# Weierstrass equations for Jacobian fibrations on a certain $K 3$ surface 

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#### Abstract

In this paper we give the Weierstrass equations for Jacobian fibrations on the $K 3$ surface that is the minimal resolution of the double covering of $\mathbf{P}^{2}$ ramified along generic six lines.


## 1. Introduction

1.1. Problem setting. Let $X$ be a $K 3$ surface defined over an algebraically closed field $k$ with $\operatorname{char}(k) \neq 2,3$. Suppose $f: X \rightarrow \mathbf{P}^{1}$ is a Jacobian fibration, that is, an elliptic fibration on $X$ with a section $O: \mathbf{P}^{1} \rightarrow X$. Let $t$ be an affine coordinate of $\mathbf{P}^{1}$. Then $f^{*}(t)$ defines a non-constant rational function on $X$, which is called an elliptic parameter for the Jacobian fibration $f$. We also denote $f^{*}(t)$ by $t$ and regard $t$ as a rational function on $X$. The generic fiber of $f$ defines an elliptic curve $E$ over the rational function field $k(t)$.

Kuwata and Shioda [7] proposed the following problems.
Problem 1. Given a K3 surface $X / k$ and a Jacobian fibration $f$, determine (i) the elliptic parameter $t$ for $f$, (ii) the defining equation of the elliptic curve $E / k(t)$, and (iii) the Mordell-Weil lattice (MWL) $E(k(t))$.

Problem 2. Given a K3 surface $X / k$, determine all the (essentially distinct) elliptic parameters.

Problem 2 is a combination of Problem 1 and the following standard problem:

Problem 3. Given a K3 surface $X / k$, classify the Jacobian fibration $f: X \rightarrow \mathbf{P}^{1}$ up to isomorphism.

Oguiso [8] solved Problem 3 in the case where $X$ is a Kummer surface of the product of non-isogenous elliptic curves over the complex number field. Namely he classified the configuration of singular fibers on such a Kummer

[^0]surface $X$ into eleven types $\mathscr{J}_{1}, \ldots, \mathscr{I}_{11}$, and determined the number of the isomorphism classes for each type.

Kuwata and Shioda [7] solved Problems 1 and 2 for each member of Oguiso's list. They gave elliptic parameters and Weierstrass equations for each member of Oguiso's list by using the Legendre parameters of the two elliptic curves.

Recently, Kumar [5] solved the above problems completely for the case of a generic Jacobian Kummer surface $X=\mathrm{Km}(J(C))$, for a genus 2 curve $C$ over an algebraically closed field of characteristic 0 . He showed that there are exactly 25 distinct Jacobian fibrations on such a generic Kummer surface, and gave an elliptic parameter, the Weierstrass equation and the Mordell-Weil lattice for each type.
1.2. Main results. In this paper, we focus on the case of a $K 3$ surface over $k$ that is the minimal resolution of the double covering of $\mathbf{P}^{2}$ ramified along generic six lines. When we say that six lines are generic, we mean that the rank of the Néron-Severi group $N S(X)$ is 16 .

In this case, Problem 3 has been partially solved by Kloosterman [3]. He classified all configurations of singular fibers of Jacobian fibrations on such a $K 3$ surface over a field of characteristic 0 (although most of his results hold over fields of any characteristic, not 2 or 3 ) into sixteen classes. However he did not give the classification of the isomorphism classes for each type. Thus, this gives a partial solution to Problem 3 on such a $K 3$ surface.

Table 1 shows a summary of Kloosterman's results in the generic case. The first column shows the class of Jacobian fibration following Kloosterman's notation. The second column shows the configuration of singular fibers. Here, for example, by $\mathbf{I}_{2}^{*}+8 \mathbf{I}_{2}$ we mean that the surface has two fibers of type $\mathrm{I}_{2}^{*}$ (Kodaira's notation [4]) and eight fibers of type $\mathrm{I}_{2}$. The third column shows the Mordell-Weil group (MWG) of the fibration.

Our main results are as follows: we solve (i) and (ii) of Problem 1 for each class of Table 1. We give an elliptic parameter and its Weierstrass equation for one Jacobian fibration in each class of the table, although there may exist nonisomorphic Jacobian fibrations belonging to the same class of the list. More details will be given in $\S 1.3$ after we fix the notation.
1.3. Notation. Fix generic six lines $L_{i} \subset \mathbf{P}^{2}$. Denote by $P_{i, j}$ the point of intersection of $L_{i}$ and $L_{j}$.

Let $\varphi^{\prime}: Y \rightarrow \mathbf{P}^{2}$ be the double cover ramified along the six lines $L_{i}$. Then $Y$ has 15 double points of type $A_{1}$, which correspond to $P_{i, j}$. Blowing up these points gives a $K 3$ surface $X$, with 15 exceptional divisors $\ell_{i, j}$ and a rational map $\varphi: X \rightarrow \mathbf{P}^{2}$. For a curve $C$ on $\mathbf{P}^{2}$, we call the strict transform of $\varphi^{\prime *}(C)$ the pull-back of $C$, for short. Let $\ell_{i}$ be the divisor on $X$ such that $2 \ell_{i}$

Table 1. List of possible configurations

| Class | Configuration of singular fibers | MWG |
| :---: | :---: | :---: |
| 1.1 | $\mathrm{I}_{10}+\mathrm{I}_{2}+a \mathrm{II}+b \mathrm{I}_{1}$ | $\mathbf{Z}^{4}$ |
| 1.2 | $\mathrm{I}_{8}+\mathrm{I}_{4}+a \mathrm{II}+b \mathrm{I}_{1}$ | $\mathbf{Z}^{4}$ |
| 1.3 | $2 \mathrm{I}_{6}+\mathrm{I}_{4}+a \mathrm{II}+b \mathbf{I}_{1}$ | $\mathbf{Z}^{4}$ |
| 1.4 | $\mathrm{IV}^{*}+\mathrm{I}_{4}+a \mathrm{II}+b \mathrm{I}_{1}$ | $\mathbf{Z}^{5}$ |
| 2.1 | $\mathrm{II}^{*}+6 \mathrm{I}_{2}+2 \mathrm{I}_{1}$ | $\{0\}$ |
| 2.2 | $\mathrm{III}^{*}+7 \mathrm{I}_{2}+\mathrm{I}_{1}$ | $\mathbf{Z} / 2 \mathbf{Z}$ |
| 2.3 | $\mathrm{III}^{*}+\mathrm{I}_{0}^{*}+3 \mathrm{I}_{2}+3 \mathrm{I}_{1}$ | $\{0\}$ |
| 2.4 | $\mathrm{I}_{6}^{*}+4 \mathrm{I}_{2}+4 \mathrm{I}_{1}$ | $\{0\}$ |
| 2.5 | $\mathrm{I}_{4}^{*}+6 \mathrm{I}_{2}+2 \mathrm{I}_{1}$ | $\mathbf{Z} / 2 \mathbf{Z}$ |
| 2.6 | $\mathrm{I}_{4}^{*}+\mathrm{I}_{0}^{*}+2 \mathrm{I}_{2}+4 \mathrm{I}_{1}$ | $\{0\}$ |
| 2.7 | $\mathrm{I}_{2}^{*}+8 \mathrm{I}_{2}$ | $(\mathbf{Z} / 2 \mathbf{Z})^{2}$ |
| 2.8 | $\mathrm{I}_{2}^{*}+\mathrm{I}_{0}^{*}+4 \mathrm{I}_{2}$ | $\mathbf{Z} / 2 \mathbf{Z}$ |
| 2.9 | $2 \mathrm{I}_{2}^{*}+2 \mathrm{I}_{2}+4 \mathrm{I}_{1}$ | $\{0\}$ |
| 2.10 | $\mathrm{I}_{2}^{*}+2 \mathrm{I}_{0}^{*}+8 \mathbf{I}_{1}$ | $\{0\}$ |
| 2.11 | $2 \mathrm{I}_{0}^{*}+6 \mathrm{I}_{2}$ | $(\mathbf{Z} / 2 \mathbf{Z})^{2}$ |
| 2.12 | $3 \mathrm{I}_{0}^{*}+2 \mathrm{I}_{2}+2 \mathrm{I}_{1}$ | $\mathbf{Z} / 2 \mathbf{Z}$ |

is the pull-back of $L_{i}$. Let $\mu_{k, m}^{i, j}$ be the pull-back of the line $M_{k, m}^{i, j}$ connecting $P_{i, j}$ and $P_{k, m}$ with $i, j, k, m$ pairwise distinct. With this notation, which is the same as Kloosterman [3], divisors $\ell_{i}, \ell_{i, j}, \mu_{k, m}^{i, j}$ are ( -2 )-curves on $X$. We have the following intersection numbers.

$$
\begin{align*}
& \ell_{i} \cdot \ell_{j}=\left\{\begin{array}{ll}
-2 & i=j \\
0 & i \neq j
\end{array}, \quad \ell_{i, j} \cdot \ell_{k, m}= \begin{cases}-2 & \{i, j\}=\{k, m\} \\
0 & \text { otherwise }\end{cases} \right. \\
& \ell_{i} \cdot \ell_{k, m}=\left\{\begin{array}{ll}
1 & i \in\{k, m\} \\
0 & \text { otherwise }
\end{array}, \quad \ell_{p} \cdot \mu_{k, m}^{i, j}= \begin{cases}1 & p \notin\{i, j, k, m\} \\
0 & \text { otherwise }\end{cases} \right.
\end{align*},
$$

Moreover, for some rational plane curves $C$, we name the pull-back of $C$ as in the following table. Note that all of them are $(-2)$-curves on $X$.

| Divisor | C |
| :---: | :---: |
| $\eta_{\left(i_{4} j_{4}\right)\left(i_{5} j_{5}\right)}^{\left(i_{1} j_{1}\right)\left(i_{2} j_{2}\right)\left(i_{3} j_{3}\right)}$ | the conic curve through $P_{i_{1}, j_{1}}, \ldots, P_{i_{5}, j_{5}}$ |
| $\left.\left.\xi_{\left(i_{j} j_{5}\right)\left(i_{i} j_{j}\right)}^{\left(\bar{i}_{2}\right)\left(i_{i}, j_{j}\right)}\right)\right)$ | the cubic curve through $P_{i_{1}, j_{1}}, \ldots, P_{i_{7}, j_{7}}$ with a double point at $P_{i_{1}, j_{1}}$ |
|  | the quartic curve through $P_{i_{1}, j_{1}}, \ldots, P_{i_{8}, j_{8}}$ with a double point at $P_{i_{1}, j_{1}}, P_{i_{2}, j_{2}}, P_{i_{3}, j_{3}}$ |
|  | the quintic curve through $P_{i_{1}, j_{1}}, \ldots, P_{i_{9}, j_{9}}$ with a triple point at $P_{i_{1}, j_{1}}$ and a double point at $P_{i_{2}, j_{2}}, P_{i_{3}, j_{3}}, P_{i_{4}, j_{4}}$ |

We may suppose that the lines $L_{i}$ are defined by the following equations

$$
\begin{align*}
& L_{1}: u=0, \quad L_{2}: u=1, \quad L_{3}: v=0, \quad L_{4}: v=1 \\
& L_{5}: a u+b v-1=0, \quad L_{6}: c u+d v-1=0, \tag{1.2}
\end{align*}
$$

where $u, v$ are the affine parameters of projective plane. We consider the six lines in the generic case, that is, the Néron-Severi group $N S(X)$ is 16 .

Then the singular affine model of $X$ is given by

$$
\begin{equation*}
w^{2}=u(u-1) v(v-1)(a u+b v-1)(c u+d v-1) . \tag{1.3}
\end{equation*}
$$



Fig. 1. six lines on projective plane

Under the above notation, we see that the divisors of typical functions are as follows.

$$
\begin{align*}
& (u)=2 \ell_{1}+\ell_{1,3}+\ell_{1,4}+\ell_{1,5}+\ell_{1,6}-\left(\mu_{3,4}^{1,2}+\ell_{3,4}\right) \\
& (u-1)=2 \ell_{2}+\ell_{2,3}+\ell_{2,4}+\ell_{2,5}+\ell_{2,6}-\left(\mu_{3,4}^{1,2}+\ell_{3,4}\right) \\
& (v)=2 \ell_{3}+\ell_{1,3}+\ell_{2,3}+\ell_{3,5}+\ell_{3,6}-\left(\mu_{3,4}^{1,2}+\ell_{1,2}\right) \\
& (v-1)=2 \ell_{4}+\ell_{1,4}+\ell_{2,4}+\ell_{4,5}+\ell_{4,6}-\left(\mu_{3,4}^{1,2}+\ell_{1,2}\right) \\
& (a u+b v-1)= \\
& \begin{aligned}
& 2 \ell_{5}+\ell_{1,5}+\ell_{2,5}+\ell_{3,5}+\ell_{4,5}+\ell_{5,6} \\
& \quad-\left(\mu_{3,4}^{1,2}+\ell_{1,2}+\ell_{3,4}\right)
\end{aligned}  \tag{1.4}\\
& \begin{array}{r}
(c u+d v-1)= \\
\quad \\
\left.\quad 2 \ell_{6}+\ell_{1,6}+\ell_{2,6}+\ell_{3,6}+\ell_{4,6}+\ell_{5,6}^{1,2}+\ell_{1,2}+\ell_{3,4}\right)
\end{array} \\
& (w)=\sum_{i=1}^{6} \ell_{i}+\sum_{1 \leq i<j \leq 6} \ell_{i, j}-3\left(\mu_{3,4}^{1,2}+\ell_{1,2}+\ell_{3,4}\right) \\
& \text { (the equation of } \left.M_{k, m}^{i, j}\right)=\mu_{k, m}^{i, j}+\ell_{i, j}+\ell_{k, m}-\left(\mu_{3,4}^{1,2}+\ell_{1,2}+\ell_{3,4}\right) .
\end{align*}
$$

For each class of Table 1, we compute a Weierstrass equation for one Jacobian fibration belonging to the class. Theoretically, constructing a Jacobian fibration on a $K 3$ surface is to find a divisor that has the same type as a singular fiber in the Kodaira's list (see [4]). In practice, however, we need to find two divisors, one for the fiber at $t=0$, and the other for the fiber at $t=\infty$, to write down an actual elliptic parameter. Once an elliptic parameter is found, we would like to find a change of variables that converts to the defining equation to a Weierstrass form. In most cases, we encounter an equation of the form $y^{2}=$ (quartic polynomial). Then we can transform it to a Weierstrass form by using a standard algorithm (see for example [2] or [1]).

In our case, we use two methods to convert defining equation (1.3) to the form $y^{2}=$ (quartic polynomial). The first method is an elimination. Since an elliptic parameter $t$ is a rational map, we can put $t=f / g$ for some $f, g \in$ $k[u, v, w]$. Thus, we can eliminate one variable from (1.3) and the equation $g t-f=0$. If such an equation can be converted to $y^{2}=$ (quartic polynomial) by a simple coordinate change, we can get a Weierstrass equation. We will call this method classical method in this paper.

The other method is a 2-neighbor step, which is designed by Noam Elkies. This is the technique to transform a Weierstrass equation of a Jacobian
fibration to a Weierstrass equation of a distinct Jacobian fibration. Using this, we can get an unknown Weierstrass equation of a class from a known class. We describe a 2 -neighbor step in $\S 3$.
1.4. Results. We state our main theorem.

Theorem 1. Let $X$ be a $K 3$ surface over an algebraically closed field $k$ with $\operatorname{char}(k) \neq 2,3$ that is the minimal resolution of the double covering of $\mathbf{P}^{2}$ ramified along generic six lines. Suppose that the rank of the Néron-Severi group $N S(X)$ is 16. Under the singular affine model (1.3) of $X$, for each class in Table 1, an elliptic parameter and a Weierstrass equation of a Jacobian fibration belonging to the class is given by Table 2,..., Table 17.

In the case of using a classical method, we give an elliptic parameter $t$ in terms of $u, v, w$, a Weierstrass equation and a picture of the configuration of singular fibers. Moreover, for the class $2 . x x$, we also give the correspondence between the divisors and the torsion sections.

In the case of using a 2 -neighbor step from another class (in fact, we only use a 2 -neighbor step from the classes 2.7 or 2.5 ), we give an elliptic parameter $s$ in terms of $t, x, y$ used in the equation of the source class, a picture that shows the way to construct of the divisor corresponding to the fiber at $s=\infty$ and a picture of the configuration of singular fibers. In this case, however, we omit Weierstrass equations, since they are all too long to print in this paper.

We explain the detail of some computation. In $\S 2$, we will give an elliptic parameter and a Weierstrass equation of a Jacobian fibration of the class 2.7 by a classical method. In §3, we explain a 2-neighbor step. In §4, we will give an elliptic parameter and a Weierstrass equation of the class 2.10 by a 2 -neighbor step from the class 2.7. In $\S 5$, we use a 2-neighbor step from the class 2.5 for the computation of the class 2.4 .

## 2. Class 2.7

An elliptic parameter for the class 2.7 is given by

$$
\begin{equation*}
t=\frac{u v}{c u+b v-1} . \tag{2.1}
\end{equation*}
$$

It is easy to verify that the divisor of $t$ is given by

$$
\begin{equation*}
(t)=\ell_{1,6}+\ell_{1,4}+2\left(\ell_{1}+\ell_{1,3}+\ell_{3}\right)+\ell_{2,3}+\ell_{3,5}-\left(\mu_{3,6}^{1,5}+\mu_{3,4}^{1,2}\right) . \tag{2.2}
\end{equation*}
$$

Then the fiber at $t=0$ is of type $\mathrm{I}_{2}^{*}$ and $t=\infty$ fiber is of type $\mathrm{I}_{2}$.


Table 2. Class 1.1
Eliminating the variable $v$ from (1.3) and (2.1), and making a simple coordinate change, we obtain an equation of the form $y^{2}=$ (quartic polynomial). Choosing $\ell_{2}$ as the zero section of the group structure, we have the Weierstrass equation for the Jacobian fibration

$$
\begin{align*}
y^{2}= & (x-t(b t+c t-t-1)(a b t-b c t-a+1)) \\
& (x-a t(b t-d t-1)(b t+c t-t-1)) \\
& (x-t(1-c t)(b t-d t-1)(a b t-b c t-a+1)), \tag{2.3}
\end{align*}
$$

| Method | Class 1.2 |
| :---: | :---: |
| Elliptic parameter | $t=\frac{1}{(a u+b v-1)(c u+d v-1)}$ |

Table 3. Class 1.2
where the change of variables is given by

$$
\begin{align*}
& x=\frac{t(b t-d t-1)(b t+c t-t-1)(a b t-b c t-a+1)}{u-1} \\
& y=\frac{t(b t-d t-1)(b t+c t-t-1)(a b t-b c t-a+1)(u-b t)^{2} w}{u(c u-1)(u-1)^{2}} . \tag{2.4}
\end{align*}
$$

The remaining of divisors $\ell_{4}, \ell_{5}$ and $\ell_{6}$ are 2 -torsion sections. The correspondence between the divisors and the sections are as follows.

$$
\begin{align*}
& \ell_{4} \leftrightarrow(x, y)=(t(1-c t)(a b t-b c t-a+1)(b t-d t-1), 0) \\
& \ell_{5} \leftrightarrow(x, y)=(a t(b t-d t-1)(b t+c t-t-1), 0)  \tag{2.5}\\
& \ell_{6} \leftrightarrow(x, y)=(t(a b t-b c t-a+1)(b t+c t-t-1), 0)
\end{align*}
$$



Table 4. Class 1.3

We can check types and positions of other singular fibers by Tate algorithm [10]. For example, the fiber at $t=\frac{1}{b c}$ is a singular fiber of type $\mathrm{I}_{2}$. Since we obtain

$$
\begin{equation*}
\left(t-\frac{1}{b c}\right)=\left(\frac{(c u-1)(b v-1)}{(c u+b c-1) c b}\right)=\mu_{3,6}^{1,2}+\mu_{3,4}^{1,5}-\left(\mu_{3,4}^{1,2}+\mu_{3,6}^{1,5}\right) \tag{2.6}
\end{equation*}
$$

by (2.1) and (1.4), we see that the fiber at $t=\frac{1}{b c}$ consists of $\mu_{3,6}^{1,2}$ and $\mu_{3,4}^{1,5}$.
Similarly, we can determine the other singular fibers. In the following Figure 2, we show the complete configuration of singular fibers for the class 2.7.

## 3. 2-neighbor step

In this section, we explain "2-neighbor step". The following description is based on [5].

Let $X$ be a $K 3$ surface over a field $k$ with an Jacobian fibration over $\mathbf{P}^{1}$ with a zero section $O$, which defines an elliptic curve $E$ over $k(t)$. Let $P, Q$ be other sections. Let $F$ be the class of a fiber. We call $F$ an elliptic divisor of this fibration. Then for an effective divisor $F^{\prime}=m O+n P+k Q+G$, where $G$ is an effective vertical divisor, we would like to compute the global sections of $\mathcal{O}_{X}\left(F^{\prime}\right)$. Denote the space of the global sections of $\mathcal{O}_{X}\left(F^{\prime}\right)$ by $L_{X}\left(F^{\prime}\right)$. Let


Fig. 2. configuration of singular fibers for the class 2.7
$\left\{s_{1}, \ldots, s_{r}\right\}$ be a basis of $L_{E}(m O+n P+k Q)$. Then for any $f \in L_{X}\left(F^{\prime}\right)$, there exist $b_{i}(t) \in k(t)$ such that

$$
\begin{equation*}
f=b_{1}(t) s_{1}+\cdots+b_{r}(t) s_{r} \tag{3.1}
\end{equation*}
$$

We suppose that there exists an elliptic divisor $F^{\prime}$ on $X$ with $F^{\prime} \cdot F=2$. Then decomposing $F^{\prime}$ into horizontal and vertical components $F^{\prime}=F_{h}^{\prime}+F_{v}^{\prime}$, we see that $F_{h}^{\prime} \cdot F+F_{v}^{\prime} \cdot F=F^{\prime} \cdot F=2$. Since $F_{v}^{\prime} \cdot F=0$, we have $F_{h}^{\prime} \cdot F=2$. Therefore the possibilities are $F_{h}^{\prime}=2 P$ or $F_{h}^{\prime}=P+Q$ for sections $P, Q$. By a translation, we can take the first one to be $2 O$, and the second to be $O+T$ with a 2-torsion section $T$ or $O+P$ with a non 2-torsion section $P$ depending on whether the class of the section $[P-Q]$ is 2-torsion or not.

We suppose that a Weierstrass equation of $E / k(t)$ is given by

$$
\begin{equation*}
y^{2}=x^{3}+a_{2}(t) x^{2}+a_{4}(t) x+a_{6}(t) \tag{3.2}
\end{equation*}
$$

with $a_{i}(t) \in k[t]$ of degree at most $2 i$.
First, we consider the case $D=2 O$. Then 1 and $x$ form a basis for $L_{E}(D)$. Therefore, for a new elliptic divisor $F^{\prime}=2 O+G$ with $G$ effective and vertical, we obtain two elements 1 and $A(t)+B(t) x$ of $L_{X}\left(F^{\prime}\right)$ for some fixed $A(t), B(t) \in k(t)$. The ratio of the two global sections gives the new elliptic parameter $s=A(t)+B(t) x$. Therefore we set

$$
\begin{equation*}
x=\frac{s-A(t)}{B(t)} \tag{3.3}
\end{equation*}
$$



Table 5. Class 1.4
and substitute this into the Weierstrass equation, to get the form of

$$
\begin{equation*}
y^{2}=g(t, s) . \tag{3.4}
\end{equation*}
$$

Since the generic fiber of the fibration over $\mathbf{P}_{s}^{1}$ is a curve of genus 1, after absorbing square factors into $y^{2}, g$ must be a polynomial of degree 3 or 4 in $t$.

Next, we consider the case $D=O+P$ where $P=\left(x_{0}, y_{0}\right)$ is not a 2-torsion section. Then 1 and $\frac{y+y_{0}}{x-x_{0}}$ form a basis of $L_{E}(D)$. Therefore we obtain a new elliptic parameter

$$
\begin{equation*}
s=A(t)+B(t) \frac{y+y_{0}}{x-x_{0}} . \tag{3.5}
\end{equation*}
$$

Solving (3.5), we get

$$
\begin{equation*}
y=\frac{(s-A(t))\left(x-x_{0}\right)}{B(t)}-y_{0} . \tag{3.6}
\end{equation*}
$$

| Class 2.1 |  |
| :---: | :---: |
| Method | 2-neighbor step from the class 2.5 |
| Elliptic parameter | $s=\frac{x-(b-1)(a d-b c+b-d) t^{3}}{t^{4}}$ |
|  |  |
| $N_{i}$ : the pull-back of a rational plane curve of degree $i$ |  |

Table 6. Class 2.1
Substituting into the Weierstrass equation (3.2) we get

$$
\begin{equation*}
\left(\frac{\left(s-A(t)\left(x-x_{0}\right)\right.}{B(t)}-y_{0}\right)^{2}=x^{3}+a_{2}(t) x^{2}+a_{4}(t) x+a_{6} . \tag{3.7}
\end{equation*}
$$

Since $x_{0}$ and $y_{0}$ satisfy the Weierstrass equation (3.2), the difference of the left and right hand sides of this equation can be divided by $\left(x-x_{0}\right)$. Therefore we get an equation $g(x, t, s)=0$ that is quadratic in $x$. By completing the square, we obtain an equation of the form

$$
\begin{equation*}
x^{2}=h(t, s) \tag{3.8}
\end{equation*}
$$

Class 2.2

| Method | 2-neighbor step from the class 2.7 |
| :---: | :---: |
| Elliptic parameter | $\begin{aligned} & s=\frac{x+(1-a) t+A_{2} t^{2}-A_{3} t^{3}}{t^{3}(b t-d t-1)} \\ & A_{2}=2 a b-b c+a c-a-b-c+1 \\ & A_{3}=(b-d)(a b-b c+a c+a d-a-c-d+1) \end{aligned}$ |
|  |  |
| $\begin{aligned} & s=\infty \\ & s=0 \\ & s=(a-c)(a d-b c-a-d+1) \\ & s=(a d-b c)(1-c-d) \\ & s=(1-b-c)(a d-b c+b-d) \\ & s=b(a-c)(1-c-d) \\ & s=c(b c-a d-b+d) \\ & s=b c(c+d-a d-1) \end{aligned}$ |  |

Table 7. Class 2.2

Class 2.3

| Method | 2-neighbor step form the class 2.7 |
| :---: | :---: |
| Elliptic paramete | $s=\frac{x-(a-1) t+A_{2} t^{2}-b(c+b-1)(a-c) t^{3}}{t^{3}(b(a-c) t-a+1)}$ $A_{2}=2 a b-b c+a c-a-b-c+1$ |
|  |  |
| $s=\infty \quad s=a d-b c-a+c \quad s=c d(1-b) \quad s=(b-d)(1-c-d) \quad s=0$ |  |

Table 8. Class 2.3


Table 9. Class 2.4

| Class 2.5 |  |
| :---: | :---: |
| Method | Classical |
| Elliptic parameter | $t=\frac{-u v(u-1)}{(a+c+d-a c-1) u v+a c u^{2}-(a+c) u-d v+1}$ |
| $t=0 \quad t=t_{1} \quad t=\frac{1}{c(a+d-1)} \quad t=\frac{1}{c(a-1)} \quad t=\frac{1}{a(c+d-1)} \quad t=t_{2} \quad t=\infty$$t_{1}=\frac{a d-b c+b-d}{(a-1)(a d-b c)(c+d-1)}, t_{2}=\frac{b-1}{a(b c-c-d+1)}$ |  |
| $\begin{aligned} y^{2}= & x^{3}+t\left(a c(a-1)(c+d-1)(a c+b c-a-c-d+1) t^{3}\right. \\ & +\left(4 a c^{2}-c^{2}+a-5 a c-c d-a^{2}+2 a c d+2 b c d-a d^{2}-3 a^{2} c^{2}\right. \\ & \left.+b c^{2}-2 a^{2} c d+c+3 a b c-a b c d+a^{2} d-2 b c+4 a^{2} c-2 b c^{2} a\right) t^{2} \\ & +(a d+b c+3 a c-2 a-2 b-2 c+d+1) t-1) x^{2} \\ & -t^{4}(c(a+d-1) t-1)(a(b c-c-d+1) t-b+1) \\ & ((a-1)(a d-b c)(c+d-1) t-a d+b c-b+d) x \end{aligned}$ |  |
| Zero section | $\ell_{6}$ |
| 2-torsion section | $\ell_{4}:(x, y)=(0,0)$ |

Table 10. Class 2.5

| Class 2.6 |  |
| :---: | :---: |
| Method | 2-neighbor step from the class 2.7 |
| Elliptic parameter |  |
| $t=\frac{1}{b-d}$ |  |
| $s=\infty \quad s=\frac{(a-c)(a+b-1)}{1-a} \quad s=\frac{a d-b c-a+c}{1-d} \quad s=0$ |  |

Table 11. Class 2.6

| Class 2.7 |  |
| :---: | :---: |
| Method | Classical |
| Elliptic parameter | $\frac{u v}{c u+b v-1}$ |
| $\begin{aligned} y^{2}= & (x-t(b t+c t-t-1)(a b t-b c t-a+1)) \\ & (x-a t(b t-d t-1)(b t+c t-t-1)) \\ & (x-t(1-c t)(b t-d t-1)(a b t-b c t-a+1)), \end{aligned}$ |  |
|  |  |
| Zero section | $\ell_{2}$ |
| 2-torsion section | $\begin{aligned} & (a b t-b c t-a+1)(b t-d t-1), 0) \\ & 3 t-1)(b t+c t-t-1), 0) \\ & c t-a+1)(b t+c t-t-1), 0) \end{aligned}$ |

Table 12. Class 2.7
and, after absorbing square factors into $x$, we have that $h$ is cubic or quartic in $t$.

Finally, we consider the case $D=O+T$ where $T$ is a 2 -torsion section. In this case, we may assume that $T=(0,0)$ and $a_{6}=0$ by a translation. Then 1 and $\frac{y}{x}$ form a basis of $L_{E}(D)$. Setting a new elliptic parameter $s=A(t)+$ $B(t) \frac{y}{x}$, we obtain

$$
\begin{equation*}
y=\frac{(s-A(t)) x}{B(t)} . \tag{3.9}
\end{equation*}
$$



Table 13. Class 2.8

Substituting this into the Weierstrass equation (3.2), we have

$$
\begin{equation*}
\left(\frac{(s-A(t)) x}{B(t)}\right)^{2}=x^{3}+a_{2}(t) x^{2}+a_{4}(t) x . \tag{3.10}
\end{equation*}
$$

Dividing both sides by $x$, we obtain a quadratic equation, and we can proceed as in the previous case.

## 4. Class 2.10

To obtain the Weierstrass equation for the class 2.10, we use a 2-neighbor step from the class 2.7. We compute explicitly the elements of $L_{X}\left(F^{\prime}\right)$ where


Fig. 3. 2-neighbor step from class 2.7 to class 2.10

$$
\begin{equation*}
F^{\prime}=2 \ell_{2}+\ell_{2,3}+\ell_{2,4}+\mu_{3,6}^{1,5}+\eta_{(36)(56)}^{(12)(34)(15)} \tag{4.1}
\end{equation*}
$$

is the class of the fiber of type $\mathrm{I}_{0}^{*}$ we are considering. The linear space $L\left(F^{\prime}\right)$ is 2-dimensional, and the ratio of two linearly independent elements will be an elliptic parameter for $X$. Thus, we may find a non-constant rational function on $X$ belonging to $L\left(F^{\prime}\right)$, for which 1 is an element of $L\left(F^{\prime}\right)$. Then it will be an elliptic parameter of fibration 2.10. Let $s \in L\left(F^{\prime}\right)$ be a non-constant. Notice that $s$ has a pole of order 2 along $\ell_{2}$, which is the zero section of fibration 2.7. Also, it has a simple pole along $\ell_{2,4}$, the identity component of the fiber at $t=\frac{1}{b+c-1}$, a simple pole along $\mu_{3,6}^{1,5}$, the identity component of the fiber at $t=\infty$, and a simple pole along $\eta_{(36)(56)}^{(12)(15)(15)}$, the identity component of the fiber at $t=\frac{-1}{a d-b c}$. Therefore we can put

$$
\begin{equation*}
s=\frac{x+A_{0}+A_{1} t+A_{2} t^{2}+A_{3} t^{3}+A_{4} t^{4}}{t\left(t+\frac{1}{a d-b c}\right)\left(t-\frac{1}{b+c-1}\right)} . \tag{4.2}
\end{equation*}
$$

We may subtract a term $A_{3} t\left(t+\frac{1}{a d-b c}\right)\left(t-\frac{1}{b+c-1}\right)$ from the numerator, since 1 is an element of $L\left(F^{\prime}\right)$. Thus, assume $A_{3}=0$. To obtain other coefficients the $A_{i}$, we look at the order of vanishing along the non-identity components of fibers at $t=0, \infty, \frac{-1}{a d-b c}, \frac{1}{b+c-1}$. For example, we look at the fiber at $t=0$. The rational function $s$ does not have any pole along $\ell_{3,5}$, which intersects with the section $\ell_{5}$ of the fibration for the class 2.7 at $t=0$. Hence $s$ has no pole at $t=0$ and $x=-a t(b t-d t+1)(b t+c t-t+1)$, which

| Class 2.9 |  |
| :---: | :---: |
| Method | Classical |
| Elliptic parameter | $t=\frac{u(a u+b v-1)}{(u-1)(v-1)(a u-1)}$ |
|  |  |
| $\begin{aligned} y^{2}= & x^{3}+t(t+1)\left((a-1)(c+d-1) t^{3}-(2 a c-2 a d+2 b c-b+3 d-2) t^{2}\right. \\ & \left.+(a d-2 b c+a+2 b+c-3 d+1) t^{2}+b-d\right) x^{2} \\ & -t^{3}(t+1)^{2}(c t-t-1)((a+b-1) t+b-1)\left((2 a c+a d-a-c) t^{2}\right. \\ & -(a c-2 a d+b c+a+c) t+a d-b c) x \\ & +a c t^{6}(t+1)^{3}(c t-t-1)^{2}(a t+b t-t+b-1)^{2} \end{aligned}$ |  |
| Zero section | $\ell 3$ |

Table 14. Class 2.9
corresponds to the section $\ell_{5}$, and that gives us $A_{0}$. Similarly, other fibers give remaining coefficients. After some calculation, we get a new elliptic parameter

$$
\begin{equation*}
s=\frac{(a d-b c) x-a(a d-b c+b-d) t+a(c+b-1)(a d-b c+b-d) t^{2}}{t((a d-b c) t+1)((b+c-1) t-1)} . \tag{4.3}
\end{equation*}
$$

Solving for $x$ in terms of $s$ and substituting into right hand side of (2.3), and dividing out some square factors, which can absorb into $y^{2}$, we obtained an equation for fibration 2.10 in the form $y^{2}=f(t)$, where $f$ is a quartic in $t$ with coefficients in $k(a, b, c, d, s)$. In fact, since the quartic $f$ factors into $t$


Table 15. Class 2.10

| Class 2.11 |  |
| :---: | :---: |
| Method | Classical |
| Elliptic parameter | $t=u$ |
|  |  |
| $\begin{aligned} y^{2}= & (x-d t(t-1)(a t-1)) \\ & (x-b t(t-1)(c t-1)) \\ & (x-t(t-1)(a t-1)(c t-1)) \end{aligned}$ |  |
| Zero section | $\ell 3$ |
| 2-torsion section | $\begin{aligned} & \ell_{4}:(x, y)=(t(1-t)(c t-1)(a t-1), 0) \\ & \ell_{5}:(x, y)=(b t(t-1)(c t-1), 0) \\ & \ell_{6}:(x, y)=(\operatorname{dtt}(t-1)(a t-1), 0) \end{aligned}$ |

Table 16. Class 2.11
times cubic factor in $t$, the new equation converts to a Weierstrass form by a standard algorithm. Finally, we get a Weierstrass equation of the fibration for the class 2.7.

$$
\begin{align*}
Y^{2}= & X^{3}+(a d-b c)(s+a d-a b)(s-a b-b c)((b+c-a d+b c-1) s \\
& +b^{2} c^{2}+b^{2} c-2 a b c-a^{2} d^{2}-b c d+a^{2} d-a d-a b^{2}+a d^{2} \\
& \left.+b c^{2}+a b-a b^{2} c+a^{2} b d\right) X^{2} \\
& -(s-a b+a d)^{2}(s-a b+b c)^{3}(a d-b c)^{3}\left(s^{3}-3 b(a-c) s^{2}\right. \\
& \left.+3 b^{2}(a-c)^{2} s-b^{2}(a-c)^{3}\right) X \\
& +b c(a-c)(b-d)(a d-b c)^{4}(s-a b+a d)^{3}(s-a b+b c)^{3} . \tag{4.4}
\end{align*}
$$

| Class 2.12 |  |
| :---: | :---: |
| Method | Classical |
| Elliptic parameter | $t=\frac{u(b v+a-1)}{a u+b v-1}$ |
| $t=0 \quad t=1 \quad t=\infty \quad t=\frac{a-1}{a-c} \quad t=\frac{(d-1)(a+b-1)}{a d-b c-a+c}$ |  |
| $\begin{aligned} y^{2}= & x^{3}+9 t(t-1)((a d-b c-2 a+2 c) t-a d+2 a+b+d-2) x^{2} \\ & -81 t^{2}(t-1)^{2}(a t-c t-a+1)((a d-b c-a+c) t-(d-1)(a+b-1)) x \end{aligned}$ |  |
| Zero section | $\ell_{3}$ |
| 2-torsion section | $\ell_{4}:(x, y)=(0,0)$ |

Table 17. Class 2.12

The new variables $X, Y$ are related to the ones from fibration 2.7 by the following equation.

$$
\begin{align*}
& X=\frac{(a d-b c)(a b-a d-s)(a b-b c-s)^{2}}{t}  \tag{4.5}\\
& Y=\frac{(a d-b c)^{3}(a b-a d-s)(a b-b c-s)^{2} y}{t^{2}} .
\end{align*}
$$

The elliptic parameter $s$ for the class 2.10 is also given in terms of $u, v$ by

$$
\begin{equation*}
s=\frac{g}{(u-1)(a u d v-b v c u+c u+b v-1)(c u+b v-1)}, \tag{4.6}
\end{equation*}
$$

where $g$ is given by

$$
\begin{align*}
g= & b(a-c)(b-d)(a d-b c) u^{2} v^{2}+a c^{2}(a d-b c+b-d) u^{3} \\
& +c\left(2 a b^{2}-b^{2} c^{2}-a b d-b^{2} c+b c d-a d^{2}+a^{2} d^{2}\right) u^{2} v \\
& -b\left(b^{2} c^{2}+a d^{2}-a^{2} d^{2}+b^{2} c+a^{2} b d-b c d-a b^{2}-a b^{2} c\right) u v^{2} \\
& +c\left(b c^{2}+a b c+2 a d-2 a^{2} d-2 a b\right) u^{2}-b^{3}(a-c) v^{2} \\
& +\left(a d^{2}-2 a b^{2} c+b^{2} c-2 a b^{2}+a b d+3 b^{2} c^{2}-a^{2} d^{2}-b c d\right) u v \\
& +\left(a b-a d-2 b c^{2}+a b c+a^{2} d\right) u+2 b^{2}(a-c) v-b(a-c), \tag{4.7}
\end{align*}
$$

which defines the rational plane curve through $P_{1,2}, P_{3,4}, P_{1,5}, P_{2,5}, P_{2,6}, P_{3,6}$ and $P_{5,6}$ with double points at $P_{1,2}, P_{1,5}$ and $P_{3,6}$.

The reducible fibers are as follows:

| Position | Reducible fiber | Type |
| :--- | :---: | :---: |
| $s=\infty$ | $2 \ell_{2}+\ell_{2,4}+\mu_{3,6}^{1,5}+\eta_{(36)(56)}^{(12)(34)(15)}$ | $\mathrm{I}_{0}^{*}$ |
| $s=b(c-a)$ | $2 \ell_{5}+\ell_{4,5}+\mu_{3,6}^{1,2}+\eta_{(26)(36)}^{(12)(34)(15)}$ | $\mathrm{I}_{0}^{*}$ |
| $s=a(b-d)$ | $\ell_{1,4}+\ell_{1,3}+2\left(\ell_{1}+\ell_{1,3}+\ell_{6}\right)+\ell_{4,6}+\eta_{(25)(36)}^{(12)(34)(15)}$ | $\mathrm{I}_{2}^{*}$ |

Then $\ell_{3}$ is the zero section and $\ell_{4}$ is a tri-section.

## 5. Class 2.4

The Weierstrass equation for the class 2.4 is obtained from class 2.5 by using a 2 -neighbor step as the following.


$$
t=\frac{a d-b c+b-d}{(a-1)(a d-b c)(c+d-1)}
$$



Fig. 4. Configuration of singular fibers for the class 2.10

A new elliptic divisor is

$$
\begin{align*}
F^{\prime}= & \ell_{1,5}+\ell_{1,4}+2\left(\ell_{1}+\ell_{1,3}+\ell_{3}+\ell_{2,3}+\ell_{2}+\ell_{2,6}+\ell_{6}\right) \\
& +\mu_{3,4}^{1,2}+\xi_{(35)(36)(56)}^{(\overline{12)(34)(16)(24)}} . \tag{5.1}
\end{align*}
$$

Then, we can put a new elliptic parameter $s$ to

$$
\begin{equation*}
s=\frac{x+A_{0}+A_{1} t+A_{2} t^{2}}{t^{2}\left(t-\frac{a d-b c+b-d}{(a-1)(a d-b c)(c+d-1)}\right)} . \tag{5.2}
\end{equation*}
$$

After some scaling for $s$, looking at the order of vanishing along the nonidentity components of fibers at $t=0$ and $t=\frac{a d-b c+b-c}{(a-1)(a d-b c)(c+d-1)}$, we get

$$
\begin{equation*}
A_{0}=0, \quad A_{2}=\frac{(1-a)(a d-b c)(c+d-1)}{a d+b c+b-d} A_{1} \tag{5.3}
\end{equation*}
$$

To determine the coefficient $A_{1}$, we need look at the order of vanishing along the non-identity component $\ell_{2,5}$ of the fiber at $t=0$. Now, we denote the equation for the class 2.5 , for short, by

$$
\begin{equation*}
y^{2}=x^{3}+a_{2} x^{2}+a_{4} x . \tag{5.4}
\end{equation*}
$$

The equation for each component of the fiber at $t=0$ of type $\mathrm{I}_{4}^{*}$ is given by the following (see [9, IV §9] for detail)

where $a_{i, r}=t^{-r} a_{i}$ and $\tilde{a}_{i, r}=a_{i, r}(0)$. In this case, we see that $\tilde{a}_{2,1}=-1$, $\tilde{a}_{4,4}=(b-1)(a d-b c+b-d)$. Thus, substituting $x=t x_{1}$ and $x_{1}=-1$, we see that

$$
\begin{equation*}
s=\frac{1+A_{1}+A_{2} t^{2}}{t((a-1)(a d-b c)(c+d-1) t-(a d-b c+b-d))} \tag{5.5}
\end{equation*}
$$

on $\ell_{2,5}$. Since $s$ has a pole along $\ell_{2,5}$ unless $A_{1}=-1$, we get $A_{1}=-1$.


Fig. 5. Configuration of singular fibers for the class 2.4

As a consequence, we have a new elliptic parameter

$$
\begin{equation*}
s=\frac{(a d-b c)(x-t)+(a-1)(a d-b c)(c+d-1) t^{2}}{t^{2}((a-1)(a d-b c)(c+d-1) t-a d+b c-b+d)} . \tag{5.6}
\end{equation*}
$$

Therefore, we can compute a Weierstrass equation for the class 2.4 by using a 2-neighbor step from the class 2.5 . However, we omit it, since it is too long to write down here. The configuration of the class 2.4 is the following.

In this picture, the value of $s_{1}$ is given by

$$
\begin{equation*}
s_{1}=\frac{\left(c^{2}+2 c d-2 c-d+1\right) a+(d-1)(b c-b+d)}{c+d-1} . \tag{5.7}
\end{equation*}
$$

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