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Epimorphisms from 2-bridge link groups onto Heckoid groups (II)

In honour of J. Hyam Rubinstein and his contribution to mathematics

Donghi LEE and Makoto SAKUMA

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ABSTRACT. In Part I of this series of papers, we made Riley's definition of Heckoid groups for 2-bridge links explicit, and gave a systematic construction of epimorphisms from 2-bridge link groups onto Heckoid groups, generalizing Riley's construction. In this paper, we give a complete characterization of upper-meridian-pair-preserving epimorphisms from 2-bridge link groups onto even Heckoid groups, by proving that they are exactly the epimorphisms obtained by the systematic construction.

1. Introduction

Let $K(r)$ be the 2-bridge link of slope $r \in \mathbf{Q}$ and let n be an integer or a half-integer greater than 1. In [8], following Riley's work [12], we introduced the *Heckoid group* $G(r;n)$ *of index* n *for* $K(r)$ as the orbifold fundamental group of the *Heckoid orbifold* $\boldsymbol{S}(r;n)$ *of index* n *for* $K(r)$. According to whether n is an integer or a non-integral half-integer, the Heckoid group $G(r;n)$ and the Heckoid orbifold $\boldsymbol{S}(r;n)$ are said to be *even* or *odd*. The even Heckoid orbifold $\boldsymbol{S}(r;n)$ is the 3-orbifold such that

(i) the underlying space $|\boldsymbol{S}(r;n)|$ is the exterior, $E(K(r)) = S^3 - \operatorname{int} N(K(r))$, of $K(r)$, and
(ii) the singular set is the lower tunnel of $K(r)$, where the index of the singularity is n.

For a description of odd Heckoid orbifolds, see [8, Proposition 5.3].

In [8, Theorem 2.3], we gave a systematic construction of upper-meridian-pair-preserving epimorphisms from 2-bridge link groups onto Heckoid groups, generalizing Riley's construction in [12].

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The main purpose of this paper is to describe all upper-meridian-pair-preserving epimorphisms from 2-bridge link groups onto *even* Heckoid groups (Theorem 2.4). The theorem says that all such epimorphisms are contained in those constructed in [8, Theorem 2.3]. To prove this result, we determine those essential simple loops on a 2-bridge sphere in an even Heckoid orbifold $\boldsymbol{S}(r;n)$ which are null-homotopic in $\boldsymbol{S}(r;n)$ (Theorem 2.3). These results form an analogy of [3, Main Theorem 2.4], which describes all upper-meridian-pair-preserving epimorphisms between 2-bridge link groups, and that of [3, Main Theorem 2.3], which gives a complete characterization of those essential simple loops on a 2-bridge sphere in a 2-bridge link complement which are null-homotopic in the link complement. As in [3], the key tool is small cancellation theory, applied to two-generator and one-relator presentations of even Heckoid groups.

This paper is organized as follows. In Section 2, we describe the main results. In Section 3, we introduce a two-generator and one-relator presentation of an even Heckoid group, and review basic facts concerning its single relator established in [3]. In Section 4, we apply small cancellation theory to the two-generator and one-relator presentations of even Heckoid groups. In Section 5, we prove Theorem 2.3.

2. Main results

We quickly recall notation and basic facts introduced in [8]. The *Conway sphere* $\boldsymbol{S}$ is the 4-times punctured sphere which is obtained as the quotient of $\mathbf{R}^2 - \mathbf{Z}^2$ by the group generated by the π-rotations around the points in $\mathbf{Z}^2$. For each $s \in \hat{\mathbf{Q}} := \mathbf{Q} \cup \{\infty\}$, let α_s be the simple loop in $\boldsymbol{S}$ obtained as the projection of a line in $\mathbf{R}^2 - \mathbf{Z}^2$ of slope s. We call s the *slope* of the simple loop α_s.

For each $r \in \hat{\mathbf{Q}}$, the *2-bridge link $K(r)$ of slope r* is the sum of the rational tangle $(B^3, t(\infty))$ of slope ∞ and the rational tangle $(B^3, t(r))$ of slope r. Recall that $\partial(B^3 - t(\infty))$ and $\partial(B^3 - t(r))$ are identified with $\boldsymbol{S}$ so that α_∞ and α_r bound disks in $B^3 - t(\infty)$ and $B^3 - t(r)$, respectively. By van-Kampen's theorem, the link group $G(K(r)) = \pi_1(S^3 - K(r))$ is obtained as follows:

$$G(K(r)) = \pi_1(S^3 - K(r)) \cong \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle \cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r \rangle\rangle.$$

We call the image in the link group of the "meridian pair" of $\pi_1(B^3 - t(\infty))$ the *upper meridian pair*.

If r is a rational number and $n \geq 2$ is an integer, then by the description of the even Heckoid orbifold $\boldsymbol{S}(r;n)$ in the introduction, the even Hekoid group $G(r;n) = \pi_1(\boldsymbol{S}(r;n))$ is identified with

$$G(r;n) \cong \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r^n \rangle\rangle \cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r^n \rangle\rangle.$$

In particular, the even Heckoid group $G(r;n)$ is a two-generator and one-relator group. We call the image in $G(r;n)$ of the meridian pair of $\pi_1(B^3 - t(\infty))$ the *upper meridian pair*.

This paper and its sequel [9] are concerned with the following natural question, which is an analogy of [2, Question 1.1] that is completely solved in the series of papers [3, 4, 5, 6] and applied in [7].

Question 2.1. For r a rational number and n an integer or a half-integer greater than 1, consider the Heckoid group $G(r;n)$ of index n for the 2-bridge link $K(r)$.

(1) Which essential simple loop α_s on $\boldsymbol{S}$ determines the trivial element of $G(r;n)$?

(2) For two distinct essential simple loops α_s and $\alpha_{s'}$ on $\boldsymbol{S}$, when do they determine the same conjugacy class in $G(r;n)$?

In [8, Theorem 2.4], we gave a certain sufficient condition for each of the questions. In this paper, we prove that, for even Heckoid groups, the sufficient condition for (1) is actually a necessary and sufficient condition. This enables us to describe all upper-meridian-pair-preserving epimorphisms from 2-bridge link groups onto even Heckoid groups.

Let $\mathscr{D}$ be the *Farey tessellation* of the upper half plane $\mathbf{H}^2$. Then $\hat{\mathbf{Q}}$ is identified with the set of the ideal vertices of $\mathscr{D}$. Let Γ_∞ be the group of automorphisms of $\mathscr{D}$ generated by reflections in the edges of $\mathscr{D}$ with an endpoint ∞. For r a rational number and n an integer or a half-integer greater than 1, let $C_r(2n)$ be the group of automorphisms of $\mathscr{D}$ generated by the parabolic transformation, centered on the vertex r, by $2n$ units in the clockwise direction, and let $\Gamma(r;n)$ be the group generated by Γ_∞ and $C_r(2n)$. Suppose that r is not an integer, i.e., $K(r)$ is not a trivial knot. Then $\Gamma(r;n)$ is the free product $\Gamma_\infty * C_r(2n)$ having a fundamental domain, R, shown in Figure 1. Here, R is obtained as the intersection of fundamental domains for Γ_∞ and $C_r(2n)$, and so R is bounded by the following two pairs of Farey edges:

(1) the pair of adjacent Farey edges with an endpoint ∞ which cuts off a region in $\overline{\mathbf{H}}^2$ containing r, and

(2) a pair of Farey edges with an endpoint r which cuts off a region in $\overline{\mathbf{H}}^2$ containing ∞ such that one edge is the image of the other by a generator of $C_r(2n)$.

Let $\bar{I}(r;n)$ be the union of two closed intervals in $\partial\mathbf{H}^2 = \hat{\mathbf{R}}$ obtained as the intersection of the closure of R and $\partial\mathbf{H}^2$. (In the special case when $r \equiv \pm 1/p$ (mod $\mathbf{Z}$) for some integer $p > 1$, one of the intervals may be degenerated to a single point.) Note that there is a pair $\{r_1, r_2\}$ of boundary points of $\bar{I}(r;n)$ such that r_2 is the image of r_1 by a generator of $C_r(2n)$. Set $I(r;n) := \bar{I}(r;n) - \{r_i\}$ with $i = 1$ or 2. Note that $I(r;n)$ is the disjoint union of a closed

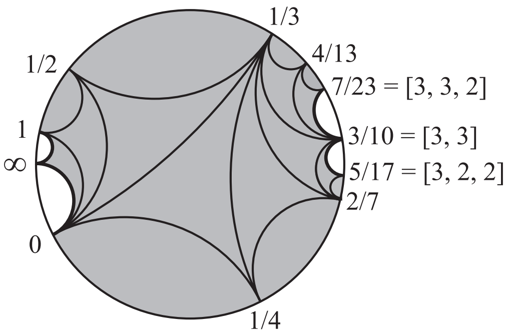

Fig. 1. A fundamental domain of $\Gamma(r;n)$ in the Farey tessellation (the shaded domain) for $r = 3/10 = \cfrac{1}{3+\cfrac{1}{3}} =: [3,3]$ and $n = 2$. In this case, $\bar{I}(r;n) = [0, 5/17] \cup [7/23, 1]$.

interval and a half-open interval, except for the special case when $r \equiv \pm 1/p \pmod{\mathbf{Z}}$.

Then we obtain the following refinement of [8, Theorem 2.4].

Theorem 2.2. *Suppose that r is a non-integral rational number and that n is an integer or a half-integer greater than* 1. *Then, for any $s \in \hat{\mathbf{Q}}$, there is a unique rational number $s_0 \in I(r;n) \cup \{\infty, r\}$ such that s is contained in the $\Gamma(r;n)$-orbit of s_0. Moreover the conjugacy classes α_s and α_{s_0} in $G(r;n)$ are equal. In particular, if $s_0 = \infty$, then α_s is the trivial conjugacy class in $G(r;n)$.*

In fact, the first assertion is proved as in [3, Lemma 7.1] by using the fact that R is a fundamental domain for the action of $\Gamma(r;n)$ on $\mathbf{H}^2$. The remaining assertions are nothing other than [8, Theorem 2.4].

The following main theorem shows that the converse to the last statement in Theorem 2.2 holds for *even* Heckoid groups.

Theorem 2.3. *Suppose that r is a non-integral rational number and that n is an integer greater than* 1. *Then α_s represents the trivial element of $G(r;n)$ if and only if s belongs to the $\Gamma(r;n)$-orbit of ∞. In other words, if $s \in I(r;n) \cup \{r\}$, then α_s does not represent the trivial element of $G(r;n)$.*

Arguing as in [8, Proof of Theorem 2.3], we see that the above theorem implies the following theorem, which says that the converse to [8, Theorem 2.3] holds for even Heckoid groups.

Theorem 2.4. *Suppose that r is a non-integral rational number and that n is an integer greater than* 1. *Then there is an upper-meridian-pair-preserving* *epimorphism from* $G(K(s))$ *to* $G(r;n)$ *if and only if* s *or* $s+1$ *belongs to the* $\Gamma(r;n)$*-orbit of* ∞.

REMARK 2.5. (1) When r is an integer, the Heckoid group $G(r;n) \cong G(0;n)$ is isomorphic to the subgroup $\langle P, SPS^{-1}\rangle$ of the classical *Hecke group* $\langle P, S\rangle$ introduced in [1], where

$$P = \begin{pmatrix} 1 & 2\cos\frac{\pi}{2n} \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Moreover, the group $\Gamma(0;n)$ is the free product of three cyclic groups of order 2 generated by the reflections in the Farey edges $\langle \infty, 0\rangle$ and $\langle \infty, 1\rangle$ and the geodesic $\overline{1, 1/n}$. (The last geodesic is a Farey edge if n is an integer, whereas it bisects a pair of adjacent Farey triangles if n is a non-integral half-integer.) The region of $\mathbf{H}^2$ bounded by these three geodesics is a fundamental domain for the action of $\Gamma(0;n)$ on $\mathbf{H}^2$. It is easy to see that Theorem 2.2 continues to be valid when r is an integer, provided that we set $I(0;n) := [1/n, n]$. It is plausible that Theorems 2.3 and 2.4 are also valid even when r is an integer. However, we cannot directly apply the arguments of this paper, and this case will be treated elsewhere.

(2) It is natural to expect that Theorems 2.3 and 2.4 also hold for odd Heckoid groups. However, we do not know how to treat these groups at this moment, because they are not one-relator groups by [8, Proposition 6.7].

3. Presentations of even Heckoid groups and review of basic facts from [3]

In the remainder of this paper, we restrict our attention to the *even* Heckoid groups $G(r;n)$. Thus n denotes an integer with $n \geq 2$. In order to describe the two-generator and one-relator presentations of even Heckoid groups to which we apply small cancellation theory, recall that

$$G(r;n) \cong \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r^n \rangle\rangle \cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r^n \rangle\rangle.$$

Let $\{a, b\}$ be the standard meridian generator pair of $\pi_1(B^3 - t(\infty), x_0)$ as described in [3, Section 3] (see also [2, Section 5]). Then $\pi_1(B^3 - t(\infty))$ is identified with the free group $F(a, b)$. For the rational number $r = q/p$, where p and q are relatively prime positive integers, let u_r be the word in $\{a, b\}$ obtained as follows. (For a geometric description, see [2, Section 5].) Set $\varepsilon_i = (-1)^{\lfloor iq/p \rfloor}$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

(1) If p is odd, then

$$u_{q/p} = a\hat{u}_{q/p} b^{(-1)^q} \hat{u}_{q/p}^{-1},$$

where $\hat{u}_{q/p} = b^{\varepsilon_1} a^{\varepsilon_2} \dots b^{\varepsilon_{p-2}} a^{\varepsilon_{p-1}}$.

(2) If p is even, then

$$u_{q/p} = a\hat{u}_{q/p}a^{-1}\hat{u}_{q/p}^{-1},$$

where $\hat{u}_{q/p} = b^{\varepsilon_1}a^{\varepsilon_2}\dots a^{\varepsilon_{p-2}}b^{\varepsilon_{p-1}}$.

Then $u_r \in F(a,b) \cong \pi_1(B^3 - t(\infty))$ is represented by the simple loop α_r, and we obtain the following two-generator and one-relator presentation of the even Heckoid group $G(r;n)$, which is used throughout the remainder of this paper:

$$G(r;n) \cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r^n \rangle\rangle \cong \langle a, b \,|\, u_r^n \rangle.$$

We recall the definition of the sequences $S(r)$ and $T(r)$ and the cyclic sequences $CS(r)$ and $CT(r)$ of slope r defined in [3], all of which are read from the word u_r defined above, and review several important properties of these sequences from [3] so that we can adopt small cancellation theory in the succeeding section. To this end, we fix some definitions and notation. Let X be a set. By a *word* in X, we mean a finite sequence $x_1^{\varepsilon_1}x_2^{\varepsilon_2}\dots x_t^{\varepsilon_t}$ where $x_i \in X$ and $\varepsilon_i = \pm 1$. Here we call $x_i^{\varepsilon_i}$ the *i-th letter* of the word. For two words u, v in X, by $u \equiv v$ we denote the *visual equality* of u and v, meaning that if $u = x_1^{\varepsilon_1}\dots x_t^{\varepsilon_t}$ and $v = y_1^{\delta_1}\dots y_m^{\delta_m}$ ($x_i, y_j \in X$; $\varepsilon_i, \delta_j = \pm 1$), then $t = m$ and $x_i = y_i$ and $\varepsilon_i = \delta_i$ for each $i = 1, \dots, t$. For example, two words $x_1x_2x_2^{-1}x_3$ and x_1x_3 ($x_i \in X$) are *not* visually equal, though $x_1x_2x_2^{-1}x_3$ and x_1x_3 are equal as elements of the free group with basis X. The length of a word v is denoted by $|v|$. A word v in X is said to be *reduced* if v does not contain xx^{-1} or $x^{-1}x$ for any $x \in X$. A word is said to be *cyclically reduced* if all its cyclic permutations are reduced. A *cyclic word* is defined to be the set of all cyclic permutations of a cyclically reduced word. By (v) we denote the cyclic word associated with a cyclically reduced word v. Also by $(u) \equiv (v)$ we mean the *visual equality* of two cyclic words (u) and (v). In fact, $(u) \equiv (v)$ if and only if v is visually a cyclic shift of u.

Definition 3.1. (1) Let v be a reduced word in $\{a, b\}$. Decompose v into

$$v \equiv v_1v_2\dots v_t,$$

where, for each $i = 1, \dots, t-1$, all letters in v_i have positive (resp., negative) exponents, and all letters in v_{i+1} have negative (resp., positive) exponents. Then the sequence of positive integers $S(v) := (|v_1|, |v_2|, \dots, |v_t|)$ is called the *S-sequence of* v.

(2) Let (v) be a cyclic word in $\{a, b\}$. Decompose (v) into

$$(v) \equiv (v_1v_2\dots v_t),$$

where all letters in v_i have positive (resp., negative) exponents, and all letters in v_{i+1} have negative (resp., positive) exponents (taking subindices modulo t). Then the *cyclic* sequence of positive integers $CS(v) := ((|v_1|, |v_2|, \dots, |v_t|))$ is called the *cyclic S-sequence of* (v). Here the double parentheses denote that the sequence is considered modulo cyclic permutations.

(3) A reduced word v in $\{a, b\}$ is said to be *alternating* if $a^{\pm 1}$ and $b^{\pm 1}$ appear in v alternately, i.e., neither $a^{\pm 2}$ nor $b^{\pm 2}$ appears in v. A cyclic word (v) is said to be *alternating* if all cyclic permutations of v are alternating. In the latter case, we also say that v is *cyclically alternating*.

Definition 3.2. For a rational number r with $0 < r \le 1$, let u_r be the word defined in the beginning of this section. Then the symbol $S(r)$ (resp., $CS(r)$) denotes the S-sequence $S(u_r)$ of u_r (resp., cyclic S-sequence $CS(u_r)$ of (u_r)), which is called the *S-sequence of slope* r (resp., the *cyclic S-sequence of slope* r).

In the remainder of this section, we suppose that r is a rational number with $0 < r \le 1$, and write r as a continued fraction expansion:

$$r = [m_1, m_2, \dots, m_k] := \cfrac{1}{m_1 + \cfrac{1}{m_2 + \ddots + \cfrac{1}{m_k}}},$$

where $k \ge 1$, $(m_1, \dots, m_k) \in (\mathbf{Z}_+)^k$ and $m_k \ge 2$ unless $k = 1$. For brevity, we write m for m_1.

Lemma 3.3 ([3, Proposition 4.3]). *The following hold.*

(1) *Suppose* $k = 1$, *i.e.,* $r = 1/m$. *Then* $S(r) = (m, m)$.

(2) *Suppose* $k \ge 2$. *Then each term of* $S(r)$ *is either* m *or* $m+1$, *and* $S(r)$ *begins with* $m+1$ *and ends with* m. *Moreover, the following hold.*

(a) *If* $m_2 = 1$, *then no two consecutive terms of* $S(r)$ *can be* (m, m), *so there is a sequence of positive integers* $(t_1, t_2, \dots, t_s)$ *such that*

$$S(r) = (t_1\langle m+1\rangle, m, t_2\langle m+1\rangle, m, \dots, t_s\langle m+1\rangle, m).$$

Here, the symbol "$t_i\langle m+1\rangle$" represents t_i successive $m+1$'s.

(b) *If* $m_2 \ge 2$, *then no two consecutive terms of* $S(r)$ *can be* $(m+1, m+1)$, *so there is a sequence of positive integers* $(t_1, t_2, \dots, t_s)$ *such that*

$$S(r) = (m+1, t_1\langle m\rangle, m+1, t_2\langle m\rangle, \dots, m+1, t_s\langle m\rangle).$$

Here, the symbol "$t_i\langle m\rangle$" represents t_i successive m's.

Definition 3.4. If $k \geq 2$, the symbol $T(r)$ denotes the sequence $(t_1, t_2, \ldots, t_s)$ in Lemma 3.3, which is called the *T-sequence of slope r*. The symbol $CT(r)$ denotes the cyclic sequence represented by $T(r)$, which is called the *cyclic T-sequence of slope r*.

Lemma 3.5 ([3, Proposition 4.4 and Corollary 4.6]). *Let $\tilde{r}$ be the rational number defined as*

$$\tilde{r} = \begin{cases} [m_3, \ldots, m_k] & \text{if } m_2 = 1; \\ [m_2 - 1, m_3, \ldots, m_k] & \text{if } m_2 \geq 2. \end{cases}$$

Then we have $CS(\tilde{r}) = CT(r)$.

Lemma 3.6 ([3, Proposition 4.5]). *The sequence $S(r)$ has a decomposition (S_1, S_2, S_1, S_2) which satisfies the following.*

(1) *Each S_i is symmetric, i.e., the sequence obtained from S_i by reversing the order is equal to S_i. (Here, S_1 is empty if $k = 1$.)*
(2) *Each S_i occurs only twice in the cyclic sequence $CS(r)$.*
(3) *The subsequence S_1 begins and ends with $m + 1$.*
(4) *The subsequence S_2 begins and ends with m.*

Lemma 3.7 ([3, Proof of Proposition 4.5]). *Let $\tilde{r}$ be the rational number defined as in Lemma 3.5. Also let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ and $S(r) = (S_1, S_2, S_1, S_2)$ be decompositions described as in Lemma 3.6. Then the following hold.*

(1) *If $m_2 = 1$ and $k = 3$, then $T_1 = \emptyset$, $T_2 = (m_3)$, and $S_1 = (m_3\langle m+1\rangle)$, $S_2 = (m)$.*
(2) *If $m_2 = 1$ and $k \geq 4$, then $T_1 = (t_1, \ldots, t_{s_1})$, $T_2 = (t_{s_1+1}, \ldots, t_{s_2})$, and*

$$S_1 = (t_1\langle m+1\rangle, m, t_2\langle m+1\rangle, \ldots, t_{s_1-1}\langle m+1\rangle, m, t_{s_1}\langle m+1\rangle),$$

$$S_2 = (m, t_{s_1+1}\langle m+1\rangle, m, \ldots, m, t_{s_2}\langle m+1\rangle, m).$$

(3) *If $m_2 \geq 2$ and $k = 2$, then $T_1 = \emptyset$, $T_2 = (m_2 - 1)$, and $S_1 = (m+1)$, $S_2 = ((m_2 - 1)\langle m\rangle)$.*
(4) *If $m_2 \geq 2$ and $k \geq 3$, then $T_1 = (t_1, \ldots, t_{s_1})$, $T_2 = (t_{s_1+1}, \ldots, t_{s_2})$, and*

$$S_1 = (m+1, t_{s_1+1}\langle m\rangle, m+1, \ldots, m+1, t_{s_2}\langle m\rangle, m+1),$$

$$S_2 = (t_1\langle m\rangle, m+1, t_2\langle m\rangle, \ldots, t_{s_1-1}\langle m\rangle, m+1, t_{s_1}\langle m\rangle).$$

By Lemmas 3.3 and 3.7, we easily obtain the following corollary.

Corollary 3.8. *Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Lemma 3.6. Then the following hold.*

(1) *If $m_2 = 1$, then $(m+1, m+1)$ appears in S_1.*
(2) *If $m_2 \geq 2$ and if $r \neq [m, 2] = 2/(2m+1)$, then (m, m) appears in S_2.*

4. Small cancellation theory

Let $F(X)$ be the free group with basis X. A subset R of $F(X)$ is said to be *symmetrized*, if all elements of R are cyclically reduced and, for each $w \in R$, all cyclic permutations of w and w^{-1} also belong to R.

Definition 4.1. Suppose that R is a symmetrized subset of $F(X)$. A nonempty word b is called a *piece* if there exist distinct $w_1, w_2 \in R$ such that $w_1 \equiv bc_1$ and $w_2 \equiv bc_2$. The small cancellation conditions $C(p)$ and $T(q)$, where p and q are integers such that $p \geq 2$ and $q \geq 3$, are defined as follows (see [10]).

(1) Condition $C(p)$: If $w \in R$ is a product of t pieces, then $t \geq p$.

(2) Condition $T(q)$: For $w_1, \dots, w_t \in R$ with no successive elements w_i, w_{i+1} an inverse pair ($i \bmod t$), if $t < q$, then at least one of the products $w_1 w_2, \dots, w_{t-1} w_t$, $w_t w_1$ is freely reduced without cancellation.

We recall the following lemma from [3], which concerns the word u_r defined in the beginning of Section 3.

Lemma 4.2 ([3, Lemma 5.3]). *Suppose that r is a rational number with $0 < r < 1$, and write $r = [m_1, m_2, \dots, m_k]$, where $k \geq 1$, $(m_1, \dots, m_k) \in (\mathbf{Z}_+)^k$ and $m_k \geq 2$. Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Lemma 3.6. Decompose*

$$u_r \equiv v_1 v_2 v_3 v_4,$$

where $S(v_1) = S(v_3) = S_1$ and $S(v_2) = S(v_4) = S_2$. Then the following hold.

(1) *If $k = 1$, then the following hold.*
   (a) *No piece can contain v_2 or v_4.*
   (b) *No piece is of the form $v_{2e} v_{4b}$ or $v_{4e} v_{2b}$, where v_{ib} and v_{ie} are nonempty initial and terminal subwords of v_i, respectively.*
   (c) *Every subword of the form v_{2b}, v_{2e}, v_{4b}, or v_{4e} is a piece, where v_{ib} and v_{ie} are nonempty initial and terminal subwords of v_i with $|v_{ib}|, |v_{ie}| \leq |v_i| - 1$, respectively.*

(2) *If $k \geq 2$, then the following hold.*
   (a) *No piece can contain v_1 or v_3.*
   (b) *No piece is of the form $v_{1e} v_2 v_{3b}$ or $v_{3e} v_4 v_{1b}$, where v_{ib} and v_{ie} are nonempty initial and terminal subwords of v_i, respectively.*
   (c) *Every subword of the form $v_{1e} v_2$, $v_2 v_{3b}$, $v_{3e} v_4$, or $v_4 v_{1b}$ is a piece, where v_{ib} and v_{ie} are nonempty initial and terminal subwords of v_i with $|v_{ib}|, |v_{ie}| \leq |v_i| - 1$, respectively.*

By using the above lemma, we establish the following key lemma concerning the cyclic word (u_r^n), where u_r^n is the single relator of the presentation $G(r; n) = \langle a, b \mid u_r^n \rangle$.

Lemma 4.3. *Suppose that r is a rational number with $0 < r < 1$, and write $r = [m_1, m_2, \dots, m_k]$, where $k \geq 1$, $(m_1, \dots, m_k) \in (\mathbf{Z}_+)^k$ and $m_k \geq 2$. Decompose $u_r \equiv v_1 v_2 v_3 v_4$ as in Lemma 4.2. Then for the relator $u_r^n \equiv (v_1 v_2 v_3 v_4)^n$, where $n \geq 2$ is an integer, the following hold.*

(1) *The cyclic word (u_r^n) is not a product of t pieces with $t \leq 4n - 1$.*

(2) *Let w be a subword of the cyclic word (u_r^n) which is a product of $4n - 1$ pieces but is not a product of t pieces with $t < 4n - 1$. Then w contains a subword, w', such that $S(w') = ((2n-1)\langle S_1, S_2 \rangle, \ell)$ or $S(w') = (\ell, (2n-1)\langle S_2, S_1 \rangle)$, where $S(r) = (S_1, S_2, S_1, S_2)$ and $\ell \in \mathbf{Z}_+$.*

Proof. For simplicity, we prove the lemma when $k \geq 2$. The case where $k = 1$ is treated similarly.

(1) Let $(u_r^n) \equiv (w_1 w_2 \dots w_t)$ be a decomposition of the cyclic word (u_r^n) into t pieces. Such a decomposition is determined by a t-tuple of "breaks" arranged in the cyclic word (u_r^n), such that w_i is the subword of (u_r^n) surrounded by the $(i-1)$-th break and the i-th break. (Here the indices are considered modulo t.) Then Lemma 4.2(2-a) and (2-b) imply the following:

(a) Each subword of the form v_1 or v_3 of (u_r^n) contains a break in its interior.

(b) Each subword of the form v_2 or v_4 of (u_r^n) contains a break in its interior or in its boundary.

Since each break is contained in either (a) the interior of a subword of the form v_1 or v_3 or (b) the interior or the boundary of a subword of the form v_2 or v_4, the above observation implies that there is a well-defined surjection, η, from the set of breaks onto the set of subwords of the form v_1, v_2, v_3 or v_4. Since the domain and the codomain of η have cardinalities t and $4n$, respectively, we have $t \geq 4n$. This completes the proof of assertion (1). Before proving (2), we note that if t is the smallest length of decompositions of (u_r^n) into pieces, then Lemma 4.2(2-c) implies that η is injective.

(2) Let $w \equiv w_1 w_2 \dots w_{4n-1}$ be a subword of the cyclic word (u_r^n), where $w_1, \dots, w_{4n-1}$ are pieces, such that w is not a product of t pieces with $t < 4n - 1$. As in the proof of (1), the decomposition $w \equiv w_1 w_2 \dots w_{4n-1}$ is determined by a $(t+1)$-tuple of breaks in (u_r^n), such that w_i is the subword of (u_r^n) surrounded by the $(i-1)$-th break and the i-th break. Lemma 4.2 implies the following:

(a) Each subword of the form v_1 or v_3 of (u_r^n) contains a unique break in its interior.

(b) Each subword of the form v_2 or v_4 of (u_r^n) contains a unique break in its interior or in its boundary.

Suppose first that the 0-th break is contained in the interior of a subword of (u_r^n) of the form v_1. Then we see from the above observations that $w \equiv v_{1e}(v_2v_3v_4v_1)^{n-1}v_2v_3v_{4b}$, where v_{1e} is a nonempty proper terminal subword of v_1 and v_{4b} is a (possibly empty or nonproper) initial subword of v_4. Let w' be the subword $v_{1e}'(v_2v_3v_4v_1)^{n-1}v_2v_3$ of w, where v_{1e}' is a nonempty positive or negative terminal subword of v_{1e}. Then we have $S(w') = (\ell, (2n-1)\langle S_2, S_1\rangle)$, where $\ell \in \mathbf{Z}_+$. Suppose next that the 0-th break is contained in the interior or the boundary of a subword of (u_r^n) of the form v_2. Then we see from the above observations $w \equiv v_{2e}(v_3v_4v_1v_2)^{n-1}v_3v_4v_{1b}$, where v_{2e} is a (possibly empty or nonproper) terminal subword of v_2 and v_{1b} is a nonempty proper initial subword of v_1. Let w' be the subword $(v_3v_4v_1v_2)^{n-1}v_3v_4v_{1b}'$ of w, where v_{1b}' is a non-empty initial positive or negative subword of v_{1b}. Then we have $S(w') = ((2n-1)\langle S_1, S_2\rangle, \ell)$, where $\ell \in \mathbf{Z}_+$. The case where the 0-th break is contained in the interior of a subword of (u_r^n) of the form v_3 and the case where 0-th break is contained in the interior or the boundary of a subword of (u_r^n) of the form v_4 are treated similarly. □

The following proposition enables us to apply small cancellation theory to our problem.

Proposition 4.4. *Suppose that r is a rational number with $0 < r < 1$ and that n is an integer with $n \geq 2$. Let R be the symmetrized subset of $F(a,b)$ generated by the single relator u_r^n of the presentation $G(r;n) = \langle a, b \mid u_r^n \rangle$. Then R satisfies $C(4n)$ and $T(4)$.*

Proof. The assertion that R satisfies $C(4n)$ is nothing other than Lemma 4.3(1). The assertion that R satisfies $T(4)$ is proved exactly as in [3, Proof of Theorem 5.1]. □

Now we want to investigate the geometric consequences of Proposition 4.4. Let us begin with necessary definitions and notation following [10]. A *map* M is a finite 2-dimensional cell complex embedded in $\mathbf{R}^2$, namely a finite collection of vertices (0-cells), edges (1-cells), and faces (2-cells) in $\mathbf{R}^2$. The boundary (frontier), ∂M, of M in $\mathbf{R}^2$ is regarded as a 1-dimensional subcomplex of M. An edge may be traversed in either of two directions. If v is a vertex of a map M, then $d_M(v)$, the *degree of* v, will denote the number of oriented edges in M having v as initial vertex. A vertex v of M is called an *interior vertex* if $v \notin \partial M$, and an edge e of M is called an *interior edge* if $e \not\subset \partial M$.

A *path* in M is a sequence of oriented edges $e_1, \dots, e_t$ such that the initial vertex of e_{i+1} is the terminal vertex of e_i for every $1 \leq i \leq t-1$. A *cycle* is a closed path, namely a path $e_1, \dots, e_t$ such that the initial vertex of e_1 is the terminal vertex of e_n. If D is a face of M, then any cycle of minimal length which includes all the edges of the boundary, ∂D, of D is called a *boundary cycle* of D. By $d_M(D)$, the *degree of* D, we denote the number of oriented edges in a boundary cycle of D.

DEFINITION 4.5. A non-empty map M is called a *$[p,q]$-map* if the following conditions hold.
(i) $d_M(v) \geq p$ for every interior vertex v in M.
(ii) $d_M(D) \geq q$ for every face D in M.

If M is connected and simply connected, then a *boundary cycle* of M is defined to be a cycle of minimal length which contains all the edges of ∂M going around once along the boundary of $\mathbf{R}^2 - M$.

DEFINITION 4.6. Let R be a symmetrized subset of $F(X)$. An *R-diagram* is a map M and a function ϕ assigning to each oriented edge e of M, as a *label*, a reduced word $\phi(e)$ in X such that the following hold.
(1) If e is an oriented edge of M and e^{-1} is the oppositely oriented edge, then $\phi(e^{-1}) = \phi(e)^{-1}$.
(2) For any boundary cycle δ of any face of M, $\phi(\delta)$ is a cyclically reduced word representing an element of R. (If $\alpha = e_1, \dots, e_t$ is a path in M, we define $\phi(\alpha) \equiv \phi(e_1) \dots \phi(e_t)$.)

In particular, if a group G is presented by $G = \langle X | R \rangle$ with R being symmetrized, then a connected and simply connected R-diagram is called a *van Kampen diagram* over the group presentation $G = \langle X | R \rangle$.

Let D_1 and D_2 be faces (not necessarily distinct) of M with an edge $e \subseteq \partial D_1 \cap \partial D_2$. Let $e\delta_1$ and $\delta_2 e^{-1}$ be boundary cycles of D_1 and D_2, respectively. Let $\phi(\delta_1) = f_1$ and $\phi(\delta_2) = f_2$. An R-diagram M is called *reduced* if one never has $f_2 = f_1^{-1}$. It should be noted that if M is reduced then $\phi(e)$ is a piece for every interior edge e of M. A *boundary label of* M is defined to be a word $\phi(\alpha)$ in X for α a boundary cycle of M. It is easy to see that any two boundary labels of M are cyclic permutations of each other.

We recall the following lemma which is a well-known classical result in combinatorial group theory (see [10]).

LEMMA 4.7 (van Kampen). *Suppose $G = \langle X | R \rangle$ with R being symmetrized. Let v be a cyclically reduced word in X. Then $v = 1$ in G if and only if there exists a reduced van Kampen diagram M over $G = \langle X | R \rangle$ with a boundary label v.*

As explained in [3, Convention 1], we may assume the following convention.

Convention 4.8. Let R be the symmetrized subset of $F(a,b)$ generated by the single relator u_r^n of the presentation $G(r;n) = \langle a, b \mid u_r^n \rangle$. For any reduced R-diagram M, we assume that M satisfies the following.

(1) Every interior vertex of M has degree at least three.

(2) For every edge e of ∂M, the label $\phi(e)$ is a piece.

(3) For a path $e_1, \dots, e_t$ in ∂M of length $n \geq 2$ such that the vertex $e_i \cap e_{i+1}$ has degree 2 for $i = 1, 2, \dots, t-1$, $\phi(e_1)\phi(e_2)\dots\phi(e_t)$ cannot be expressed as a product of less than t pieces.

The following corollary is immediate from Proposition 4.4 and Convention 4.8.

Corollary 4.9. *Suppose that r is a rational number with $0 < r < 1$ and that n is an integer with $n \geq 2$. Let R be the symmetrized subset of $F(a,b)$ generated by the single relator u_r^n of the presentation $G(r;n) = \langle a, b \mid u_r^n \rangle$. Then every reduced R-diagram is a $[4, 4n]$-map.*

We recall the following lemma obtained from the arguments of [10, Theorem V.3.1].

Lemma 4.10 (cf. [10, Theorem V.3.1]). *Let M be an arbitrary connected and simply-connected map. Then*

$$4 \leq \sum_{v \in \partial M} (3 - d_M(v)) + \sum_{v \in M - \partial M} (4 - d_M(v)) + \sum_{D \in M} (4 - d_M(D)).$$

In particular, if M is a $[4, 4n]$-map, then

$$4 \leq \sum_{v \in \partial M} (3 - d_M(v)) + \sum_{D \in M} (4 - 4n).$$

We now close this section with the following proposition which will play an important role in the proof of Theorem 2.3.

Proposition 4.11. *Let M be an arbitrary connected and simply-connected $[4, 4n]$-map such that there is no vertex of degree 3 in ∂M. Put*

$$A = \text{the number of vertices } v \text{ in } \partial M \text{ such that } d_M(v) = 2,$$

$$B = \text{the number of vertices } v \text{ in } \partial M \text{ such that } d_M(v) \geq 4.$$

Then the following inequality holds.

$$A \geq (4n - 3)B + 4n$$

PROOF. Put

$V =$ the number of vertices of M,

$E =$ the number of (unoriented) edges of M,

$F =$ the number of faces of M.

Then, since every interior vertex in M has degree at least 4, we have

$$E \geq \frac{1}{2}\{2A + 4(V - A)\} = 2V - A.$$

This inequality together with Euler's formula $1 = V - E + F$ yields $1 \leq V - (2V - A) + F$, so that

$$F \geq V - A + 1 \geq (A + B) - A + 1 = B + 1. \tag{†}$$

On the other hand, by Lemma 4.10, we have

$$4 \leq \sum_{v \in \partial M}(3 - d_M(v)) + \sum_{D \in M}(4 - 4n) = \sum_{v \in \partial M}(3 - d_M(v)) + (4 - 4n)F,$$

so that $\sum_{v \in \partial M}(3 - d_M(v)) \geq 4 + (4n - 4)F$. Here, since $A - B \geq \sum_{v \in \partial M}(3 - d_M(v))$ and since $(4n - 4)F \geq (4n - 4)(B + 1)$ by (†), we have

$$A - B \geq (4n - 4)(B + 1) + 4 = (4n - 4)B + 4n,$$

so that $A \geq (4n - 4)B + 4n + B = (4n - 3)B + 4n$, as required. □

COROLLARY 4.12. *Let r be a rational number with $0 < r < 1$ and let n be an integer with $n \geq 2$. Write $r = [m_1, m_2, \ldots, m_k]$, where $k \geq 1$, $(m_1, \ldots, m_k) \in (\mathbf{Z}_+)^k$ and $m_k \geq 2$, and let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Lemma 3.6. Suppose that v is a cyclically alternating word which represents the trivial element in $G(r; n) = \langle a, b \mid u_r^n \rangle$. Then the cyclic word (v) contains a subword w of the cyclic word $(u_r^{\pm n})$ which is a product of $4n - 1$ pieces but is not a product of less than $4n - 1$ pieces. In particular, the cyclic S-sequence $CS(v)$ of the cyclic word (v) satisfies the following conditions.*

(1) *If $k = 1$, then $CS(v)$ contains $((2n - 2)\langle m_1 \rangle)$ as a subsequence.*

(2) *If $k \geq 2$, then $CS(v)$ contains $((2n - 1)\langle S_1, S_2 \rangle)$ or $((2n - 1)\langle S_2, S_1 \rangle)$ as a subsequence.*

PROOF. By Lemma 4.7, there is a reduced connected and simply-connected diagram M over $G(r; n) = \langle a, b \mid u_r^n \rangle$ with $(\phi(\partial M)) = (v)$. By Corollary 4.9, M is a $[4, 4n]$-map over $G(r; n) = \langle a, b \mid u_r^n \rangle$. Furthermore, since $(\phi(\partial M)) = (v)$ is cyclically alternating, there is no vertex of degree 3 in ∂M.

Then by Proposition 4.11, we have $A \geq (4n-3)B + 4n$, where A and B denote the numbers of vertices v in ∂M such that $d_M(v) = 2$ and $d_M(v) \geq 4$, respectively. This implies that there are at least $4n-2$ consecutive vertices of degree 2 on ∂M. Hence, by Convention 4.8, the cyclic word $(\phi(\partial M)) = (v)$ contains a subword w of the cyclic word $(u_r^{\pm n})$ which is a product of $4n-1$ pieces but is not a product of less than $4n-1$ pieces. By Lemma 4.3(2), we may assume that $S(w) = ((2n-1)\langle S_1, S_2\rangle, \ell)$ or $S(w) = (\ell, (2n-1)\langle S_2, S_1\rangle)$, where $\ell \in \mathbf{Z}_+$. It follows that if $k = 1$, then $CS(v)$ contains $((2n-2)\langle m_1\rangle)$ as a subsequence, while if $k \geq 2$, then $CS(v)$ contains $((2n-1)\langle S_1, S_2\rangle)$ or $((2n-1)\langle S_2, S_1\rangle)$ as a subsequence. □

Remark 4.13. In [11, Theorem 3] (cf. [10, Theorem IV.5.5]), Newman gives a powerful theorem for the word problem for one relator groups with torsion, which implies that if a cyclically reduced word v represents the trivial element in $G(r;n) \cong \langle a, b \,|\, u_r^n\rangle$, then the cyclic word (v) contains a subword of the cyclic word $(u_r^{\pm n})$ of length greater than $(n-1)/n = 1 - 1/n$ times the length of u_r^n. Though the above Corollary 4.12 is applicable only when v is cyclically alternating, it imposes a stronger restriction on (v). In fact, since every piece has length less than a half of the length of u_r (see Lemma 4.2), Corollary 4.12 implies that such a cyclic word (v) contains a subword of the cyclic word $(u_r^{\pm n})$ of length greater than $1 - 1/(2n)$ times the length of u_r^n.

5. Proof of Theorem 2.3

Throughout this section, suppose that r is a rational number with $0 < r < 1$, write $r = [m_1, m_2, \ldots, m_k]$, where $k \geq 1$, $(m_1, \ldots, m_k) \in (\mathbf{Z}_+)^k$ and $m_k \geq 2$, and let n be an integer with $n \geq 2$. Recall that the region, R, bounded by a pair of Farey edges with an endpoint ∞ and a pair of Farey edges with an endpoint r forms a fundamental domain for the action of $\Gamma(r;n)$ on $\mathbf{H}^2$ (see Figure 1). Let $I_1(r;n)$ and $I_2(r;n)$ be the (closed or half-closed) intervals in $\mathbf{R}$ defined as follows:

$$I_1(r;n) = \begin{cases} [0, r_1), \text{ where } r_1 = [m_1, \ldots, m_k, 2n-2], & \text{if } k \text{ is odd,} \\ [0, r_1], \text{ where } r_1 = [m_1, \ldots, m_{k-1}, m_k - 1, 2], & \text{if } k \text{ is even,} \end{cases}$$

$$I_2(r;n) = \begin{cases} [r_2, 1], \text{ where } r_2 = [m_1, \ldots, m_{k-1}, m_k - 1, 2], & \text{if } k \text{ is odd,} \\ (r_2, 1], \text{ where } r_2 = [m_1, \ldots, m_k, 2n-2], & \text{if } k \text{ is even.} \end{cases}$$

Then we may choose a fundamental domain R so that the intersection of $\bar{R}$ with $\partial\mathbf{H}^2$ is equal to the union $\bar{I}_1(r;n) \cup \bar{I}_2(r;n) \cup \{\infty, r\}$.

Proposition 5.1. *Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Lemma 3.6. Then, for any $0 \neq s \in I_1(r;n) \cup I_2(r;n)$, the following hold.*

(1) *If* $k = 1$, *that is,* $r = 1/m = [m]$, *then* $CS(s)$ *does not contain* $((2n-2)\langle m\rangle)$ *as a subsequence.*
(2) *If* $k \geq 2$, *then* $CS(s)$ *does not contain* $((2n-1)\langle S_1, S_2\rangle)$ *nor* $((2n-1)\langle S_2, S_1\rangle)$ *as a subsequence.*

In the above proposition, we mean by a *subsequence* a subsequence without leap. Namely a sequence $(a_1, a_2, \ldots, a_p)$ is called a *subsequence* of a cyclic sequence, if there is a sequence $(b_1, b_2, \ldots, b_t)$ representing the cyclic sequence such that $p \leq t$ and $a_i = b_i$ for $1 \leq i \leq p$.

PROOF. (1) Suppose that $r = 1/m = [m]$. Then any rational number $0 \neq s \in I_1(r;n) \cup I_2(r;n) = [0, r_1) \cup [r_2, 1]$, where $r_1 = (2n-2)/((2n-2)m+1) = [m, 2n-2]$ and $r_2 = 2/(2m-1) = [m-1, 2]$, has a continued fraction expansion $s = [l_1, \ldots, l_t]$, where $t \geq 1$, $(l_1, \ldots, l_t) \in (\mathbf{Z}_+)^t$ and $l_t \geq 2$ unless $t = 1$, such that
(i) $t \geq 1$ and $1 \leq l_1 \leq m-2$;
(ii) $t = 1$ and $l_1 = m-1$;
(iii) $t \geq 2$, $l_1 = m-1$ and $l_2 \geq 2$;
(iv) $t \geq 3$, $l_1 = m$ and $l_2 = 1$;
(v) $t \geq 2$, $l_1 = m$ and $2 \leq l_2 \leq 2n-3$; or
(vi) $t \geq 1$ and $l_1 \geq m+1$.
If (i) happens, then $s = [l_1, l_2, \ldots, l_t]$ with $1 \leq l_1 \leq m-2$, so each component of $CS(s)$ is equal to $l_1 \leq m-2$ or $l_1 + 1 \leq m-1$ by Lemma 3.3. Hence the assertion holds. If (ii) happens, then $s = [m-1]$, so $CS(s) = (\!(m-1, m-1)\!)$. Hence the assertion holds. If (iii) happens, then $s = [m-1, l_2, \ldots, l_t]$ with $l_2 \geq 2$, so $CS(s)$ consists of $m-1$ and m but it does not have (m, m) as a subsequence by Lemma 3.3. Hence the assertion holds. If (iv) happens, then $s = [m, 1, l_3, \ldots, l_t]$, so $CS(s)$ consists of m and $m+1$ but it does not have (m, m) as a subsequence by Lemma 3.3. Hence the assertion holds. If (v) happens, then $s = [m, l_2, \ldots, l_t]$ with $2 \leq l_2 \leq 2n-3$, so $CS(s)$ consists of m and $m+1$ by Lemma 3.3. Also by Lemma 3.5, $\tilde{s} = [l_2 - 1, l_3, \ldots, l_t]$ and $CS(\tilde{s}) = CT(s)$. Again by Lemma 3.3, each component of $CS(\tilde{s}) = CT(s)$ is equal to $l_2 - 1 \leq 2n-4$ or $l_2 \leq 2n-3$. This implies by Definition 3.4 that $CS(s)$ contains at most $((2n-3)\langle m\rangle)$ as a subsequence, as required. Finally, if (vi) happens, then $s = [l_1, l_2, \ldots, l_t]$ with $l_1 \geq m+1$, so each component of $CS(s)$ is equal to $l_1 \geq m+1$ or $l_1 + 1 \geq m+2$ by Lemma 3.3. Hence the assertion holds.

(2) The proof proceeds by induction on $k \geq 2$. For simplicity, we write m for m_1. By Lemma 3.6, S_1 begins and ends with $m+1$, and S_2 begins and ends with m. Suppose on the contrary that there exists some $0 \neq s \in I_1(r;n) \cup I_2(r;n)$ for which $CS(s)$ contains $((2n-1)\langle S_1, S_2\rangle)$ or $((2n-1)\langle S_2, S_1\rangle)$ as a subsequence. This implies by Lemma 3.3 that $CS(s)$ consists of m and $m+1$.

So s has a continued fraction expansion $s = [l_1, \dots, l_t]$, where $t \geq 2$, $(l_1, \dots, l_t) \in (\mathbf{Z}_+)^t$, $l_1 = m$ and $l_t \geq 2$. For the rational numbers r and s, define the rational numbers $\tilde{r}$ and $\tilde{s}$ as in Lemma 3.5 so that $CS(\tilde{r}) = CT(r)$ and $CS(\tilde{s}) = CT(s)$.

We consider three cases separately.

Case 1. $m_2 = 1$.

In this case, $k \geq 3$ and, by Corollary 3.8(1), $(m+1, m+1)$ appears in S_1 as a subsequence, so in $CS(s)$ as a subsequence. Thus by Lemma 3.3, $l_2 = 1$ and so $t \geq 3$. So, we have

$$\tilde{r} = [m_3, \dots, m_k] \qquad \text{and} \qquad \tilde{s} = [l_3, \dots, l_t].$$

It follows from $0 \neq s \in I_1(r;n) \cup I_2(r;n)$ that $0 \neq \tilde{s} \in I_1(\tilde{r};n) \cup I_2(\tilde{r};n)$. At this point, we divide this case into two subcases.

Case 1.a. $k = 3$.

By Lemma 3.7(1), $S_1 = (m_3\langle m+1\rangle)$ and $S_2 = (m)$. Since $((2n-1)\langle S_1, S_2\rangle)$ or $((2n-1)\langle S_2, S_1\rangle)$ is contained in $CS(s)$ by assumption, $(S_2, (2n-2)\langle S_1, S_2\rangle)$ is contained in $CS(s)$. This implies that $CS(\tilde{s}) = CT(s)$ contains $((2n-2)\langle m_3\rangle)$ as a subsequence. But since $\tilde{r} = 1/m_3 = [m_3]$ and $0 \neq \tilde{s} \in I_1(\tilde{r};n) \cup I_2(\tilde{r};n)$, this gives a contradiction to (1).

Case 1.b. $k \geq 4$.

Let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ be the decomposition of $S(\tilde{r})$ given by Lemma 3.6. Since S_1 begins and ends with $m+1$, S_2 begins and ends with m, and since $((2n-1)\langle S_1, S_2\rangle)$ or $((2n-1)\langle S_2, S_1\rangle)$ is contained in $CS(s)$ by assumption, we see by Lemma 3.7(2) that $CS(\tilde{s}) = CT(s)$ contains, as a subsequence,

$$(t_1 + \ell', t_2, \dots, t_{s_1 - 1}, t_{s_1}, T_2, (2n-2)\langle T_1, T_2\rangle), \qquad \text{or}$$

$$((2n-2)\langle T_2, T_1\rangle, T_2, t_1, t_2, \dots, t_{s_1 - 1}, t_{s_1} + \ell''),$$

where $(t_1, t_2, \dots, t_{s_1}) = T_1$ and $\ell', \ell'' \in \mathbf{Z}_+ \cup \{0\}$. (Note that $((2n-1)\langle S_1, S_2\rangle)$ begins with $m+1$ and ends with m, whereas $((2n-1)\langle S_2, S_1\rangle)$ begins with m and ends with $m+1$.) Since $t_1 = t_{s_1} = m_3 + 1$ by Lemma 3.6, this actually implies that $\ell' = 0$ or $\ell'' = 0$ accordingly, and therefore $CS(\tilde{s})$ contains $((2n-1)\langle T_1, T_2\rangle)$ or $((2n-1)\langle T_2, T_1\rangle)$ as a subsequence. But since $\tilde{r} = [m_3, \dots, m_k]$ and $0 \neq \tilde{s} \in I_1(\tilde{r};n) \cup I_2(\tilde{r};n)$, this gives a contradiction to the induction hypothesis.

Case 2. $k = 2$ *and* $m_2 = 2$.

In this case, $r = [m, 2]$, so by Lemma 3.7(3), $S_1 = (m+1)$ and $S_2 = (m)$. Since $((2n-1)\langle S_1, S_2\rangle)$ or $((2n-1)\langle S_2, S_1\rangle)$ is contained in $CS(s)$ by assumption, both $(m+1,(2n-2)\langle m,m+1\rangle)$ and $((2n-2)\langle m,m+1\rangle,m)$ are contained in $CS(s)$. This implies that $CS(\tilde{s})=CT(s)$ contains $((2n-2)\langle 1\rangle)$ as a subsequence. Moreover, we can see that this subsequence is proper, i.e., it is not equal to the whole cyclic sequence $CS(\tilde{s})=CT(s)$. As described below, this in turn implies that s has the form either $s=[m,1,1,l_4\dots,l_t]$ or $s=[m,2,l_3,\dots,l_t]$ with $l_3\geq 2n-2$. If $l_2=1$, then $\tilde{s}=[l_3,\dots,l_t]$ and so l_3 is the minimal component of $CS(\tilde{s})$ (see Lemma 3.3). Hence we must have $l_3=1$, i.e., $s=[m,1,1,l_4\dots,l_t]$, because $CS(\tilde{s})$ contains 1 as a component. On the other hand, if $l_2\geq 2$, then $\tilde{s}=[l_2-1,\dots,l_t]$ and so l_2-1 is the minimal component of $CS(\tilde{s})$ (see Lemma 3.3). Since $CS(\tilde{s})$ contains 1 as a component, we have $l_2-1=1$, i.e., $l_2=2$. Since $CS(\tilde{s})$ contains $((2n-2)\langle 1\rangle)$ as a subsequence, we see that $CS(\tilde{\tilde{s}})=CT(\tilde{s})$ contains a component $\geq 2n-2$. Since the subsequence $((2n-2)\langle 1\rangle)$ of $CS(\tilde{s})$ is proper, we see $t\geq 3$ and $l_3\geq 2$. Thus $\tilde{\tilde{s}}=[l_3-1,\dots,l_t]$ and therefore l_3-1 is the minimal component of $CS(\tilde{\tilde{s}})$. Hence we must have $l_3=(l_3-1)+1\geq 2n-2$ and so $s=[m,2,l_3,\dots,l_t]$ with $l_3\geq 2n-2$.

But then s cannot belong to the interval $I_1(r;n)\cup I_2(r;n)=[0,r_1]\cup(r_2,1]$, where $r_1=[m,1,2]$ and $r_2=[m,2,2n-2]$, a contradiction to the hypothesis.

Case 3. *Either both $k=2$ and $m_2\geq 3$ or both $k\geq 3$ and $m_2\geq 2$.*

In this case, by Corollary 3.8(2), (m,m) appears in S_2 as a subsequence, so in $CS(s)$ as a subsequence. Thus $l_2\geq 2$ by Lemma 3.3, and so we have

$$\tilde{r}=[m_2-1,m_3,\dots,m_k] \qquad \text{and} \qquad \tilde{s}=[l_2-1,l_3,\dots,l_t].$$

It follows from $0\neq s\in I_1(r;n)\cup I_2(r;n)$ that $0\neq\tilde{s}\in I_1(\tilde{r};n)\cup I_2(\tilde{r};n)$. At this point, we consider two subcases separately.

Case 3.a. *$k=2$ and $m_2\geq 3$.*

By Lemma 3.7(3), $S_1=(m+1)$ and $S_2=((m_2-1)\langle m\rangle)$. Since $((2n-1)\langle S_1,S_2\rangle)$ or $((2n-1)\langle S_2,S_1\rangle)$ is contained in $CS(s)$ by assumption, $(S_1,(2n-2)\langle S_2,S_1\rangle)$ is contained in $CS(s)$. This implies that $CS(\tilde{s})=CT(s)$ contains $((2n-2)\langle m_2-1\rangle)$ as a subsequence. But since $\tilde{r}=1/(m_2-1)=[m_2-1]$ and $0\neq\tilde{s}\in I_1(\tilde{r};n)\cup I_2(\tilde{r};n)$, this gives a contradiction to (1).

Case 3.b. *$k\geq 3$ and $m_2\geq 2$.*

Let $S(\tilde{r})=(T_1,T_2,T_1,T_2)$ be the decomposition of $S(\tilde{r})$ given by Lemma 3.6. Since S_1 begins and ends with $m+1$, S_2 begins and ends with m, and since $((2n-1)\langle S_1,S_2\rangle)$ or $((2n-1)\langle S_2,S_1\rangle)$ is contained in $CS(s)$ by assumption, we see by Lemma 3.7(4) that $CS(\tilde{s})=CT(s)$ contains, as a subsequence,

$$((2n-2)\langle T_2, T_1\rangle, T_2, t_1, t_2, \dots, t_{s_1-1}, t_{s_1} + \ell'), \qquad \text{or}$$

$$(t_1 + \ell'', t_2, \dots, t_{s_1-1}, t_{s_1}, T_2, (2n-2)\langle T_1, T_2\rangle),$$

where $(t_1, t_2, \dots, t_{s_1}) = T_1$ and $\ell', \ell'' \in \mathbf{Z}_+ \cup \{0\}$. Since $t_1 = t_{s_1} = (m_2 - 1) + 1 = m_2$ by Lemma 3.6, this actually implies that $\ell' = 0$ or $\ell'' = 0$ accordingly, and therefore $CS(\tilde{s})$ contains $((2n-1)\langle T_1, T_2\rangle)$ or $((2n-1)\langle T_2, T_1\rangle)$ as a subsequence. But since $\tilde{r} = [m_2 - 1, m_3, \dots, m_k]$ and $0 \neq \tilde{s} \in I_1(\tilde{r}; n) \cup I_2(\tilde{r}; n)$, this gives a contradiction to the induction hypothesis.

The proof of Proposition 5.1 is completed. □

We are now in a position to prove Theorem 2.3.

Proof of Theorem 2.3. Suppose on the contrary that there exists a rational number $s \in I(r; n) \cup \{r\} = I_1(r; n) \cup I_2(r; n) \cup \{r\}$ for which α_s is null-homotopic in $\boldsymbol{S}(r; n)$. Then u_s equals the identity in $G(r; n)$. Since u_r is a non-trivial torsion element in $G(r; n) = \langle a, b \mid u_r^n\rangle$ by [10, Theorem IV.5.2], we may assume $s \in I_1(r; n) \cup I_2(r; n)$. By Corollary 4.12, the cyclic word (u_s) contains a subword w of the cyclic word $(u_r^{\pm n})$ which is a product of $4n-1$ pieces but is not a product of less than $4n-1$ pieces. Since $4n-1 \geq 7$, the length of such a subword w is greater or equal to 7. So s cannot be zero, because the word $u_0 = ab$ cannot contain such a subword w. By Corollary 4.12 again, if $r = 1/m$, then $CS(u_s) = CS(s)$ contains $((2n-2)\langle m\rangle)$ as a subsequence, while if $r \neq 1/m$, then $CS(s)$ contains $((2n-1)\langle S_1, S_2\rangle)$ or $((2n-1)\langle S_2, S_1\rangle)$ as a subsequence, where $S(r) = (S_1, S_2, S_1, S_2)$ is as in Lemma 3.6. This contradicts Proposition 5.1. □

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Donghi Lee
Department of Mathematics
Pusan National University
San-30 *Jangjeon-Dong, Geumjung-Gu, Pusan*
609-735, *Republic of Korea*
E-mail: donghi@pusan.ac.kr

Makoto Sakuma
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima, 739-8526, *Japan*
E-mail: sakuma@math.sci.hiroshima-u.ac.jp