM 10. *Assume f and T satisfy the hypotheses of* DEFINITION 8. *Then the operator $f(T)$ defined be the Taylor functional calculus equals the operator defined by* (8.1).

PROOF. Let $F \in \text{ball } H_m^\infty$ be defined by (3.5), so $F(\delta(\lambda)) = f(\lambda)$. As the Taylor functional calculus respects composition, it is enough to prove that $F(S)$ defined by the Taylor calculus and $F(S)$ defined by

$$aI_{\mathscr{H}} + R_\beta^* S_P [I_{\mathscr{H}} \otimes I_{\mathscr{M}} - (I_{\mathscr{H}} \otimes D) S_P]^{-1} R_\gamma \tag{8.2}$$

agree. By the argument in the proof of PROPOSITION 8, the contraction $(I_{\mathscr{H}} \otimes D)S_P$ has spectral radius less than 1, so the Neumann series expansion of (8.2) converges absolutely. The partial sums form a sequence of polynomials in S that converge to $F(S)$ in norm, and by continuity they also converge to $F(S)$ defined by the Taylor functional calculus.

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