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Classification of spherical tilings by congruent quadrangles over pseudo-double wheels (I)—a special tiling by congruent concave quadrangles

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Abstract. Every simple quadrangulation of the sphere is generated by a graph called a pseudo-double wheel with two local expansions (Brinkmann et al. "Generation of simple quadrangulations of the sphere." Discrete Math., Vol. 305, No. 1–3, pp. 33–54, 2005). So, toward a classification of the spherical tilings by congruent quadrangles, we propose to classify those with the tiles being convex and the graphs being pseudo-double wheels. In this paper, we verify that a certain series of assignments of edge-lengths to pseudo-double wheels does not admit a tiling by congruent convex quadrangles. Actually, we prove the series admits only one tiling by twelve congruent *concave* quadrangles such that the symmetry of the tiling has only three perpendicular 2-fold rotation axes, and the tiling seems to be new.

1. Introduction

A *spherical tiling by congruent polygons* is, by definition, a covering of the unit sphere by congruent spherical polygons such that (i) none of the polygons share their inner point, (ii) an edge of a spherical polygon matches an edge of another spherical polygon, and (iii) each vertex is incident to more than two edges. Each spherical polygon is called a *tile*.

If there is a spherical tiling by p-gons, then p is either 3, 4 or 5 ([10]), because of Euler's formula $V - E + F = 2$. In [11], Ueno-Agaoka classified all the spherical tilings by congruent triangles. By using their classification, we can complete the classification of all the spherical tilings by congruent quadrangles where the quadrangle can be divided into two congruent triangles [3]. Interestingly, there is a spherical tiling by 24 congruent kites with the graph being the same as *deltoidal icositetrahedron*, a *Catalan-solid* [6], while there is another spherical tiling by the congruent kites with different symmetry.

As for spherical tilings by congruent quadrangles, if the quadrangle cannot be divided into two congruent triangles, then the edge-lengths of the tile is

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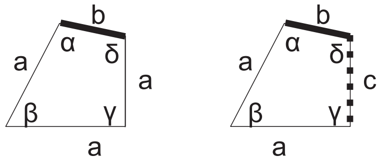

Fig. 1. A quadrangle of type 2 (left) and a quadrangle of type 4 (right). Designation of edge-lengths and angles. The variables a, b, c are for edge-lengths and they have different values. The variables α, β, γ, δ are for inner angles.

necessarily one of two types of Figure 1, because an edge of a tile should match an edge of another tile with the same length and because Euler's formula implies the existence of a 3-valent vertex [10]. Following [10], we call the two types *type 2* and *type 4*. They gave a somehow unexpected, sporadic, 4-fold rotational symmetric spherical tiling by 16 congruent quadrangles of type 2 [10, Theorem 2]. As a necessary condition for a spherical quadrangle of type 2 to exist, they provided inequalities involving trigonometric functions, in terms of the inner angles of the tile ([10, Proposition 3]). In [10, Figure 11], they presented spherical tilings by congruent *concave* quadrangles of type 2, and suggested that the class of spherical tilings by congruent concave quadrangles is difficult to classify.

Among the spherical monohedral quadrangular tilings, we are concerned with the following spherical tilings consisting of $F \geq 6$ congruent quadrangles, having $F/2$-fold rotational symmetry. See Figure 2 and the first three spherical tilings in [10, Figure 11], for example. The pair of the vertices and the

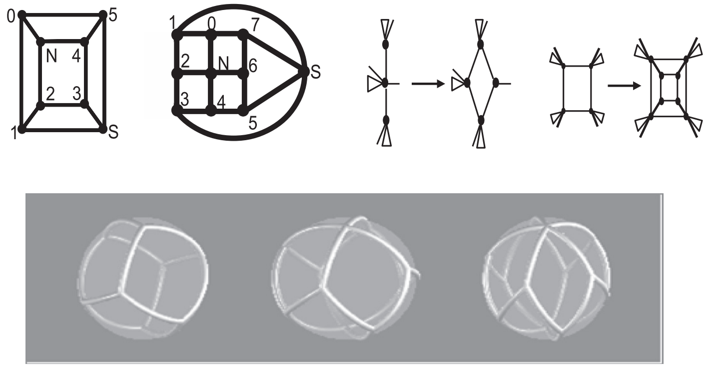

Fig. 2. The pictures in the upper row are, from left to right, a pseudo-double wheel of 6 faces, a pseudo-double wheel of 8 faces, and two expansions of spherical quadrangulations to increase the number of faces (Brinkmann et al. [5]). The graphs in this paper should be interpreted using Convention 1 stated at the end of Section 2. The lower pictures are spherical tilings by congruent quadrangles over pseudo-double wheel of F faces ($F = 6, 8, 10$, from left to right).

edges of the tiling forms a graph, which is obtained from the cycle consisting of F vertices, by adjoining the north and the south poles alternately to the vertices of the cycle. Such a graph falls into the class of *pseudo-double wheels*. According to Brinkmann et al. [5], every spherical quadrangulation is obtained from a pseudo-double wheel of even or odd faces through finite applications of two local expansions (see Figure 2 (upper right)).

In [2], we prove that for every spherical tiling by congruent *convex* quadrangles of type 2 or type 4, the edge-lengths of the graph are none of two graphs in Figure 3. As a direct consequence, we can prove the following statement: If a spherical tiling by congruent convex quadrangles of type 2 or type 4 has a pseudo-double wheel as a graph, then the length-assignment of the graph must be $F/2$-fold or an $F/6$-fold "rotationally symmetric."

The main theorem of this paper is: If a spherical tiling by congruent quadrangles of type 2 or type 4 over a pseudo-double wheel has an $F/6$-fold "rotationally symmetric" length-assignment, then the tiling necessarily consists of twelve congruent *concave* quadrangles. Such a tiling actually exists uniquely up to mirror image, and the symmetry of the tiling is low compared to the number of tiles. By this theorem, we can complete the classification of all the spherical tilings by congruent *convex* quadrangles over pseudo-double wheels ([2]).

This paper is organized as follows: In the next section, we present basic relevant notions of spherical tilings by congruent polygons. In Section 3, we present our main theorem, the background and the organization of the proof. The rest of the sections are devoted to the proof of the main theorem.

2. Preliminaries

Throughout this paper, by the sphere we mean the sphere with the center being the origin and the radius being 1, and say two figures on the sphere are *congruent* if there is an orthogonal transformation between them.

By a *spherical triangle* (resp. *spherical quadrangle*), we mean a nonempty, simply connected, closed subset T of the sphere with the area less than 2π such that the boundary is the union of three (resp. four) distinct geodesic lines but is not the union of any two (resp. three) distinct geodesic lines.

To prove that a spherical triangle actually exists, we use the following:

Proposition 1 ([4, p. 62]). *If $0 < A, B, C < \pi$, $A + B + C > \pi$, $-A + B + C < \pi$, $A - B + C < \pi$, $A + B - C < \pi$, then there exists uniquely up to orthogonal transformation a spherical triangle such that all the edges are geodesic lines and the inner angles are A, B and C.*

We formalize relevant combinatorial notions of spherical tilings by congruent polygons. Refer [8] for the terminology of the graph theory.

Definition 1. A *map* is a triple $M = (V, E, \{O_v \mid v \in V\})$ such that (V, E) is a graph (see [8, Section 1.1]) and each O_v is a cyclic order for the edges incident to the vertex v. For a vertex v of M, the set A_v of *angles* around v is

$$A_v := \{(v_1, v, v_2), (v_2, v, v_3), \dots, (v_{n-1}, v, v_n), (v_n, v, v_1)\}$$

where the list $vv_1, vv_2, \dots, vv_n$ is the enumeration of $\{vu \mid vu \in E\}$ without repetition by the cyclic order O_v. We write an inner angle (u, v, w) by $\angle uvw$. The *mirror* of a map M is the map M^R with the cyclic orders reversed. Thus $\angle uvw$ is an inner angle of the original map, if and only if $\angle wvu$ is an inner angle of the mirror of the map.

We recall a *pseudo-double wheel* [5].

Definition 2. For an even number F greater than or equal to 6, a *pseudo-double wheel* pdw_F with F faces is a map such that

- the graph is obtained from a cycle $(v_0, v_1, v_2, \dots, v_{F-1})$, by adjoining a new vertex N to each v_{2i} $(0 \le i < F/2)$ and then by adjoining a new vertex S to each v_{2i+1} $(0 \le i < F/2)$. We identify the suffix i of the vertex v_i modulo F.
- The cyclic order at the vertex N is defined as follows: the edge Nv_{2i+2} is next to the edge Nv_{2i}. The cyclic order at the vertex v_{2i} $(0 \le i \le F/2)$ is: the edge $v_{2i}N$ is next to the edge $v_{2i}v_{2i+1}$, which is next to the edge $v_{2i}v_{2i-1}$. The cyclic order at the vertex S is: the edge Sv_{2i-1} is next to the edge Sv_{2i+1}. The cyclic order at the vertex v_{2i+1} $(0 \le i < F/2)$ is: the edge $v_{2i+1}S$ is next to the edge $v_{2i+1}v_{2i}$, which is next to the edge $v_{2i+1}v_{2i+2}$.

We call each edge Nv_{2i} *northern*, each edge Sv_{2i+1} *southern*, and the other edges *non-meridian*. The number of edges is $2F$. See Figure 2 (lower) for pdw_6, pdw_8 and pdw_{10}.

We use frequently the following notion:

Definition 3. A *chart* is, by definition, a triple $\mathscr{A} = (M, L, K)$ such that

- M is a map. Let V be the vertex set of M;
- L is a *length-assignment*, which is a function from the edge set E of M to $\{a, b, c\}$ with $a, b, c \in (0, 2\pi)$;
- K is an *angle-assignment*, which is a function from the set $\bigcup_{v \in V} A_v$ of the angles to the set of affine combinations of the variables α, β, γ, δ

over **R** subject to the equation

$$\sum_{\psi \in A_v} K(\psi) = 2\pi \qquad \text{for each vertex } v \in V. \tag{1}$$

Here A_v is the set of angles around the vertex v, as in defined in Definition 1. For a vertex $v \in V$ and an angle $\psi \in A_v$, $K(\psi)$ is called the *type* of the angle ψ, and $\sum_{\psi \in A_v} K(\psi)$ is called the *vertex type of the vertex* v; and

- The pair of K and L satisfies the constraint described by Figure 1 (left) ("type 2") for each face or the constraint described by Figure 1 (right) ("type 4") for each face.

We say a spherical tiling $\mathscr{T}$ by polygons *realizes* a chart $\mathscr{A}$, provided that there is an embedding G from $\mathscr{A}$ to the sphere such that

- G is a bijection from the vertex set of the chart $\mathscr{A}$ to that of the tiling $\mathscr{T}$,
- G is a bijection from the edge set of the chart $\mathscr{A}$ to that of the tiling $\mathscr{T}$, such that (i) if two edges uv and $u'v'$ of $\mathscr{A}$ has intersection $\{w\}$, then the intersection of two edges $G(uv)$ and $G(u'v')$ is $\{G(w)\}$; and (ii) if uv and $u'v'$ is disjoint, then $G(uv)$ and $G(u'v')$ do not intersect.
- G preserves the cyclic order O_v of each vertex v of the chart $\mathscr{A}$ to the orientation of the sphere at $G(v)$. In other words, for any vertex v of the chart, if we rotate a screw at the vertex $G(v)$ according to the cyclic order O_v, then the screw goes outbound from the center of the sphere.
- For any angle $\angle uvw$ of the chart $\mathscr{A}$, $\angle G(u)G(v)G(w)$ is an angle of the tiling $\mathscr{T}$ and is $K(\angle uvw)$.
- For any edge uv of the chart $\mathscr{A}$, $G(u)G(v)$ is an edge of the tiling $\mathscr{T}$ and has length $L(uv)$.

The *mirror* $\mathscr{A}^R$ of a chart $\mathscr{A} = (M, L, K)$ is, by definition, a chart (M^R, L, K^R) where $K(\angle wvu) = K^R(\angle uvw)$. Thus if a tiling $\mathscr{T}$ realizes $\mathscr{A}$, then the mirror image of $\mathscr{T}$ does the mirror $\mathscr{A}^R$.

The charts in Figure 2 and Figure 3 are subject to the following convention as in [5].

Convention 1. *Each displayed vertex is distinct from the others; Edges that are completely drawn must occur in the cyclic order given in the picture; Half-edges indicate that an edge must occur at this position in the cyclic order around the vertex; A triangle indicates that one or more edges may occur at this position in the cyclic order around the vertex* (*but they need not*)*; If neither a half-edge nor a triangle is present in the angle between two edges in the picture, then these two edges must follow each other directly in the cyclic order of edges around that vertex.*

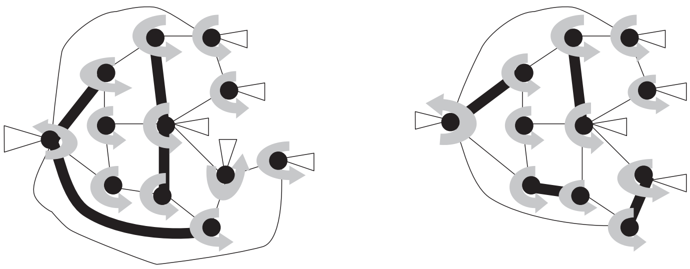

Fig. 3. Charts forbidden for a spherical tiling by ten or more congruent *convex* quadrangles of type $t = 2, 4$. See Theorem 1. The figures are subject to Convention 1.

3. Main theorem and the background

In [9], Sakano classified the spherical tilings by six or eight congruent quadrangles of type 2 or of type 4. His argument is generalized to the case in which the number of the tiles is more than ten, as follows: Recall quadrangles of type 2 and those of type 4, by Figure 1.

Theorem 1 ([2]). *Two charts in Figure* 3 *and their mirrors do not occur as the chart of a spherical tiling by ten or more congruent* convex *quadrangles of type t for each* $t = 2, 4$. *Thick edges are of length b while the other edges are of length a or c. The charts should be interpreted by Convention* 1.

Let the leftmost vertex of the chart in Figure 3 be the vertex S of the pseudo-double wheel and let the central vertex incident to at least five edges be the vertex N. In the left chart of Figure 3, an upper, designated, northern edge of length b is immediately followed by another northern edge of length b. In the right chart, the designated northern edge of length b is followed immediately by two consecutive non-meridian edges of length b. By the repeated applications of Theorem 1 to a slightly rotated pseudo-double wheel pdw_F around the vertices N and S, we can observe that all the possibilities of the length-assignment of pdw_F $(F \geq 10)$ is as follows:

- (p) the tile is a convex quadrangle, and no northern edges of pdw_F have length b; or
- (a) the tile is a convex quadrangle. The northern edges and non-meridian edges of pdw_F alternatingly have length b. See Assumption (II) of Theorem 2, and Figure 4; or
- (a^R) (a) with pdw_F replaced by the mirror $(\mathrm{pdw}_F)^R$.

As a consequence of the following Theorem 2, the possibilities (a) and (a^R) are impossible for any even number $F \geq 10$ of tiles. In other words, for any even number $F \geq 10$, there is no spherical tiling by F congruent *convex* quadrangles over pdw_F such that (a) or (a^R) holds. The only possibility (p) of the length-assignment of pdw_F is actually realized with a spherical tiling by F congruent quadrangles. Such a spherical tiling is obtained from a spherical tiling by congruent rhombic tiles, such as Figure 2 (lower), by deforming the tiles to quadrangles of type 2 or of type 4 with the tiling kept $F/2$-fold rotational symmetric. For details, see [1, 2]. So Theorem 2, which is rather a long statement about tiles of type 2 or type 4, leads to a complete classification of the spherical tilings by congruent *convex* quadrangles of type 2 or of type 4 over pseudo-double wheels.

THEOREM 2. *Assume a chart $\mathscr{A}$ satisfies the following assumptions:*

(I) *the map of the chart is* pdw_F *for some* $F \geq 10$*, and*

(II) *there is* $b > 0$ *such that all the edges* Nv_{6i}*,* $v_{6i+1}S$ *and* $v_{6i+3}v_{6i+4}$ *have length* b *for each nonnegative integer* $i < F/6$ *while the other edges do lengths* $\neq b$*.*

Then there exists a spherical tiling $\mathscr{T}$ by congruent quadrangles, uniquely up to special orthogonal transformation. Moreover the tile is a concave *quadrangle of type* 2*, and the following three properties hold:*

(1) *The tiling $\mathscr{T}$ realizes $\mathscr{A}$, where the length-assignment and the angle-assignment are Figure* 4 *(*$F = 12$ *in particular) with the following equations:*

$$a = \arccos \frac{1}{3}, \tag{2}$$

$$b = \arccos \frac{-5}{9}, \tag{3}$$

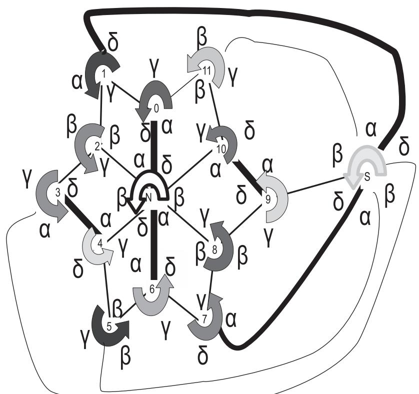

Fig. 4. The chart $\mathscr{A}$ of the tiling. Thick (resp. thin) edges of the chart correspond to edges of length b (resp. a) of the tiling (see Theorem 2).

$$\alpha = \arccos \frac{-1}{2\sqrt{7}}, \tag{4}$$

$$\beta = \frac{\pi}{3}, \tag{5}$$

$$\gamma = \frac{4\pi}{3}, \tag{6}$$

$$\delta = \arccos \frac{5}{2\sqrt{7}}. \tag{7}$$

(2) *Define the spherical polar coordinate of a point p on the unit sphere to be the pair* (θ, φ) *of the length* θ *of the geodesic line Np and the* longitude φ *of p. The longitude is the angle from the geodesic line* Nv_0 *to Np, defined consistently with the cyclic order of N. Let*

$$\phi = \arccos \frac{13}{14}. \tag{8}$$

Then the spherical polar coordinates of the vertices v_{i+6} *is* $(\theta, \rho + \pi)$ *for* $v_i = (\theta, \rho)$ $(i = 0, 1, 2, 3, 4, 5)$. *Moreover* $S = (\pi, 0)$, $v_0 = (b, 0)$, $v_1 = (\pi - b, \phi)$, $v_2 = (a, \alpha)$, $v_3 = (\pi - a, \phi + \delta)$, $v_4 = (a, \alpha + \beta)$, *and* $v_5 = (\pi - a, \phi + \delta + \beta)$.

(3) *The tiling* $\mathscr{T}$ *has only three perpendicular* 2*-fold rotation axes but no mirror planes. In other words, the Schönflies symbol of the tiling is* D_2. *See Figure* 5 *for the views of* $\mathscr{T}$ *from the three axes.*

The proof of Theorem 2 is organized as follows: In Section 4, by assuming the existence of such a tiling $\mathscr{T}$, we prove the assertions (1) and (3) concerning the inner angle α and the number of tiles F, and prove that the tile

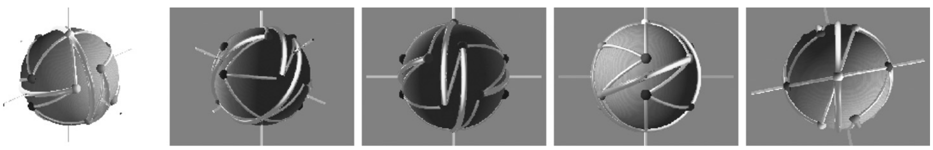

Fig. 5. The tiling of twelve congruent *concave* quadrangles over a pseudo-double wheel and the three 2-fold rotation axes. The upper left figure is the view from a general position, and the upper right is from the antipodal of the first viewpoint. The thick edges are of length b. The lower left figure is the view from a 2-fold rotation axis through the midpoint between the vertices v_0 and v_1. The lower middle figure is from a 2-fold rotation axis through the midpoint between the vertices v_3 and v_4. The lower right is from the other 2-fold rotation axis through the poles. In the figure (lower left), two great circles containing the two middle thick edges forms an angle $\phi = \arccos(13/14)$ (see (8)).

is a concave quadrangle of type 2. In Section 5, we determine the absolute values of all inner angles, all edge-lengths of the tile, and F. In Section 6, by those values, we prove that the tile actually exists as a spherical quadrangle, by using Proposition 1. In Section 7, we prove the assertion (2). In Section 8, we complete the proof of Theorem 2 by establishing the assertion (3).

4. Which angles and which edges are equal?

We answer the question in this section by manipulations of systems of equations of the form (1) and by elementary geometry.

By an *automorphism* of a map M, we mean any automorphism [8, Section 1.1] h of the graph that preserves the cyclic orders of the vertices. Here we let h send any angle $\angle uvw$ to $\angle h(u)h(v)h(w)$. For a chart $\mathscr{A} = (M, L, K)$ and an automorphism h of the map M, let $h(\mathscr{A})$ be a chart $(M, L \circ h^{-1}, K \circ h^{-1})$.

Definition 4. Let $\mathscr{A} = (M, L, K)$ be a chart satisfying the assumptions of Theorem 2, define two automorphisms f and g on the map M of $\mathscr{A}$ as follows:

$$f(N) = S, \quad f(S) = N, \quad f(v_i) = v_{1-i} \qquad (0 \le i \le F-1); \tag{9}$$

$$g(N) = N, \quad g(S) = S, \quad g(v_i) = v_{i+6} \qquad (0 \le i \le F-1). \tag{10}$$

Here the suffix i of the vertex v_i is understood modulo F as before. Then we can prove that both of f and g are indeed automorphisms.

Lemma 1. *Suppose some tiling $\mathscr{T}$ realizes a chart $\mathscr{A} = (M, L, K)$ satisfying the assumptions of Theorem* 2. *Then,*

(1) *The number F of faces is a multiple of* 6 *greater than or equal to* 12.
(2) *The tile is a quadrangle of type* 2.
(3) $\mathscr{A} = f(\mathscr{A}) = g(\mathscr{A})$.
(4) *The inner angles satisfy* $\beta = \frac{4\pi}{F}$, $\gamma = 2\pi - \frac{8\pi}{F} > \pi$, *and* $\delta = \frac{8\pi}{F} - \alpha$.
(5) *K has the same assignment as in Figure* 4 *on the angles* $\angle v_{-2}Nv_0$, $\angle v_0Nv_2$, $\angle v_2Nv_4$ *and on all angles around the vertices* v_0, v_2, v_3. *Especially,*

$$\angle v_3v_2N = 2\pi - \frac{8\pi}{F} > \pi, \qquad \angle v_2Nv_4 = \frac{4\pi}{F}, \qquad \angle Nv_4v_3 = \alpha. \tag{11}$$

To prove the assertion (1), assume F is not a multiple of 6. Then the edge Nv_2 or Nv_4 should have length b, which contradicts against Assumption (II) of Theorem 2. By this and Assumption (I) of Theorem 2, the assertion (1) of Lemma 1 follows.

4.1. Basic general lemmas. To prove the other assertions of Lemma 1, we associate to the chart in question a system of equations and inequalities about the inner angles α, β, γ, δ of a tile. The system consists of the equations (1), inequalities $0 < \alpha, \beta, \gamma, \delta < 2\pi$, and an equation

$$\alpha + \beta + \gamma + \delta - 2\pi = \frac{4\pi}{F}. \tag{12}$$

The last equation holds because all F tiles are congruent quadrangles and because the area of a spherical triangle is the sum of the inner angles subtracted by π according to [4].

If the system of equations and inequalities associated to a chart $\mathscr{A}$ is unsolvable, no tiling realizes $\mathscr{A}$. To show that such systems are unsolvable, we use the following lemmas.

Lemma 2. *In a quadrangle of type* 2, *no inner angle is equal to the opposite inner angle* (*i.e.* $\beta \neq \delta$ *and* $\alpha \neq \gamma$). *In a quadrangle of type* 4, *we have* $\alpha \neq \gamma$.

Proof. Draw a diagonal line in the tiles, and then argue with the isosceles triangle(s). □

Lemma 3. *If a tiling by congruent quadrangle of type* 4 *realizes a chart, then, for each* $Z \in \{\alpha, \beta, \gamma, \delta\}$, *the chart has no* 3-*valent vertex of type* $Z + 2\delta$ *or* $Z + \beta + \delta$.

Proof. Recall δ is the type of an angle between an edge of length b and that of length c. Because $b \neq c$, if a 3-valent vertex has vertex type $2\delta + Z$ for some type Z, then Z is the type of an angle between two edges of length b, or is that of an angle between two edges of length c. But such an angle does not exists in any quadrangles of type 4.

β is the type of an angle between two edges of length a. δ is the type of an angle between an edge of length b and that of length c. By $c \neq a \neq b$, no edge of the angle of type β does not match an edge of the angle of type δ. Thus an angle of type δ cannot be adjacent to an angle of type β in a quadrangle of type 4. Hence a vertex of type $Z + \beta + \delta$ is impossible. □

Lemma 4. *For any quadrangle of type* 2, $\alpha = \delta$ *if and only if* $\beta = \gamma$.

Proof. Assume $\alpha = \delta$. We identify the vertex of the quadrangle with the inner angle. Because the quadrangle $\alpha\beta\gamma\delta$ is of type 2, the two edges $\alpha\beta$ and $\gamma\delta$ have length a, and then the triangle $\alpha\beta\delta$ is congruent to the triangle $\delta\gamma\alpha$. Thus $\angle\beta\delta\alpha = \angle\gamma\alpha\delta$. As $\alpha = \delta$, we have $\angle\beta\alpha\gamma = \angle\gamma\delta\beta$. Hence $\angle\gamma\beta\delta = \angle\gamma\delta\beta = \angle\beta\alpha\gamma = \angle\beta\gamma\alpha$. Thus $\beta = \angle\alpha\beta\delta + \angle\gamma\beta\delta = \angle\delta\gamma\alpha + \angle\beta\gamma\alpha = \gamma$. Conversely, assume the inner angle β is equal to the inner angle γ. As the triangle $\alpha\beta\gamma$ is congruent to the triangle $\delta\gamma\beta$, we have $\angle\beta\alpha\gamma = \angle\beta\gamma\alpha = \angle\gamma\beta\delta = \angle\gamma\delta\beta$. Let the point P be shared by the two diagonal segments $\alpha\gamma$ and $\beta\delta$ of the quadrangle. As

$\angle P\beta\gamma = \angle P\gamma\beta$, the length of the edge βP is equal to that of the edge γP. Since the length of the diagonal $\alpha\gamma$ is equal to that of the other diagonal $\delta\beta$, we have $P\alpha = P\delta$, by which the triangle $\alpha P\delta$ is an isosceles triangle. So $\angle\beta\alpha\gamma = \angle\gamma\delta\beta$. Thus the inner angle α is equal to the inner angle δ. □

Lemma 5. *Suppose a tiling by congruent quadrangles of type* 2 *realizes a chart such that* (1) *there is a vertex incident to only three edges of length a, and* (2) *there is a* 3*-valent vertex incident to two edges of length a and to one edge of length b. Then* $\alpha \neq \delta$ *and* $\beta \neq \gamma$.

Proof. Assume otherwise. By Lemma 4, we have $\alpha = \delta$ and $\beta = \gamma$. So the assumption (1) implies $\beta = \gamma = 2\pi/3$. By the assumption (2), there exists a vertex of type $X + X' + Y$ $(X, X' \in \{\alpha, \delta\},\ Y \in \{\beta, \gamma\})$. Thus $\alpha = \beta = \gamma = \delta = 2\pi/3$. This contradicts against Lemma 2. □

4.2. The proof of the assertions (2), (3), (4) and (5) of Lemma 1.

Lemma 6. *If a tiling by congruent quadrangles of type* 2 *realizes a chart satisfying the assumptions of Theorem* 2*, then* $\alpha \neq \delta$ *and* $\beta \neq \gamma$.

Proof. Since the tile $Nv_0v_1v_2$ of the pseudo-double wheel pdw_F in question is of type 2 and has an edge Nv_0 of length b, other edges v_1v_2 and v_2N of the same tile have lengths a. The tile $Sv_1v_2v_3$ is of type 2, and has an edge Sv_1 of length b because of Assumption (II) of Theorem 2. Thus the opposite edge v_2v_3 of the same tile $Sv_1v_2v_3$ has length a. Thus the vertex v_2 is incident to only three edges of length a. On the other hand, the vertex v_1 is incident to one edge of length b and to two edges v_0v_1 and v_0v_{F-1} of length a. By Lemma 5, we have $\alpha \neq \delta$ and $\beta \neq \gamma$. □

We can easily verify the following Lemma for automorphisms f, g defined in Definition 4.

Lemma 7. *If* $\mathscr{A}$ *is any chart satisfying the assumptions of Theorem* 2*, then*
(1) $f(\mathscr{A})$ *and* $g(\mathscr{A})$ *are charts satisfying the assumptions of Theorem* 2.
(2) *Whenever we can prove a property* $P(\mathscr{A})$*, we can prove the properties* $P(f(\mathscr{A}))$ *and* $P(g(\mathscr{A}))$.

Definition 5. Fix a chart. For any property $P(\alpha, \beta, \gamma, \delta)$ for the inner angles α, β, γ, δ of the tile, the *conjugate property* $P^*(\alpha, \beta, \gamma, \delta)$ is, by definition, a property $P(\delta, \gamma, \beta, \alpha)$.

Lemma 8. *Suppose a tiling by congruent quadrangles realizes a chart satisfying the assumptions of Theorem* 2*. Then the tile is of type* 2*, and the chart satisfies the following properties* (A) *and* (B)*, or satisfies the conjugate properties* (A)* *and* (B)*.

(A) *For each integer* $0 \le i < F/6$, *the types of* $\angle Sv_{6i+1}v_{6i+2}$ *and* $\angle Nv_{6i}v_{6i-1}$ *are* δ, *those of* $\angle v_{6i}v_{6i+1}S$ *and* $\angle v_{6i+1}v_{6i}N$ *are* α, *and those of* $\angle v_{6i+2}v_{6i+1}v_{6i}$ *and* $\angle v_{6i-1}v_{6i}v_{6i+1}$ *are* β.

(B) *For each integer* $0 \le i < F/6$, *the types of* $\angle v_{6i+2}Nv_{6i+4}$ *and* $\angle v_{6i+5}Sv_{6i+3}$ *are all* γ.

Proof. It is sufficient to prove the following properties and the conjugate properties Property 1^* and Property 2^*:

Property 1. It is not the case that: for some nonnegative integer $i < F/6$, both of $\angle v_{6i+1}v_{6i}N$ and $\angle Nv_{6i}v_{6i-1}$ have type δ and $\angle v_{6i-1}v_{6i}v_{6i+1}$ has type β.

Property 2. It is not the case that: for some nonnegative integer $i < F/6$, both of $\angle v_{6i+1}v_{6i}N$ and $\angle Nv_{6i}v_{6i-1}$ have type α and $\angle v_{6i-1}v_{6i}v_{6i+1}$ has type β.

We first verify the sufficiency. Assume we can prove Property 1, Property 2, Property 1^* and Property 2^*. Then for every nonnegative integer $i < F/6$, the type of the vertex v_{6i} of pdw_F is $\alpha+\beta+\delta$ or $\alpha+\gamma+\delta$, and thus the type of the vertex v_{6i+1} of pdw_F is so, because the automorphism f satisfies $f(v_{6i}) = v_{6i+1}$ and by Lemma 7.

Because every edge $v_{6i+3}v_{6i+4}$ has length b, each angle mentioned in (B) is β or γ.

First consider the case where all the vertices v_{6i} and v_{6i+1} have type $\alpha+\gamma+\delta$. Then all the angles $\angle v_{6i+2}Nv_{6i+4}$ and $\angle v_{6i+5}Sv_{6i+3}$ do β. Otherwise, $\alpha+\gamma+\delta < 2\pi$ because α and δ appear as types of angles around the poles and $F/2 \ge 5$. Hence we have $(\mathrm{B})^*$.

Each $\angle v_{6i+1}v_{6i}N$ has type δ. Otherwise $\angle v_{6i+1}v_{6i}N$ has type α and thus $\angle v_{6i+2}v_{6i+1}v_{6i}$ does β, which contradicts against the vertex type of v_{6i+1} being $\alpha+\gamma+\delta$. Similarly we can prove $\angle v_{6i}v_{6i+1}S$ has type δ. So we have $(\mathrm{A})^*$.

Thus each $\angle v_{6i+4}Nv_{6i+6}$ has type δ. If the tile is of type 4, then we have contradiction against $(\mathrm{B})^*$, by considering the length of the edge v_4N. So the tile is of type 2.

Next consider the case where all the vertices v_{6i} and v_{6i+1} have type $\alpha+\beta+\delta$. In this case we can prove (A) and (B) by the same argument as above but with $(\alpha,\beta) \leftrightarrow (\delta,\gamma)$ swapped.

In the remaining case, we have a vertex of type $\alpha+\beta+\delta = 2\pi$ and that of type $\alpha+\gamma+\delta = 2\pi$. The former vertex leads to a contradiction against Lemma 3, when the tile is of type 4. If the tile is of type 2, this case implies $\beta = \gamma$, which contradicts against Lemma 6.

Hence to verify Lemma 8, we prove Property 1, Property 2 and their conjugate properties below.

Property 1^* holds when the tile is of type 4, by the following argument. If both of $\angle v_{6i+1}v_{6i}N$ and $\angle Nv_{6i}v_{6i-1}$ have type α, then the length of the edge $v_{6i}v_{6i+1}$ and that of edge $v_{6i}v_{6i-1}$ are both a. So $\angle v_{6i-1}v_{6i}v_{6i+1}$ has type β.

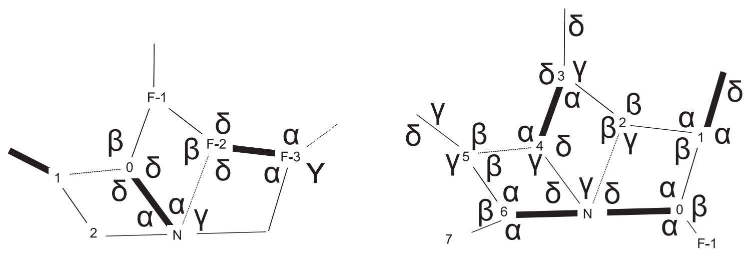

Fig. 6. The proof that Property 1 holds when the tile is of type 2 (left), and the proof of Claim 1 (right). The thick edges have length b.

Property 1 and Property 2^* hold when the tile is of type 4, by Lemma 3.

Property 1 holds when the tile is of type 2. Otherwise, $\beta + 2\delta = 2\pi$. See Figure 6 (left). The type of $\angle v_{6i-1}v_{6i-2}N$ is β. By Assumption (II) of Theorem 2, the edge $v_{6i-2}v_{6i-3}$ has length b, and the type of the vertex v_{6i-2} is one of $\alpha + \beta + \delta$, $\beta + 2\alpha$, and $\beta + 2\delta$. For the first two cases, we have $\alpha = \delta$ which contradicts against Lemma 6. Therefore among the vertex type of the vertex N, the type γ occurs. However the vertex type of v_{6i-3} is $2\alpha + Y$ for some $Y \in \{\beta, \gamma\}$. In case $Y = \beta$, we have $\alpha = \delta$, which contradicts against Lemma 6. In the other case $Y = \gamma$, the vertex type of v_{6i-3} is a proper subtype of N, which is a contradiction.

We can prove Property 1^* holds when the tile is of type 2, by swapping $(\alpha, \beta) \leftrightarrow (\delta, \gamma)$ in the proof above.

Claim 1. *Property* 2 *holds when the tile is of type* 2 *or of type* 4.

Proof. Assume otherwise. See Figure 6 (right). Since the edge Nv_{6i} has length b, the type of the angle $\angle v_{6i+2}v_{6i+1}v_{6i}$ is β. Since the edge Sv_{6i+1} has length b, the type of $\angle v_{6i}v_{6i+1}S$ is α. Assume the type of $\angle Sv_{6i+1}v_{6i+2}$ is δ. When the tile is of type 4, we have a contradiction against Lemma 3. When the tile is of type 2, we have a contradiction against Lemma 6 because $\alpha = \delta$ follows by comparing the vertex types of v_{6i} and v_{6i+1}. Therefore the vertex types of v_{6i} and v_{6i+1} are both

$$2\alpha + \beta = 2\pi, \tag{13}$$

and thus the type of $\angle v_{6i+1}v_{6i+2}v_{6i+3}$ is β. If the vertex type of v_{6i+2} is $\beta + 2\gamma$, then $\alpha = \gamma$ by comparing the vertex types of v_{6i+1} and v_{6i+2}. This contradicts against Lemma 2, whether the tile is of type 2 or of type 4. Thus the vertex type of v_{6i+2} is

$$2\beta + \gamma = 2\pi. \tag{14}$$

Because the type of $\angle v_{6i+2}v_{6i+3}S$ is γ, if the vertex type of v_{6i+3} is $2\alpha+\gamma$, then (13) and (14) implies $\alpha=\beta=\gamma=2\pi/3$, which contradicts against Lemma 2. Therefore the vertex type of v_{6i+3} is

$$\alpha+\gamma+\delta=2\pi. \tag{15}$$

Then the vertex type of v_{6i+4} is (15), too. Otherwise $\alpha+\beta+\delta=2\pi$. This contradicts against Lemma 6 and the equation (15). Hence the type of $\angle Nv_{6i+6}v_{6i+5}$ is α. The type of $\angle Sv_{6i+5}v_{6i+6}$ is γ. Otherwise it is β. Then (14) implies $\beta=\gamma$, which contradicts against Lemma 6. Thus the type of $\angle v_{6i+5}v_{6i+6}v_{6i+7}$ is β.

Actually, the type of $\angle v_{6i+7}v_{6i+6}N$ is α. Otherwise it is δ, and the vertex type of v_{6i+6} is $\alpha+\beta+\delta$. This contradicts against Lemma 6 and (15).

To sum up, from the negation of Property 2 we derived the negation of Property 2 with the indices increased by 6. This increase of the indices by 6 contributes $\gamma+2\delta$ to the vertex type of the vertex N, and three tiles to the northern part of the tiling. Therefore $\gamma+2\delta=2\pi/(F/6)=12\pi/F$. By solving the system of this equation, (14), (15), and (12), we have $0<\delta=(10-F)\pi/F$, which is, however, be negative from $F\geq 10$. This ends the proof of Claim 1. □

We can prove Property 2* holds when the tile is of type 2, by the same argument as above with $(\alpha,\beta)\leftrightarrow(\delta,\gamma)$ swapped. We can prove Property 2* holds when the tile is of type 4 by Lemma 3. This completes the proof of Lemma 8. □

We now resume the proof of Lemma 1. The readers are kindly advised to follow the argument with Figure 4.

Proof. By Lemma 8, the assertion (2) of Lemma 1 follows. Hence $L(e)=L(f(e))=L(g(e))$ for every edge e of the chart.

First consider the case where the properties (A)* and (B)* of Lemma 8 hold. By the property (A)*, for each nonnegative integer $i<F/6$,

$$\text{the type of } \angle v_{6i}Nv_{6i+2} \text{ is } \alpha \tag{16}$$

and the type of $\angle v_{6i+2}v_{6i+1}v_{6i}$ is γ. By Lemma 8,

$$\text{the type of } \angle v_{6i-1}v_{6i}v_{6i+1} \text{ is } \gamma. \tag{17}$$

By the property (B)*,

$$\text{the type of } \angle v_{6i+2}Nv_{6i+4} \text{ is } \beta. \tag{18}$$

By (16),

$$\text{the type of } \angle Nv_{6i+2}v_{6i+1} \text{ is } \beta. \tag{19}$$

By (17), the type of $\angle v_{6i}v_{6i+1}S$ is δ. By Lemma 8, the type of $\angle Sv_{6i+1}v_{6i+2}$ is α. Thus $K(\psi) = K(f(\psi))$ if ψ is an angle around v_{6i} or v_{6i+1}. Observe

$$\text{the type of } \angle v_{6i+1}v_{6i+2}v_{6i+3} \text{ is } \beta, \tag{20}$$

and thus

$$\text{the type of } \angle v_{6i+2}v_{6i+3}S \text{ is } \gamma. \tag{21}$$

By (18),

$$\text{the type of } \angle v_{6i+3}v_{6i+2}N \text{ is } \gamma. \tag{22}$$

Thus the vertex type of v_{6i+2} is (14). Note the type of $\angle Nv_{6i+4}v_{6i+3}$ is α, and

$$\text{the type of } \angle v_{6i+4}v_{6i+3}v_{6i+2} \text{ is } \delta. \tag{23}$$

By the property (A)*, we have the equation (15). By this, (21), and (23), if the type of $\angle Sv_{6i+3}v_{6i+4}$ is δ, then we have $\alpha = \delta$ which contradicts against Lemma 6. So

$$\text{the type of } \angle Sv_{6i+3}v_{6i+4} \text{ is } \alpha, \tag{24}$$

and

$$\text{that of } \angle v_{6i+3}v_{6i+4}v_{6i+5} \text{ is } \delta. \tag{25}$$

By the property (A)*, the type of $\angle v_{6i+5}v_{6i+4}N$ is γ. Hence $K(\psi) = K(f(\psi))$ if ψ is an angle around v_{6i+3} or v_{6i+4}. Therefore the vertex type of v_{6i+3} is (15). The property (A)* furthermore implies that the type of $\angle v_{6i+6}v_{6i+5}v_{6i+4}$ is β. By (25), the type of $\angle v_{6i+4}v_{6i+5}S$ is γ. By (17), the type of $\angle Sv_{6i+5}v_{6i+6}$ is β. Therefore $K(\psi) = K(f(\psi))$ if ψ is an angle around v_{6i+2} or v_{6i+5}. This completes the proof of the assertion (3) of Lemma 1. By the property (A)*,

$$\text{the type of } \angle Nv_{6i}v_{6i-1} \text{ is } \alpha \text{ and the type of } \angle v_{6i-2}Nv_{6i} \text{ is } \delta. \tag{26}$$

Because of (16), (18) and (26), the vertex type of the north pole is

$$(\alpha + \beta + \delta)\frac{F}{6} = 2\pi.$$

From this and the equations (12) and (15), we obtain the assertion (4) of Lemma 1. The claim $\gamma > \pi$ follows from Lemma 1 (1).

We prove the assertion (5) of Lemma 1. The assertion states the following four statements: (i) the three inner angles $\angle v_{-2}Nv_0$, $\angle v_0Nv_2$, and $\angle v_2Nv_4$ are mapped by the angle-assignment K, to δ, α, and β; (ii) the inner angles around the vertex v_0 are $\angle Nv_0v_{-1}$, $\angle v_{-1}v_0v_1$ and $\angle v_1v_0N$, which K maps to α, γ and δ; (iii) the inner angles around v_2 are $\angle Nv_2v_1$, $\angle v_1v_2v_3$ and $\angle v_3v_2N$, which K maps to β, β and γ; and (iv) the inner angles around v_3 are $\angle v_4v_3v_2$, $\angle v_2v_3S$ and $\angle Sv_3v_4$, which K maps to δ, γ and α. The statement (i) follows from (26), (16), and (18); the statement (ii) does from the property (A)*; the statement (iii) does from (19), (20), and (22); and the statement (iv) does from (23), (21), and (24).

Next consider the case where the properties (A) and (B) of Lemma 8 hold. By the same argument as above but with $(\alpha, \beta) \leftrightarrow (\delta, \gamma)$ being swapped, we obtain a chart from Figure 4 with $(\alpha, \beta) \leftrightarrow (\delta, \gamma)$ swapped and obtain the assertion (4)*. But these data provide exactly the same angle-assignment for the map pdw_F which we have just obtained in the case the properties (A)* and (B)* hold. Hence the assertions (1), (2), (3), (4) and (5) of Lemma 1 hold also. □

5. The absolute values of the angles, the edges and the number of tiles

By trigonometry argument, we finish the proof of Assertion (1) of Theorem 2. In the rest of the paper, we use the spherical polar coordinate system introduced in Assertion (2) of Theorem 2. The vertex S of the chart is located at the south pole $(\pi, 0)$. To see it, first we can compute the longitude φ of the vertex S by applying trigonometry arguments along the path Nv_0v_1S. By rotating the path Nv_0v_1S by $(\alpha + \beta + \delta) = 12\pi/F$ radian, we obtain a path Nv_6v_7S. By applying the same trigonometry arguments along the latter path, the longitude of S becomes $\pi + \varphi$. Thus the vertex S should be the north pole or the south pole of the spherical polar coordinate system. If the vertex is the north pole, then some tile containing the vertex N and some tile containing the vertex S shares the interior, which contradicts against the definition of the tiling.

For the length a of an edge of the tile, $\cos a$ is positive. To see it, assume $a \geq \pi/2$. Then the two points v_2 and v_4 are located on the southern hemisphere while the point v_3 on the northern hemisphere. So $\gamma = \angle v_3v_2N$ is less than or equal to π, which contradicts against the assertion (4) of Lemma 1.

Consider the triangle Nv_2v_3. If we join the edge Nv_3 of the triangle with the edge v_3S of the tiling, we obtain a geodesic line from the north pole to the south pole. Because the edge v_3S has length a,

$$Nv_3 = \pi - a.$$

The first equation of (11) of Lemma 1 implies $\angle v_3v_2N = \gamma > \pi$. So the angle between two edges Nv_2 and v_2v_3 is $2\pi - \gamma = 8\pi/F$. By the spherical cosine theorem, we obtain a quadratic equation $\cos(\pi - a) = \cos^2 a + \sin^2 a \cos(8\pi/F)$ of $\cos a$. Since $\cos a > 0$, the solution is

$$\cos a = -\frac{\cos\left(\frac{8\pi}{F}\right)}{1 - \cos\left(\frac{8\pi}{F}\right)} > 0. \tag{27}$$

Thus by the premise $F \geq 10$ of Theorem 2 and Lemma 1 (1), the number of tiles is $F = 12$. From this and (27), the length a is as in the equation (2) of Theorem 2.

By $F = 12$ and the assertion (4) of Lemma 1, the inner angles β and γ are as in the equations (5) and (6) of Theorem 2. By the spherical cosine theorem applied to the isosceles triangle v_2Nv_4, the equations (2) and (5) imply

$$\cos v_4v_2 = \cos^2 a + \sin^2 a \cos\beta = \frac{1}{3^2} + \left(1 - \frac{1}{3^2}\right)\cos\frac{\pi}{3} = \frac{5}{9}. \tag{28}$$

The length b of an edge is less than π, because otherwise two edges Nv_0 and Nv_6 of length b share more than two points. We compute the length b by considering a triangle Nv_0v_1. If we join the edge Nv_1 of the triangle with the edge v_1S of the tiling, we obtain a geodesic line from the north pole to the south pole. Moreover the edge v_1S has length b. Therefore the edge Nv_1 of the triangle Nv_0v_1 has length $\pi - b$. Because two tiles $Sv_{11}v_0v_1$ and $v_4Nv_2v_3$ are congruent, the segment v_2v_4 has length $\pi - b$. By (28), the length b is as in the equation (3) of Theorem 2.

Then we compute the other inner angles α and δ of the tile. To compute α, we apply the spherical cosine theorem to the triangle Nv_3v_4. By the equations (2), (3) and (11), we have $\cos(\pi - a) = \cos b \cos a + \sin b \sin a \cos\alpha$. Here $0 < \alpha < \pi$ because α occurs twice in the vertex type of the vertex N. Hence the inner angles α and δ are as in the equations (4) and (7) of Theorem 2.

By the assertion (5) and the assertion (3) of Lemma 1, all angles of the chart $\mathscr{A}$ are as in Assertion (1) of Theorem 2.

6. The existence of the tile

To prove that the tiling $\mathscr{T}$ actually exists, we verify that the quadrangle $Nv_2v_3v_4$ actually exists. It is sufficient to show the following three assertions:

(i) Let u be the value of $\cos v_2v_4$ computed by applying the spherical cosine theorem to the triangle $v_2v_3v_4$. Then u is equal to the value $5/9$ of $\cos v_2v_4$ we have computed already in (28) by applying the spherical cosine theorem to the triangle v_2Nv_4.

(ii) The triangles Nv_2v_4 and $v_2v_3v_4$ actually exist.

(iii) The different edges of the tile $Nv_0v_1v_2$ do not have a common inner point.

To prove the assertion (i), we observe that $\sin a = 2\sqrt{2}/3$ and $\sin b = 2\sqrt{14}/9$, because $0 < a, b < \pi$ and the equations (2) and (3) hold. Then $u = \cos v_3v_2 \cos v_3v_4 + \sin v_3v_2 \sin v_3v_4 \cos \angle v_4v_3v_2 = \cos a \cos b + \sin a \sin b \cos\delta = 5/9$ as desired.

To prove the assertion (ii), we employ Proposition 1. First we verify the assumptions of Proposition 1 for the triangle $v_2v_3v_4$. Let ϕ be $\angle v_0Nv_1$. By applying the spherical cosine theorem to the triangle v_0Nv_1, we have $\cos a = \cos b\cos(\pi - b) + \sin b \sin(\pi - b)\cos\phi$. By the equation (3), we have $\phi = \pm\arccos(13/14)$. We prove the equation (8), that is, $\phi = \arccos(13/14)$. If ϕ is $-\arccos(13/14)$, then the spherical polar coordinate of v_1 and that of v_2 are $(\theta_1, \varphi_1) := (\pi - b, -\arccos(13/14))$ and $(\theta_2, \varphi_2) := (a, \alpha)$ respectively. For the length a of the geodesic line between two points v_1 and v_2, the value $\cos a = 1/3$ should be equal to the inner product of the cartesian coordinates of v_1 and v_2, that is, $\sin(\theta_1)\sin(\theta_2)\cos(\varphi_2 - \varphi_1) + \cos(\theta_1)\cos(\theta_2) = -5/21$. This is a contradiction. Hence we have (8). Thus

$$\angle v_2v_4v_3 = \angle v_0Nv_1 = \phi = \arccos\frac{13}{14} = 0.380251\ldots. \tag{29}$$

By the spherical cosine theorem applied to the triangle $v_2v_3v_4$, we have $\cos b = \cos a\cos(\pi - b) + \sin a\sin(\pi - b)\cos\angle v_3v_2v_4$. By the equations (2), (3), and (27), we have $\cos\angle v_3v_2v_4 = -5/(2\sqrt{7})$. If $\angle v_3v_2v_4$ is greater than π, then the edges v_3v_4 and Nv_2 of the tile $Nv_2v_3v_4$ has a common inner point, which is a contradiction. Hence

$$\angle v_3v_2v_4 = \arccos\frac{-5}{2\sqrt{7}} = 2.80812\ldots. \tag{30}$$

For a geodesic line from N to S through v_1, we have $\angle v_0v_1N = \angle v_3v_2v_4$. So

$$\angle v_4v_3v_2 = \delta = \pi - \angle v_3v_2v_4 = \arccos\frac{5}{2\sqrt{7}} = 0.3334373\ldots. \tag{31}$$

By (29), (30) and (31), the assumptions of Proposition 1 hold for the triangle $v_2v_3v_4$.

Next we verify the assumptions of Proposition 1 for the triangle v_2Nv_4. By the equations (6), (30), (5), (4), and (8), the inner angles of the triangle v_2Nv_4 are

$$\gamma - \angle v_3v_2v_4 = \frac{4\pi}{3} - \arccos\frac{-5}{2\sqrt{7}} = 1.38067\ldots, \quad \beta = \frac{\pi}{3} = 1.0472\ldots,$$

$$\alpha - \phi = \arccos\frac{-1}{2\sqrt{7}} - \arccos\frac{13}{14} = 1.38067\ldots.$$

These also satisfy the assumptions of Proposition 1. Thus the triangle v_2Nv_4 actually exists.

The assertion (iii) holds because the longitudes of the vertices v_0, v_1 and v_2 are, respectively, 0, $\phi = \arccos(13/14)$, and $\alpha = \arccos(-1/(2\sqrt{7}))$, increasing between 0 and π. To sum up, the quadrangle $Nv_2v_3v_4$ actually exists.

7. The uniqueness up to special orthogonal transformation

We prove Assertion (2) of Theorem 2. Observe that the spherical polar coordinate of each vertex v_{2i} (resp. v_{2i+1}) of the tiling $\mathscr{T}$ is determined from Assertion (1) of Theorem 2 by considering the angle around the north pole N (resp. around the south pole S with the equation (8) of Theorem 2). The length of the geodesic line between N and v_{2i+1} is π subtracted by the length of the edge Sv_{2i+1}.

We have a freedom to set a spherical polar coordinate system, so long that the north pole is the vertex N and the "Greenwich meridian" of the spherical polar coordinate system contains the edge Nv_0. However, the spherical coordinate system must respect the cyclic orders of the vertices of the chart. So each vertex is unique up to special orthogonal transformation. In fact, the mirror image of the tiling cannot be transformed to the original tiling, by a special orthogonal transformation.

8. The symmetry

To complete the proof of Theorem 2, we prove Assertion (3) of Theorem 2: Schönflies symbol of the tiling $\mathscr{T}$ is D_2. The Schönflies symbol is determined according to [7, p. 55] and Step 1, Step 2, ... of [7, Figure 3.10 (p. 56)]. As for Step 1 of Figure 3.10, the tiling $\mathscr{T}$ does not have continuous rotational symmetry, and so the Schönflies symbol is neither $C_{\infty v}$ nor $D_{\infty h}$. Moreover the tiling has none of 3-fold and 5-fold symmetry, so the Schönflies symbol is none of T, T_h, T_d, O, O_h, I, I_h.

Observe that the tiling $\mathscr{T}$ has the following three 2-fold rotation axes perpendicular to each other (cf. Figure 5):

- The axis through the both poles. The rotation around this axis by π-radian maps a point (θ, φ) on the sphere to a point $(\theta, \varphi + \pi)$. The action of the rotation on the vertices and the edges of the tiling $\mathscr{T}$ is the automorphism g defined in (10).
- The axis through the ball's center and the midpoint of v_0 and v_1. The rotation around this axis by π-radian maps a point (θ, φ) on the sphere to a point $(\pi - \theta, \phi - \varphi)$ where ϕ is defined in the equation (8). The action of the rotation on the vertices and the edges of the tiling $\mathscr{T}$ is the automorphism f defined in (9).
- The axis through the ball's center and the midpoint of v_3 and v_4. The 2-fold rotation around this axis corresponds to the automorphism $g \circ f$.

Because of the three perpendicular 2-fold axes, we can skip the Step 2, and we see in Step 3 that the Schönflies symbol is none of $S_4, S_6, S_8, \ldots$. Hence, we go to Step 5 and see that the symbol is one of D_2, D_{2d}, D_{2h}. But by Figure 5, the tiling has no mirror plane perpendicular to a 2-fold axis, the symbol is not D_{2h}. Moreover the tiling has no mirror plane between two 2-fold axes, so the symbol is not D_{2d} either. Therefore the Schönflies symbol is D_2. This completes the proof of Theorem 2.

Three movies of the tiling $\mathscr{T}$ spinning around the rotation axes are available at the web-site http://www.math.tohoku.ac.jp/akama/stcq.

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