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The values of the generalized matrix functions of 3×3 matrices

Ryo Tabata

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Abstract. When A is a 3×3 positive semi-definite Hermitian matrix, Schur's inequality and the permanental dominance conjecture are known to hold. In [5], we determined the possible positions of the normalized generalized matrix functions relative to the determinant and the permanent except in the case that the order of the subgroup is 2. The purpose of this paper is to determine the possible positions in the last open case.

1. Introduction

Let $M_n(\mathbf{C})$ be the set of $n \times n$ complex matrices. *The generalized matrix function* on $M_n(\mathbf{C})$ associated to G and χ is defined to be

$$d_\chi^G(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where G is a subgroup of the symmetric group $\mathfrak{S}_n$, and χ a character of G. When $G = \mathfrak{S}_n$ and χ is an irreducible character, d_χ^G is called an *immanant*. The determinant and permanent, which are well-known functions on matrices, are examples of immanants: These are the special cases where χ are the alternating character and the trivial character of $\mathfrak{S}_n$.

If the domain is restricted to positive semi-definite Hermitian matrices, then each d_χ^G is a real-valued function. We also define the normalized generalized matrix function as $\bar{d}_\chi^G = d_\chi^G/\chi(\mathrm{id})$, where id is the identity element of G, hence $\chi(\mathrm{id})$ is the dimension of the corresponding representation. In 1918, Schur [4] proved an interesting inequality on generalized matrix functions.

Theorem 1 (Schur [4]). *If A is an $n \times n$ positive semi-definite Hermitian matrix, then*

$$\bar{d}_\chi^G(A) \geq \det A.$$

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Namely, the determinant is the smallest normalized generalized matrix function. On the other hand, the permanent is conjectured to be the largest normalized generalized matrix function:

CONJECTURE (Lieb [2]). *If A is an $n \times n$ positive semi-definite Hermitian matrix, then*

$$\operatorname{per} A \geq \bar{d}_\chi^G(A).$$

The conjecture holds for immanants with $n \leq 13$ ([3]), and for all generalized matrix functions with $n = 3$ ([1]). Hence one can write

$$\bar{d}_\chi^G(A) = t \operatorname{per} A + (1 - t) \det A$$

for some $t \in [0, 1]$. In [5], the possible values of t are determined when $G = \mathfrak{S}_3$, $\{\mathrm{id}\}$ or $\mathfrak{A}_3$. Let $R(G, \chi)$ denote the set of all possible values $t \in [0, 1]$ such that

$$\bar{d}_\chi^G(A) = t \operatorname{per} A + (1 - t) \det A$$

for some 3×3 positive semi-definite Hermitian matrices A with $\operatorname{per} A \neq \det A$.

THEOREM 2 ([5]). *Let χ_λ be the character of $\mathfrak{S}_3$ corresponding to the partition λ of* 3, triv *the trivial character, and ω a non-trivial irreducible character of $\mathfrak{A}_3$ with $\bar{\omega}$ its conjugate. Then*

(1) $R(\mathfrak{S}_3, \chi_{(3)}) = \{1\}$.
(2) $R(\mathfrak{S}_3, \chi_{(1,1,1)}) = \{0\}$.
(3) $R(\mathfrak{S}_3, \chi_{(2,1)}) = \left[0, \frac{3}{4}\right]$.
(4) $R(\{(\mathrm{id})\}, \mathrm{triv}) = \left[\frac{1}{6}, \frac{2}{3}\right]$.
(5) $R(\mathfrak{A}_3, \mathrm{triv}) = \left\{\frac{1}{2}\right\}$.
(6) $R(\mathfrak{A}_3, \omega) = R(\mathfrak{A}_3, \bar{\omega}) = \left[0, \frac{1}{\sqrt[3]{2}}\right]$.

The goal of this paper is to complete this table:

THEOREM 3 (Main Theorem). *Let $G \subset \mathfrak{S}_3$ be a subgroup of order 2, and $\chi_+ : G \to \mathbf{C}^*$ be the trivial character and $\chi_- : G \to \mathbf{C}^*$ the non-trivial irreducible character of G. Then*

$$R(G, \chi_+) = \left[\frac{1}{3}, 1\right] \quad \textit{and} \quad R(G, \chi_-) = \left[0, \frac{1}{\sqrt{3}}\right].$$

2. Proof of main theorem

In this section, we work with the 3×3 positive semi-definite Hermitian matrix

$$A = \begin{pmatrix} a & b & c \\ \bar{b} & d & e \\ \bar{c} & \bar{e} & f \end{pmatrix}.$$

The variables a, b, c, d, e, and f always refer to the entries of A.

If $\det A = \operatorname{per} A$, then we have $\bar{d}_\chi^G(A) = \det A = \operatorname{per} A$ for any subgroup G and its character χ, and nothing interesting happens. Throughout this paper we assume $\det A < \operatorname{per} A$, namely $(\operatorname{per} A - \det A)/2 = a|e|^2 + d|c|^2 + f|b|^2 > 0$. We denote the set of 3×3 positive semi-definite Hermitian matrices with $a|e|^2 + d|c|^2 + f|b|^2 > 0$ by $\mathscr{H}_3^+(\mathbf{C})$. For such A, following [5], we define the function T as follows.

Definition 1. For $A \in \mathscr{H}_3^+(\mathbf{C})$, define

$$T(A) = \frac{b\bar{c}e}{a|e|^2 + d|c|^2 + f|b|^2}.$$

The value of $T(A)$ determines the value of $t \in [0,1]$ such that $\bar{d}_\chi^G(A) = t \operatorname{per} A + (1-t) \det A$ for all $G \subset \mathfrak{S}_3$ and χ, except in the case of $|G| = 2$ (See [5]).

Proposition 1 ([5], Lemma 2). *Writing $T(A) = x + yi$ $(x, y \in \mathbf{R})$, the possible values of $T(A)$ are given by*

$$\begin{cases} 54x(x^2+y^2) - 27(x^2+y^2) + 1 \geq 0, \\ -\frac{1}{6} \leq x \leq \frac{1}{3}. \end{cases}$$

In [5], Theorem 2 was deduced from Proposition 1.

Definition 2. Define real-valued functions X, u_1, u_2 and u_3 on $\mathscr{H}_3^+(\mathbf{C})$ by

$$X = X(A) = a|e|^2 + d|c|^2 + f|b|^2,$$

$$u_1 = u_1(A) = \frac{a|e|^2}{X}, \qquad u_2 = u_2(A) = \frac{d|c|^2}{X}, \qquad \textit{and} \qquad u_3 = u_3(A) = \frac{f|b|^2}{X}.$$

Hence $u_i \geq 0$ and $u_1 + u_2 + u_3 = 1$. Also, define $K(A) = u_1 u_2 u_3$.

Proposition 2. *The possible values of $K(A)$ are $0 \leq K(A) \leq 1/27$. More precisely, for $\lambda \in [0, 1/27]$ and $x + yi \in \mathbf{C}$ $(x, y \in \mathbf{R})$, there exists a positive semi-definite Hermitian matrix A such that*

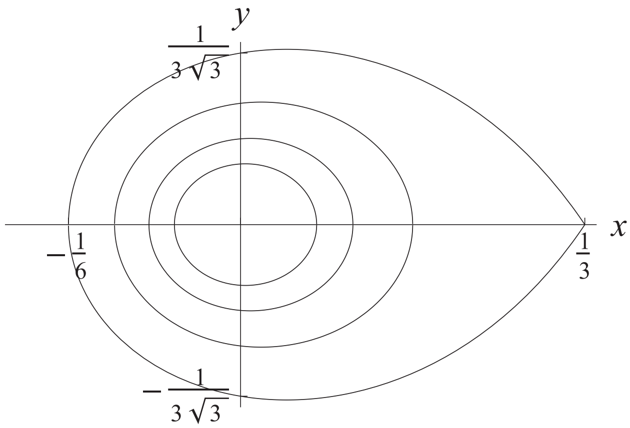

Fig. 1. (Contour plot of $2x(x^2+y^2)-(x^2+y^2)+\lambda=0$ for $\lambda=1/216, 1/108, 1/54, 1/27$)

$$T(A)=x+yi \qquad \textit{and} \qquad K(A)=u_1u_2u_3=\lambda$$
$$u_1,u_2,u_3\geq 0, \qquad u_1+u_2+u_3=1$$

if and only if

$$(*)\begin{cases}\lambda\geq -2x(x^2+y^2)+(x^2+y^2),\\ x\leq \frac{1}{3}.\end{cases}$$

The region of $x, y\in\mathbf{R}$ satisfying $(*)$ for a few values of λ are shown in Figure 1.

Proof. $K(A)=(\sqrt[3]{u_1u_2u_3})^3\leq((u_1+u_2+u_3)/3)^3=1/27$, hence $0\leq K(A)\leq 1/27$. Also since $0\leq \det A=adf+2\,\mathrm{Re}(b\bar{c}e)-(a|e|^2+d|c|^2+f|b|^2)$,

$$\begin{aligned}K(A)&=\frac{adf|bce|^2}{X^3}\\ &\geq\frac{(-2\,\mathrm{Re}(b\bar{c}e)+a|e|^2+d|c|^2+f|b|^2)|bce|^2}{X^3}\\ &=\frac{-2\,\mathrm{Re}(b\bar{c}e)|bce|^2}{X^3}+\frac{|bce|^2}{X^2}.\end{aligned}$$

When $T(A)=x+yi=(b\bar{c}e)/X$ and $K(A)=\lambda$, since $|b\bar{c}e|^2/X^2=|T(A)|^2=x^2+y^2$ and $\mathrm{Re}(b\bar{c}e)/X=x$, we have

$$\lambda\geq -2x(x^2+y^2)+(x^2+y^2).$$

By Proposition 1, $x\leq 1/3$ follows.

Conversely, for $x+yi$ and λ as above, if $\lambda = 0$, which implies $x = y = 0$, we can find $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \geq 0$. If $\lambda > 0$, let A be the matrix

$$\begin{pmatrix} 1 & \frac{1}{\sqrt[3]{\lambda}}(x+yi) & \frac{1}{\sqrt[3]{\lambda}}(x+yi) \\ \frac{1}{\sqrt[3]{\lambda}}(x-yi) & 1 & \frac{1}{\sqrt[3]{\lambda}}(x+yi) \\ \frac{1}{\sqrt[3]{\lambda}}(x-yi) & \frac{1}{\sqrt[3]{\lambda}}(x-yi) & 1 \end{pmatrix}.$$

Then $(x^2+y^2) \leq \lambda/(1-2x)$, therefore $1-(x^2+y^2)/\sqrt[3]{\lambda^2} \geq 1-\sqrt[3]{\lambda}/(1-2x) \geq 1-(1/27^{1/3})/(1-2/3) = 0$, which means the 2×2 principal minor of A is non-negative. Combining with the fact that $\det A = 1+2x(x^2+y^2)/\lambda - (x^2+y^2)/\sqrt[3]{\lambda^2} = (\lambda + 2x(x^2+y^2) - 3\sqrt[3]{\lambda}(x^2+y^2)) \geq 0$, we can conclude that A is a positive semi-definite Hermitian matrix satisfying $T(A) = x+yi$ and $K(A) = \lambda$.

Now let us consider the case of $G = \{(1),(12)\} \subset \mathfrak{S}_3$. Let χ_+ be the trivial character and χ_- the other irreducible character of G. The character table of G is the following:

G	(1)	(12)
χ_+	1	1
χ_-	1	-1

Writing

$$\bar{d}^G_{\chi_\pm}(A) = F_\pm(A)\ \mathrm{per}\ A + (1-F_\pm(A))\ \det A,$$

we have

$$F_+(A) = \frac{1}{2}u_3 - \mathrm{Re}\ T(A) + \frac{1}{2},$$

$$F_-(A) = -\frac{1}{2}u_3 - \mathrm{Re}\ T(A) + \frac{1}{2}.$$

Note that the values of $F_\pm(A)$ depend on the values of u_3 and $x = \mathrm{Re}\ T(A)$. If we fix u_3, the possible values of $K(A) = u_1u_2u_3$ are $0 \leq K(A) \leq u_3(1-u_3)^2/4$. Proposition 2 says that the possible values of x are

$$\begin{cases} 2x^3 - x^2 \geq -K(A) + y^2(1-2x), \\ x \leq \frac{1}{3}. \end{cases}$$

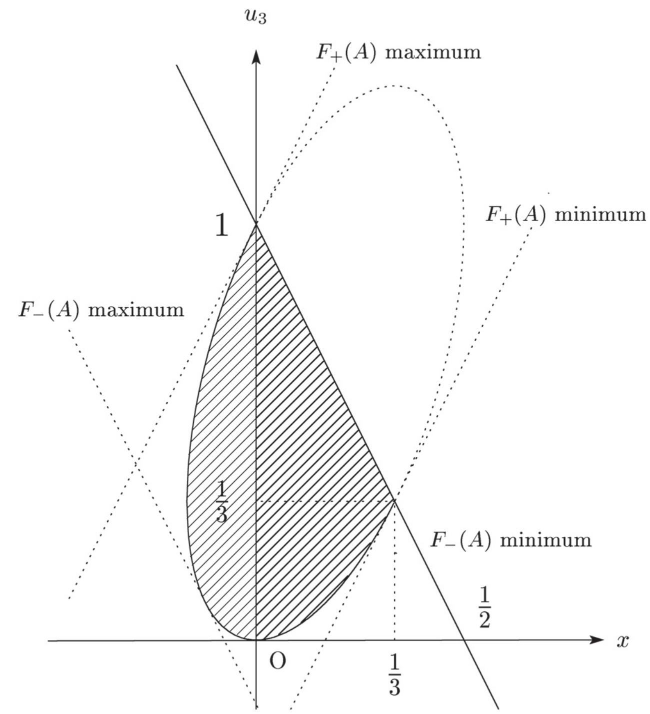

Fig. 2

Thus, $K(A) = u_3(1-u_3)^2/4$ and $y = 0$ give the largest range for the values of x. Hence it is enough to give the possible values for $F_{\pm}(A)$ under the assumption

$$\begin{cases} 2x^3 - x^2 + \dfrac{u_3(1-u_3)^2}{4} = (2x + u_3 - 1)\left(x^2 - \dfrac{u_3}{2}x + \dfrac{1}{4}(u_3^2 - u_3)\right) \geq 0, \\ x \leq \dfrac{1}{3}, \end{cases}$$

whose region is in Figure 2.

We state a theorem that implies Theorem 3 below.

Theorem 4. *If A is a 3×3 positive semi-definite Hermitian matrix, and χ_+ and χ_- are the trivial and non-trivial irreducible characters of a subgroup G of $\mathfrak{S}_3$ with order 2, then*

$$\frac{1}{3}\operatorname{per} A + \frac{2}{3}\det A \leq \bar{d}^G_{\chi_+}(A) \leq \operatorname{per} A.$$

$$\det A \leq \bar{d}^G_{\chi_-}(A) \leq \frac{1}{\sqrt{3}}\operatorname{per} A + \left(1 - \frac{1}{\sqrt{3}}\right)\det A.$$

PROOF. All we need to calculate is the possible values of

$$F_+(A) = \frac{1}{2}u_3 - x + \frac{1}{2} \qquad \text{and} \qquad F_-(A) = -\frac{1}{2}u_3 - x + \frac{1}{2}.$$

in Figure 2. One easily sees that

$$F_+(A) \text{ is maximum at } (x, u_3) = (0, 1) \text{ with } F_+(A) = 1,$$

$$\text{and minimum at } (x, u_3) = \left(\frac{1}{3}, \frac{1}{3}\right) \text{ with } F_+(A) = \frac{1}{3}.$$

Also,

$$F_-(A) \text{ is maximum at } (x, u_3) = \left(\frac{1-\sqrt{3}}{6}, \frac{2-\sqrt{3}}{3}\right) \text{ with } F_-(A) = \frac{1}{\sqrt{3}},$$

$$\text{and minimum on the line segment } (0,1)\text{-}\left(\frac{1}{3}, \frac{1}{3}\right) \text{ with } F_-(A) = 0.$$

REMARK 1. *Let* $A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *and*

$$A_3 = \begin{pmatrix} 1 & -2+\sqrt{3} & \sqrt{\frac{\sqrt{3}-1}{2}} \\ -2+\sqrt{3} & 1 & \sqrt{\frac{\sqrt{3}-1}{2}} \\ \sqrt{\frac{\sqrt{3}-1}{2}} & \sqrt{\frac{\sqrt{3}-1}{2}} & 1 \end{pmatrix}.$$

Then

$$F_+(A_1) = \frac{1}{3}, \qquad F_+(A_2) = 1,$$

$$F_-(A_1) = F_-(A_2) = 0, \qquad F_-(A_3) = \frac{1}{\sqrt{3}}.$$

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Ryo Tabata
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739-8526, *Japan*
E-mail: tabata-ryo@hiroshima-u.ac.jp