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A weighted weak type estimate for the fractional integral operator on spaces of homogeneous type

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Abstract. Let $(\mathscr{X}, d, \mu)$ be a space of homogeneous type in the sense of Coifman and Weiss. In this paper, we give a sufficient condition on the pair of weights (u, v) so that the fractional integral operator on spaces of homogeneous type is bounded from $L^p(\mathscr{X}, v)$ to weak $L^q(\mathscr{X}, u)$ with $1 < p \le q < \infty$.

1. Introduction

Let $\mathscr{X}$ be a set endowed with a positive Borel regular measure μ and a quasi-metric d satisfying that there exists a constant $\kappa \ge 1$ such that for all $x, y, z \in \mathscr{X}$,

$$d(x, y) \le \kappa[d(x, z) + d(y, z)]. \tag{1}$$

The triplet $(\mathscr{X}, d, \mu)$ is said to be a space of homogeneous type in the sense of Coifman and Weiss [6], if μ satisfies the following doubling condition: there exists a constant $C \ge 1$ such that for all $x \in \mathscr{X}$ and $r > 0$,

$$\mu(B(x, 2r)) \le C\mu(B(x, r)) < \infty. \tag{2}$$

Moreover, if C is the smallest constant for which the measure μ verifies the doubling condition (2), then $D = \log_2 C$ is called the doubling order of μ and we have that

$$\frac{\mu(B_1)}{\mu(B_2)} \le C_\mu \left(\frac{r_{B_1}}{r_{B_2}}\right)^D, \qquad \text{for all balls } B_2 \subset B_1 \subset \mathscr{X}, \tag{3}$$

where r_{B_i} denotes the radius of B_i, $i = 1, 2$, and C_μ is the constant that is dependent of the parameter μ.

We remark that although all balls defined by d satisfy the axioms of complete system of neighborhoods in $\mathscr{X}$, and therefore induce a (separated)

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topology in $\mathscr{X}$, the balls $B(x, r)$ for $x \in \mathscr{X}$ and $r > 0$ need not to be open with respect to this topology. However, Macías and Segovia in [12] showed that there are other quasi-metric $\tilde{d}$ on $\mathscr{X}$ and a number $\theta \in (0,1)$ such that $\tilde{d}$ is equivalent to d and for any $x, x', y \in \mathscr{X}$,

$$|\tilde{d}(x, y) - \tilde{d}(x', y)| \le C\tilde{d}(x, x')^{\theta}(\tilde{d}(x, y) + \tilde{d}(x', y))^{1-\theta}. \tag{4}$$

Moreover, the $\tilde{d}$-balls are open in the $\tilde{d}$-topology.

We consider the function $d' : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ defined by

$$d'(x, y) = \begin{cases} \frac{1}{2}[\mu(B(x, d(x, y))) + \mu(B(y, d(x, y)))], & \text{if } x \ne y, \\ 0, & \text{if } x = y. \end{cases}$$

It is easy to check that d' is a quasi-metric on $\mathscr{X}$. Let η be a continuous quasi-metric equivalent to d' and satisfy (4). For $\alpha \in (0,1)$, define the fractional integral operator I_α as

$$I_\alpha f(x) = \int_{\mathscr{X}} Q_\alpha(x, y) f(y) \mathrm{d}\mu(y)$$

with the kernel

$$Q_\alpha(x, y) = \begin{cases} \eta(x, y)^{\alpha-1}, & \text{if } x \ne y, \\ \mu(\{x\})^{\alpha-1}, & \text{if } x = y \text{ and } \mu(\{x\}) > 0. \end{cases}$$

There are well known properties related to the boundedness of I_α on spaces of homogeneous type, shortly, I_α is bounded from $L^p(\mathscr{X})$ to $L^q(\mathscr{X})$ with $1 < p < q < \infty$ and $1/q = 1/p - \alpha$ (see [4]), and I_α is of weak type $(1, (1-\alpha)^{-1})$ (see [3]). Moreover, there are versions of these results with different weights. The result of Bernardis et al. [2] states that for any fixed $p \in (1, \infty)$, there is a constant $C > 0$ such that for any weight w,

$$\int_{\mathscr{X}} |I_\alpha f(x)|^p w(x) \mathrm{d}\mu(x) \le C \int_{\mathscr{X}} |f(x)|^p M_{\alpha p}(M^{[p]} w)(x) \mathrm{d}\mu(x),$$

where and in the sequel, by a weight w, we mean that w is a nonnegative and locally integrable function, $[p]$ denotes the biggest integer not more than p, M_α is the fractional maximal operator (see the definition below), M is the standard Hardy-Littlewood maximal operator and for any positive integer k, M^k is the operator M iterated k times. Martell [13] proved the operator I_α is bounded from $L^p(\mathscr{X}, v)$ to weak $L^q(\mathscr{X}, u)$ with $1 < p \le q < \infty$, provided that the pair of weights (u, v) verifies a Muckenhoupt condition with a "power-bump" on the weight u. Li et al. [11] gave sufficient conditions in terms of Orlicz bumps for the two-weight strong type (p, q) inequalities $(1 < p \le q < \infty)$ for the commutators of potential integral operators, which is more general than the fractional integral operator.

The purpose of this paper is to improve Martell's result on the two-weight weak type estimate for the fractional integral operator. We will prove that if the pair of weights (u, v) satisfies a Muckenhoupt condition with a "Orlicz-bump" on the weight u, then I_α is bounded from $L^p(\mathscr{X}, v)$ to weak $L^q(\mathscr{X}, u)$ for any $1 < p \le q < \infty$. To state our result, we first recall some notation.

Let Φ be a Young function, that is to say, $\Phi : [0, \infty) \to [0, \infty)$ is a continuous, convex and, increasing function and satisfies $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. Let E be a measurable set with $\mu(E) < \infty$, define the Luxemburg norm of f over E as

$$\|f\|_{\Phi,E} = \inf\left\{\lambda > 0 : \frac{1}{\mu(E)}\int_E \Phi\left(\frac{|f(x)|}{\lambda}\right)\mathrm{d}\mu(x) \le 1\right\}.$$

The main Young function that we will use is $\Phi(t) = t\log(\mathrm{e}+t)^\delta$ for some $\delta > 0$. For this Young function, we denote the mean Luxemburg norm of f over E by $\|f\|_{L(\log L)^\delta, E}$.

Our main result can be stated as follows.

Theorem 1. *Let $1 < p \le q < \infty$ and $\alpha \in (0,1)$. Suppose that (u, v) is a pair of weights such that there exists $\gamma > 0$ such that for any ball $B \subset \mathscr{X}$,*

$$[\mu(B)]^{\alpha+1/p'-1/q'}\|u\|^{1/q}_{L(\log L)^{2q-1+\gamma}, B}\left(\frac{1}{\mu(B)}\int_B v(x)^{-p'/p}\mathrm{d}\mu(x)\right)^{1/p'} \le C < \infty.$$

Then for any bounded function f with bounded support,

$$\sup_{\lambda>0} \lambda u(\{x \in \mathscr{X} : |I_\alpha f(x)| > \lambda\})^{1/q} \le C\left(\int_{\mathscr{X}} |f(x)|^p v(x)\mathrm{d}\mu(x)\right)^{1/p}.$$

Remark 1. *A result analogous to Theorem* 1 *for the Calderón-Zygmund singular integral operators on Euclidean spaces was proved by Cruz-Uribe and Pérez in* [7]. *And for a version of this result in the Euclidean setting when $p = q$ see* [10]. *As far as we know, our result is new even in the case of Euclidean spaces.*

Throughout this paper, C denotes the constant that is independent of the main parameters involved but whose values may differ from line to line. Constants with subscript such as c_1, do not change in different occurrences. For a measurable set E and a weight ω, χ_E denotes the characteristic function of E, $\omega(E) = \int_E \omega(x)\mathrm{d}\mu(x)$. Given $\lambda > 0$ and a ball B, r_B denotes the radius of B, λB denotes the ball with the same center as B and whose radius is λ times that of B. For a fixed $p \in (1, \infty)$, p' denotes the dual exponent of p, namely, $p' = p/(p-1)$. For a locally integrable function f on $\mathscr{X}$ and a bounded measurable set E, $m_E(f)$ denotes the mean value of f over E, that is,

$$m_E(f) = \frac{1}{\mu(E)} \int_E f(x) \mathrm{d}\mu(x).$$

For a locally integrable function f, define the Fefferman-Stein sharp maximal function $M^{\#}f$ as

$$M^{\#}f(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y) - m_B(f)| \mathrm{d}\mu(y),$$

where the supremum is taken over all the balls B containing x. For fixed $q \in (0,1)$, the sharp maximal function $M_q^{\#}f$ is defined by

$$M_q^{\#}f(x) = (M^{\#}(|f|^q)(x))^{1/q}.$$

We then give a few facts about Orlicz spaces. Given a Young function Φ and $\alpha \in [0,1)$, define the fractional Orlicz maximal operator $M_{\alpha,\Phi}$ by

$$M_{\alpha,\Phi}f(x) = \sup_{B \ni x} [\mu(B)]^{\alpha} \|f\|_{\Phi,B},$$

where the supremum is taken over all the balls B containing x. If $\alpha = 0$, we denote $M_{0,\Phi}$ by M_Φ simply. If $\Phi(t) = t$, $M_{\alpha,\Phi}$ is just the classical fractional maximal operator M_α defined by

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{[\mu(B)]^{1-\alpha}} \int_B |f(y)| \mathrm{d}\mu(y).$$

A Young function Φ is said to be doubling if there exists $C > 0$ such that for all $t \geq 0$, $\Phi(2t) \leq C\Phi(t)$. Pradolini and Salinas [17] proved that if a doubling Young function Φ satisfies the B_p $(p \in (1,\infty))$ condition, that is, for some constant $c > 0$,

$$\int_c^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t} < \infty,$$

then M_Φ is bounded on $L^p(\mathscr{X})$.

The Lorentz space $L^{p,1}(\mathscr{X}, w)$ will be useful in our discussion. For a weight w and a measurable function f, let f^* be the decreasing rearrangement of f defined by

$$f^*(t) = \inf\{s > 0 : w(\{x \in \mathscr{X} : |f(x)| > s\}) \leq t\}.$$

For $p, q \in (0,\infty)$, let

$$\|f\|_{L^{p,q}(\mathscr{X},w)} = \begin{cases} \left(\displaystyle\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t} \right)^{1/q}, & \text{if } q < \infty; \\ \displaystyle\sup_{t>0} t^{1/p} f^*(t), & \text{if } q = \infty. \end{cases}$$

The set of all f with $\|f\|_{L^{p,q}(\mathscr{X},w)} < \infty$ is denoted by $L^{p,q}(\mathscr{X},w)$ and is called the Lorentz space with indices p and q. It is obvious that $L^{p,\infty}(\mathscr{X},w)$ is just the standard weak L^p space with weight w. For $p \in (1,\infty)$, we know that

$$\|f\|_{L^{p,\infty}(\mathscr{X},w)} \le C \sup_{\|h\|_{L^{p',1}(\mathscr{X},w)} \le 1} \left| \int_{\mathscr{X}} f(x)h(x)w(x)\mathrm{d}\mu(x) \right|, \tag{5}$$

see [8] for details.

2. A two-weight estimate for fractional Orlicz maximal operator

This section is devoted to a weighted norm inequality for the fractional Orlicz maximal operator $M_{\alpha,\Phi}$. We will prove that

THEOREM 2. *Given $1 < p \le q < \infty$ and $\alpha \in [0,1)$. Let Φ, Ψ and Θ be Young functions such that for any $t > 0$, $\Psi^{-1}(t)\Theta^{-1}(t) \le \Phi^{-1}(t)$, and Θ be doubling satisfying the B_p condition. (u,v) is a pair of weights such that for every ball B,*

$$[\mu(B)]^{\alpha+1/q-1/p} \left(\frac{1}{\mu(B)} \int_B u(x)\mathrm{d}\mu(x) \right)^{1/q} \|v^{-1/p}\|_{\Psi,B} \le C < \infty.$$

Then for any function $f \in L^p(\mathscr{X},v)$,

$$\left(\int_{\mathscr{X}} [M_{\alpha,\Phi} f(x)]^q u(x)\mathrm{d}\mu(x) \right)^{1/q} \le C \left(\int_{\mathscr{X}} |f(x)|^p v(x)\mathrm{d}\mu(x) \right)^{1/p}.$$

For the case that $\Phi(t) = t$, the related result in Euclidian spaces was proved by Pérez (see Theorem 2.11 in [15]). To prove Theorem 2, we need the following dyadic sets on spaces of homogeneous type given by Sawyer and Wheeden in [18], which have a lot of properties in common with the dyadic cubes in the Euclidean spaces.

LEMMA 1. *Let $(\mathscr{X},d,\mu)$ be a space of homogeneous type. Fix $\rho = 8\kappa^5$. For every (large negative) integer m, there exist a collection of points $\{x_j^k\}$ and a family of sets $\mathcal{D}_m = \{\mathcal{E}_j^k\}_{j,k}$ with $k = m, m+1, \dots$ and $j = 1, 2, \dots$ such that*

- (A_1) *$B(x_j^k, \rho^k) \subset \mathcal{E}_j^k \subset B(x_j^k, \rho^{k+1})$*
- (A_2) *For every $k \ge m$, the sets $\{\mathcal{E}_j^k\}_j$ are pairwise disjoint in j, and $\mathscr{X} = \bigcup_j \mathcal{E}_j^k$.*
- (A_3) *If $m \le k < l$, then either $\mathcal{E}_j^k \cap \mathcal{E}_i^l = \varnothing$ or $\mathcal{E}_j^k \subset \mathcal{E}_i^l$.*

We will refer to $\mathcal{D} = \bigcup_m \mathcal{D}_m$ as a dyadic cube decomposition of $\mathscr{X}$ and the sets in $\mathcal{D}$ as dyadic cubes. For every integer $k \ge m$, set $\mathcal{D}_m^k = \{\mathcal{E}_j^k\}_j$. A dyadic cube will be written as Q, and Q^* will denote the ball that contains Q in such a way that if $Q = \mathcal{E}_j^k$, then $Q^* = B(x_j^k, \rho^{k+1})$. Associated with the dyadic cubes of $\mathcal{D}_m$, Young function Φ and $\alpha \in [0,1)$, we define the maximal operators as

$$M_{\alpha,\Phi,m}^d f(x) = \sup_{Q \ni x,\, Q \in \mathcal{D}_m} [\mu(Q)]^\alpha \|f\|_{\Phi,Q},$$

where the supremum is taken over all the dyadic cubes $Q \in \mathcal{D}_m$ containing x, and

$$M_{\alpha,\Phi,m} f(x) = \sup_{B \ni x,\, r_B \ge \rho^m} [\mu(B)]^\alpha \|f\|_{\Phi,B},$$

where the supremum is taken over all the balls B containing x and $r_B \ge \rho^m$. Corresponding to the maximal operators $M_{\alpha,\Phi,m}^d$ and $M_{\alpha,\Phi,m}$, the following lemma is a generalized version of the dyadic version of Calderón-Zygmund decomposition.

LEMMA 2. *Let $\alpha \in [0,1)$, Φ be a Young function and f be a nonnegative function such that $\int_{\mathcal{X}} \Phi(f(x))\mathrm{d}\mu(x) < \infty$. Let $\tau_{\mathcal{X}} = 0$ if $\mu(\mathcal{X}) = \infty$ and $\tau_{\mathcal{X}} = [\mu(\mathcal{X})]^\alpha \|f\|_{\Phi,\mathcal{X}}$ if $\mu(\mathcal{X}) < \infty$. Given $\sigma > C_\mu \rho^{2D}$, for each integer l with $\sigma^l > \tau_{\mathcal{X}}$, we have*

$$\{x \in \mathcal{X} : M_{\alpha,\Phi,m} f(x) > \sigma^l\} \subset \bigcup_{Q \in \mathcal{F}_l} 3\kappa^2 Q^*,$$

where $\mathcal{F}_l \subset \mathcal{D}_m$ is a family of maximal disjoint dyadic cubes satisfying that there exist positive constants c_1 and c_2 which only depend on the space $\mathcal{X}$, ρ and α, such that

$$\Omega_l^d = \{x \in \mathcal{X} : M_{\alpha,\Phi,m}^d f(x) > c_1 \sigma^l\} = \bigcup_{Q \in \mathcal{F}_l} Q$$

and for any $Q \in \mathcal{F}_l$,

$$c_1 \sigma^l < [\mu(Q)]^\alpha \|f\|_{\Phi,Q} \le c_2 \sigma^l. \tag{6}$$

PROOF. We will employ the ideas used in the proof of Lemma 4.1 in [16]. Note that if there exists a dyadic cube $Q \in \mathcal{D}_m$ such that $[\mu(Q)]^\alpha \|f\|_{\Phi,Q} > c_1\sigma^l$, then it is contained in a dyadic cube of this type which is maximal with respect to inclusion. Let $\mathcal{F}_l = \{P_i\}_i \subset \mathcal{D}_m$ be the family of maximal disjoint dyadic cubes satisfying $[\mu(P_i)]^\alpha \|f\|_{\Phi,P_i} > c_1\sigma^l$. According to Lemma 1, for each fixed P_i, we know that there exist $j_i \in \mathbf{N}$, $k_i \ge m$ such that $P_i = \mathcal{E}_{j_i}^{k_i} \subset \bigcup_j \mathcal{E}_j^{k_i+1}$. Then for some $j_i' \in \mathbf{N}$,

$$B(x_{j_i}^{k_i}, \rho^{k_i}) \subset P_i \subset \mathcal{E}_{j_i'}^{k_i+1} \subset B(x_{j_i'}^{k_i+1}, \rho^{k_i+2}).$$

The maximality of the dyadic cube P_i together with the inequality (3) gives us that

$$\begin{aligned}
&\frac{1}{\mu(P_i)}\int_{P_i}\Phi\left(\frac{f(x)[\mu(P_i)]^{\alpha}}{c_1\sigma^l}\right)\mathrm{d}\mu(x)\\
&\quad\le \frac{\mu(B(x_{j_i'}^{k_i+1},\rho^{k_i+2}))}{\mu(P_i)}\frac{1}{\mu(\mathcal{E}_{j_i'}^{k_i+1})}\int_{\mathcal{E}_{j_i'}^{k_i+1}}\Phi\left(\frac{f(x)[\mu(\mathcal{E}_{j_i'}^{k_i+1})]^{\alpha}}{c_1\sigma^l}\right)\mathrm{d}\mu(x)\\
&\quad\le \frac{\mu(B(x_{j_i'}^{k_i+1},\rho^{k_i+2}))}{\mu(B(x_{j_i}^{k_i},\rho^{k_i}))}\\
&\quad\le C_\mu\rho^{2D}.
\end{aligned}$$

Consequently,

$$c_1\sigma^l < [\mu(P_i)]^{\alpha}\|f\|_{\Phi,P_i} \le C_\mu\rho^{2D}c_1\sigma^l.$$

For any $x\in\{x\in\mathscr{X}: M_{\alpha,\Phi,m}f(x)>\sigma^l\}$, there exists a ball B satisfying $x\in B$, $r_B\ge\rho^m$ and

$$[\mu(B)]^{\alpha}\|f\|_{\Phi,B}>\sigma^l.$$

Choose the integer $k\ge m$ such that $\rho^k\le r_B<\rho^{k+1}$, then there is a collection of dyadic cubes $\{J_i\}_{i=1}^{c_3}\subset\mathcal{D}_m^k$ verifying $J_i^*\cap B\neq\emptyset$ for $i\in[1,c_3]$. Remark 2.5 in [13] tells us that

$$c_3\le C_\mu\kappa^D\left(\frac{r_B}{\rho^k}+2\rho\kappa\right)^D\le C_\mu\rho^D(\kappa+2\kappa^2)^D.$$

In what follows, set $c_3=C_\mu\rho^D(\kappa+2\kappa^2)^D$. We claim that there exists at least one of these cubes, say J_1, such that $J_1\cap B\neq\emptyset$ and

$$[\mu(B)]^{\alpha}\|\chi_{J_1}f\|_{\Phi,B}>\sigma^l/c_3.$$

In fact, if it were not true, that is, for any $i\in[1,c_3]$, $[\mu(B)]^{\alpha}\|\chi_{J_i}f\|_{\Phi,B}\le\sigma^l/c_3$, then

$$[\mu(B)]^{\alpha}\|f\|_{\Phi,B}=[\mu(B)]^{\alpha}\|\chi_{\bigcup_{i=1}^{c_3}J_i}f\|_{\Phi,B}\le\sum_{i=1}^{c_3}[\mu(B)]^{\alpha}\|\chi_{J_i}f\|_{\Phi,B}\le\sigma^l,$$

which is a contradiction to the fact that $[\mu(B)]^{\alpha}\|f\|_{\Phi,B}>\sigma^l$. It is easy to check that $B\subset(\kappa+2\kappa^2)J_1^*$. A straightforward computation via the inequality (3) shows that

$$\frac{1}{\mu(J_1)}\int_{J_1}\Phi\left(\frac{c_3 f(x)[\mu(J_1)]^\alpha}{\sigma^l}\right)\mathrm{d}\mu(x)$$
$$> \frac{\mu(B)}{\mu(J_1)}\frac{1}{\mu(B)}\int_{J_1\cap B}\Phi\left(\frac{c_3 f(x)[\mu(B)]^\alpha}{\sigma^l[C_\mu(\kappa+2\kappa^2)^D\rho^D]^\alpha}\right)\mathrm{d}\mu(x)$$
$$> \frac{\mu(B)}{\mu((\kappa+2\kappa^2)J_1^*)}\frac{1}{[C_\mu(\kappa+2\kappa^2)^D\rho^D]^\alpha}$$
$$> \frac{1}{[C_\mu(\kappa+2\kappa^2)^D\rho^D]^{1+\alpha}}.$$

It follows that $[\mu(J_1)]^\alpha\|f\|_{\Phi,J_1} > c_1\sigma^l$ with $c_1^{-1} = [C_\mu(\kappa+2\kappa^2)^D\rho^D]^{2+\alpha}$. Then there exists a family of maximal disjoint dyadic cubes $\mathcal{F}_l \subset \mathcal{D}_m$ satisfying that

$$\Omega_l^d = \bigcup_{Q\in\mathcal{F}_l} Q$$

and for any $Q\in\mathcal{F}_l$,

$$c_1\sigma^l < \|f\|_{\Phi,Q} \le c_2\sigma^l,$$

where $c_2 = C_\mu\rho^{2D}c_1$. On the other hand, we observe that there exists some $Q\in\mathcal{F}_l$ such that $J_1\subset Q$, and then $B\cap Q\neq\emptyset$. Thus for any $x\in\{x\in\mathcal{X} : M_{\alpha,\Phi,m}f(x) > \sigma^l\}$,

$$x\in B\subset(\kappa+2\kappa^2)Q^*\subset 3\kappa^2Q^*,$$

which in turn implies that

$$\{x\in\mathcal{X} : M_{\alpha,\Phi,m}f(x) > \sigma^l\} \subset \bigcup_{Q\in\mathcal{F}_l} 3\kappa^2Q^*.$$

Lemma 3. *Under the hypotheses of Lemma 2, for every $Q\in\mathcal{F}_l$, set $\tilde{Q} = Q\backslash(Q\cap\Omega_{l+1}^d)$. Then $\{\tilde{Q}\}$ is a family of pairwise disjoint sets which satisfies that*

$$\mu(Q) < \frac{1}{1 - C_\mu\rho^{2D}\sigma^{-1}}\mu(\tilde{Q}).$$

Proof. The family $\{\tilde{Q}\}$ is clearly pairwise disjoint. Applying the inequality (6), we get for every $Q\in\mathcal{F}_l$,

$$\frac{1}{\mu(Q)}\int_Q\Phi\left(\frac{f(x)[\mu(Q)]^\alpha}{c_1\sigma^l}\right)\mathrm{d}\mu(x) > 1$$

and

$$\frac{1}{\mu(Q)}\int_Q\Phi\left(\frac{f(x)[\mu(Q)]^\alpha}{c_2\sigma^l}\right)\mathrm{d}\mu(x) \le 1.$$

It is obvious that $\Omega_{l+1}^d \subseteq \Omega_l^d$. A trivial computation gives that

$$\begin{aligned}\mu(Q \cap \Omega_{l+1}^d) &= \sum_{\{Q' \in \mathcal{F}_{l+1} : Q' \subseteq Q\}} \mu(Q') \\ &\le \sum_{\{Q' \in \mathcal{F}_{l+1} : Q' \subseteq Q\}} \int_{Q'} \Phi\left(\frac{f(x)[\mu(Q')]^{\alpha}}{c_1 \sigma^{l+1}}\right) d\mu(x) \\ &\le C_\mu \rho^{2D} \sigma^{-1} \int_{Q \cap \Omega_{l+1}^d} \Phi\left(\frac{f(x)[\mu(Q)]^{\alpha}}{c_2 \sigma^{l}}\right) d\mu(x) \\ &\le C_\mu \rho^{2D} \sigma^{-1} \mu(Q).\end{aligned}$$

It follows that

$$\mu(\tilde{Q}) = \mu(Q) - \mu(Q \cap \Omega_{l+1}^d) > (1 - C_\mu \rho^{2D} \sigma^{-1}) \mu(Q).$$

This leads to our desired estimate.

We also need the following generalization of the Hölder inequality (see [14]).

Lemma 4. *Let Φ, Ψ and Θ be Young functions such that for any $t > 0$, $\Psi^{-1}(t)\Theta^{-1}(t) \le \Phi^{-1}(t)$, then for any suitable functions f, g and any measurable set E with $\mu(E) < \infty$,*

$$\|fg\|_{\Phi, E} \le C \|f\|_{\Psi, E} \|g\|_{\Theta, E}. \tag{7}$$

Proof (Proof of Theorem 2). By a standard density argument we may assume that f is a bounded function with bounded support. Note that for any $x \in \mathcal{X}$,

$$M_{\alpha,\Phi,m} f(x) \le M_{\alpha,\Phi,m-1} f(x) \le \cdots \quad \text{and} \quad \lim_{m \to -\infty} M_{\alpha,\Phi,m} f(x) = M_{\alpha,\Phi} f(x).$$

The monotone convergence theorem shows that

$$\lim_{m \to -\infty} \int_{\mathcal{X}} (M_{\alpha,\Phi,m} f(x))^q u(x) d\mu(x) = \int_{\mathcal{X}} [M_{\alpha,\Phi} f(x)]^q u(x) d\mu(x).$$

Then it suffices to prove that for any large enough negative integer m,

$$\left(\int_{\mathcal{X}} [M_{\alpha,\Phi,m} f(x)]^q u(x) d\mu(x)\right)^{1/q} \le C \left(\int_{\mathcal{X}} |f(x)|^p v(x) d\mu(x)\right)^{1/p}. \tag{8}$$

Fix a constant $\sigma > C_\mu \rho^{2D}$. For each integer l with $\sigma^l > \tau_{\mathcal{X}}$, where $\tau_{\mathcal{X}} = 0$ if $\mu(\mathcal{X}) = \infty$ and $\tau_{\mathcal{X}} = [\mu(\mathcal{X})]^{\alpha} \|f\|_{\Phi,\mathcal{X}}$ if $\mu(\mathcal{X}) < \infty$, set

$$\Omega_l = \{x \in \mathcal{X} : \sigma^l < M_{\alpha,\Phi,m} f(x) \le \sigma^{l+1}\}.$$

By Lemma 2, there exists a family of maximal disjoint dyadic cubes $\mathcal{F}_l \subset \mathcal{D}_m$ such that

$$\Omega_l \subset \bigcup_{Q \in \mathcal{F}_l} 3\kappa^2 Q^* \qquad \text{and} \qquad [\mu(Q)]^{\alpha} \|f\|_{\Phi, Q} > c_1 \sigma^l.$$

For the case $\mu(\mathscr{X}) = \infty$, a direct computation along with the inequality (7) gives us that for $q \in (1, \infty)$,

$$\begin{aligned}
&\int_{\mathscr{X}} [M_{\alpha,\Phi,m} f(x)]^q u(x) \mathrm{d}\mu(x) \\
&\quad = \sum_l \int_{\Omega_l} [M_{\alpha,\Phi,m} f(x)]^q u(x) \mathrm{d}\mu(x) \\
&\quad \le \sum_l \sigma^{(l+1)q} u(\Omega_l) \\
&\quad \le \sum_l \sum_{Q \in \mathcal{F}_l} \sigma^{(l+1)q} u(3\kappa^2 Q^*) \\
&\quad \le C \sum_l \sum_{Q \in \mathcal{F}_l} [\mu(Q)]^{\alpha q} \|f\|_{\Phi, Q}^q u(3\kappa^2 Q^*) \\
&\quad \le C \sum_l \sum_{Q \in \mathcal{F}_l} [\mu(Q)]^{\alpha q} \|f v^{1/p}\|_{\Theta, Q}^q \|v^{-1/p}\|_{\Psi, Q}^q u(3\kappa^2 Q^*).
\end{aligned}$$

It is easy to verify that $\|v^{-1/p}\|_{\Psi, Q} \le C_\mu (3\kappa^2 \rho)^D \|v^{-1/p}\|_{\Psi, 3\kappa^2 Q^*}$. Applying Lemma 3 and the L^p-boundedness of M_Θ, we obtain that for $1 < p \le q < \infty$,

$$\begin{aligned}
&\left(\int_{\mathscr{X}} [M_{\alpha,\Phi,m} f(x)]^q u(x) \mathrm{d}\mu(x) \right)^{p/q} \\
&\quad \le C \sum_l \sum_{Q \in \mathcal{F}_l} [\mu(Q)]^{\alpha p} \|f v^{1/p}\|_{\Theta, Q}^p \|v^{-1/p}\|_{\Psi, 3\kappa^2 Q^*}^p \\
&\qquad \times \left(\frac{\mu(\tilde{Q})}{\mu(3\kappa^2 Q^*)} \int_{3\kappa^2 Q^*} u(x) \mathrm{d}\mu(x) \right)^{p/q} \\
&\quad \le C \sum_l \sum_{Q \in \mathcal{F}_l} \inf_{x \in \tilde{Q}} [M_\Theta (f v^{1/p})(x)]^p \mu(\tilde{Q}) \\
&\quad \le C \int_{\mathscr{X}} [M_\Theta (f v^{1/p})(x)]^p \mathrm{d}\mu(x) \\
&\quad \le C \int_{\mathscr{X}} |f(x)|^p v(x) \mathrm{d}\mu(x).
\end{aligned}$$

For the case $\mu(\mathscr{X}) < \infty$, write

$$\begin{aligned}\int_{\mathscr{X}} &[M_{\alpha,\Phi,m}f(x)]^q u(x)\mathrm{d}\mu(x)\\ &= \int_{\{x\in\mathscr{X}:M_{\alpha,\Phi,m}f(x)\le\tau_{\mathscr{X}}\}}[M_{\alpha,\Phi,m}f(x)]^q u(x)\mathrm{d}\mu(x)\\ &\quad + \int_{\{x\in\mathscr{X}:M_{\alpha,\Phi,m}f(x)>\tau_{\mathscr{X}}\}}[M_{\alpha,\Phi,m}f(x)]^q u(x)\mathrm{d}\mu(x)\\ &= \mathrm{I} + \mathrm{II}.\end{aligned}$$

The estimate of the term II is similar to the previous case. To estimate the term I, note that $\mu(\mathscr{X}) < \infty$ implies that $\mathscr{X}$ is bounded, that is, there exist $x_0 \in \mathscr{X}$ and $R > 0$ such that $\mathscr{X} = B(x_0, R)$. Then

$$[\mu(\mathscr{X})]^{\alpha q - q/p}\|v^{-1/p}\|^q_{\Psi,\mathscr{X}}u(\mathscr{X}) \le C$$

and

$$\|fv^{1/p}\|_{\Theta,\mathscr{X}} \le \inf_{x\in\mathscr{X}} M_\Theta(fv^{1/p})(x).$$

It follows from the inequality (7) and the L^p-boundedness of M_Θ that

$$\begin{aligned}\mathrm{I} \le \tau^q_{\mathscr{X}}u(\mathscr{X}) &= [\mu(\mathscr{X})]^{\alpha q}\|fv^{1/p}v^{-1/p}\|^q_{\Phi,\mathscr{X}}u(\mathscr{X})\\ &\le [\mu(\mathscr{X})]^{\alpha q}\|fv^{1/p}\|^q_{\Theta,\mathscr{X}}\|v^{-1/p}\|^q_{\Psi,\mathscr{X}}u(\mathscr{X})\\ &\le C[\mu(\mathscr{X})]^{q/p}\inf_{x\in\mathscr{X}}[M_\Theta(fv^{1/p})(x)]^q\\ &\le C\left(\int_{\mathscr{X}}[M_\Theta(fv^{1/p})(x)]^p\mathrm{d}\mu(x)\right)^{q/p}\\ &\le C\left(\int_{\mathscr{X}}|f(x)|^p v(x)\mathrm{d}\mu(x)\right)^{q/p}.\end{aligned}$$

Combining the estimates for the cases $\mu(\mathscr{X}) = \infty$ and $\mu(\mathscr{X}) < \infty$ yields the inequality (8), and then completes the proof of Theorem 2.

3. An endpoint estimate for fractional integral operator

In this section, we will establish the following weak type estimate with general weights for fractional integral operator I_α. This estimate plays an important role in the proof of Theorem 1 and is of independent interest. It should be pointed out that for the Eculidean space, this result was proved in [5].

THEOREM 3. *Let $\alpha \in (0,1)$ and $\varepsilon > 0$, then there exists a constant $C > 0$ depending only on α and ε, such that for any weight w and any bounded function f with a bounded support,*

$$\|I_\alpha f\|_{L^{1,\infty}(\mathscr{X},w)} \leq C \int_{\mathscr{X}} |f(x)| M_{\alpha, L(\log L)^{1+\varepsilon}} w(x) \mathrm{d}\mu(x).$$

To prove Theorem 3, we will invoke some preliminary lemmas.

LEMMA 5 (see [1]). *Let $(\mathscr{X}, d, \mu)$ be a space of homogeneous type, $\mathcal{B} = \{\mathcal{B}_\tau : \tau \in \Lambda\}$ be a family of balls in $\mathscr{X}$ such that $E = \bigcup_{\tau \in \Lambda} \mathcal{B}_\tau$ is measurable and $\mu(E) < \infty$. Then there exists a disjoint sequence $\{B(x_j, r_j)\}_j \subset \mathcal{B}$, such that $E \subset \bigcup_j B(x_j, c_4 r_j)$ with c_4 a positive constant depending only on κ (the constant appearing in the inequality (1)). Moreover, for any $\tau \in \Lambda$, $\mathcal{B}_\tau$ is contained in some $B(x_j, c_4 r_j)$.*

LEMMA 6 (see [9]). *There is a constant $C > 0$ such that for any weight w and any nonnegative function f with $\mu(\{x \in \mathscr{X} : f(x) > \lambda\}) < \infty$ for any $\lambda > 0$,*
(i) *if $\mu(\mathscr{X}) = \infty$, then*

$$\int_{\mathscr{X}} f(x) w(x) \mathrm{d}\mu(x) \leq C \int_{\mathscr{X}} M^{\#} f(x) M w(x) \mathrm{d}\mu(x);$$

(ii) *if $\mu(\mathscr{X}) < \infty$, then*

$$\int_{\mathscr{X}} f(x) w(x) \mathrm{d}\mu(x) \leq C \int_{\mathscr{X}} M^{\#} f(x) M w(x) \mathrm{d}\mu(x) + C w(\mathscr{X}) m_{\mathscr{X}}(f).$$

LEMMA 7. *Let $\alpha \in (0,1)$ and $q \in (0,1)$. Then there exists a constant $C > 0$ such that for any $x \in \mathscr{X}$ and any function f satisfying that $I_\alpha f$ is locally integrable,*

$$M_q^{\#}(I_\alpha f)(x) \leq C M_\alpha f(x).$$

This lemma follows the similar argument in the proof of Lemma 5.1 in [2]. We omit the details for brevity.

LEMMA 8. *If $\alpha \in (0,1)$ and $q \in (0,1)$, then for any weight w and any bounded function f with a bounded support,*

$$\int_{\mathscr{X}} |I_\alpha f(x)|^q w(x) \mathrm{d}\mu(x) \leq C \int_{\mathscr{X}} [M_\alpha f(x)]^q M w(x) \mathrm{d}\mu(x). \tag{9}$$

PROOF. For the case of $\mu(\mathscr{X}) = \infty$, the inequality (9) follows from Lemma 6 and Lemma 7 immediately. For the case of $\mu(\mathscr{X}) < \infty$, since I_α is of weak type $(1, (1-\alpha)^{-1})$, the Kolmogorov's inequality yields that for

$q \in (0,1)$,

$$m_{\mathscr{X}}(|I_\alpha f|^q) \le C\left(\frac{1}{\mu(\mathscr{X})^{1-\alpha}}\int_{\mathscr{X}}|f(x)|\mathrm{d}\mu(x)\right)^q \le C\inf_{x\in\mathscr{X}}[M_\alpha f(x)]^q.$$

Therefore, again by Lemma 6 and Lemma 7, we can deduce that for $q \in (0,1)$,

$$\begin{aligned}\int_{\mathscr{X}}|I_\alpha f(x)|^q w(x)\mathrm{d}\mu(x) &\le C\int_{\mathscr{X}}[M_q^{\#}(I_\alpha f)(x)]^q Mw(x)\mathrm{d}\mu(x)\\ &\quad + Cw(\mathscr{X})m_{\mathscr{X}}(|I_\alpha f|^q)\\ &\le C\int_{\mathscr{X}}[M_\alpha f(x)]^q Mw(x)\mathrm{d}\mu(x)\\ &\quad + C\int_{\mathscr{X}}[M_\alpha f(x)]^q w(x)\mathrm{d}\mu(x)\\ &\le C\int_{\mathscr{X}}[M_\alpha f(x)]^q Mw(x)\mathrm{d}\mu(x).\end{aligned}$$

Lemma 9. *Let* $\alpha \in (0,1)$ *and* $\varepsilon > 0$*, then for any weight* w *and any bounded function* f *with a bounded support,*

$$\|I_\alpha f\|_{L^{1,\infty}(\mathscr{X},w)} \le C\|M_\alpha f\|_{L^{1,\infty}(\mathscr{X},M_{L(\log L)^\varepsilon}w)}.$$

Proof. We will employ the ideas used in the proof of Theorem 3.2 in [5]. Set $p \in (1,\infty)$ which will be chosen later. The inequality (5) via Lemma 8 tells us that

$$\begin{aligned}\|I_\alpha f\|_{L^{1,\infty}(\mathscr{X},w)}^{1/p} &= \|(I_\alpha f)^{1/p}\|_{L^{p,\infty}(\mathscr{X},w)}\\ &\le C\sup_{g\ge 0,\,\|g\|_{L^{p',1}(\mathscr{X},w)}\le 1}\int_{\mathscr{X}}|I_\alpha f(x)|^{1/p}g(x)w(x)\mathrm{d}\mu(x)\\ &\le C\sup_{g\ge 0,\,\|g\|_{L^{p',1}(\mathscr{X},w)}\le 1}\int_{\mathscr{X}}[M_\alpha f(x)]^{1/p}M(gw)(x)\mathrm{d}\mu(x).\end{aligned}$$

For any $\delta > 0$, weight w and function h, define the operator S by

$$Sh = \frac{M(hw)}{M_{L(\log L)^{p-1+2\delta}}w}.$$

As in the proof of Theorem 3.2 in [5], we can prove that S is bounded from $L^{p',1}(\mathscr{X},w)$ to $L^{p',1}(\mathscr{X},M_{L(\log L)^{p-1+2\delta}}w)$. Then it follows from the Hölder inequality for Lorentz spaces that

$$\begin{aligned}
&\int_{\mathscr{X}} [M_\alpha f(x)]^{1/p} M(gw)(x) \mathrm{d}\mu(x) \\
&\quad = \int_{\mathscr{X}} (M_\alpha f(x))^{1/p} \frac{M(gw)(x)}{M_{L(\log L)^{p-1+2\delta}} w(x)} M_{L(\log L)^{p-1+2\delta}} w(x) \mathrm{d}\mu(x) \\
&\quad \leq C \|(M_\alpha f)^{1/p}\|_{L^{p,\infty}(\mathscr{X}, M_{L(\log L)^{p-1+2\delta}} w)} \\
&\qquad \times \left\| \frac{M(gw)(x)}{M_{L(\log L)^{p-1+2\delta}} w(x)} \right\|_{L^{p',1}(\mathscr{X}, M_{L(\log L)^{p-1+2\delta}} w)} \\
&\quad \leq C \|M_\alpha f\|^{1/p}_{L^{1,\infty}(\mathscr{X}, M_{L(\log L)^{p-1+2\delta}} w)} \|g\|_{L^{p',1}(\mathscr{X}, w)}.
\end{aligned}$$

Choosing δ, p such that $0 < 2\delta < \varepsilon$ and $p = 1 + \varepsilon - 2\delta$ gives us the desired conclusion.

LEMMA 10. *Let* $\alpha \in [0, 1)$ *and* $\varepsilon > 0$. *Then there exists a constant* $C > 0$ *such that for any nonnegative function* f *satisfying that* $M_{L(\log L)^\varepsilon} f$ *is locally integrable and any* $x \in \mathscr{X}$,

$$M_\alpha(M_{L(\log L)^\varepsilon} f)(x) \leq C M_{\alpha, L(\log L)^{1+\varepsilon}} f(x). \tag{10}$$

PROOF. Assume that $M_{\alpha, L(\log L)^{1+\varepsilon}} f$ is finite almost everywhere, for otherwise there is nothing to prove. We first claim that if there exists a ball B such that $\operatorname{supp} f \subset B$, then

$$\frac{1}{\mu(B)} \int_B M_{L(\log L)^\varepsilon} f(y) \mathrm{d}\mu(y) \leq C \|f\|_{L(\log L)^{1+\varepsilon}, B}. \tag{11}$$

In fact, by a homogeneity argument we may assume that $\|f\|_{L(\log L)^{1+\varepsilon}, B} = 1$, which means that

$$\int_B f(y) \log^{1+\varepsilon}(\mathrm{e} + f(y)) \mathrm{d}\mu(y) \leq \mu(B).$$

For each fixed $\lambda > 0$, set

$$\Omega_\lambda = \{x \in B : M_{L(\log L)^\varepsilon} f(x) > \lambda\}.$$

Then for any $x \in \Omega_\lambda$, there exists a ball B_x such that $\|f\|_{L(\log L)^\varepsilon, B_x} > \lambda$. Applying Lemma 5, we obtain a sequence of disjoint balls $\{B_j\}_j$ such that

$$\Omega_\lambda \subset \bigcup_j c_4 B_j \qquad \text{and} \qquad \|f\|_{L(\log L)^\varepsilon, B_j} > \lambda.$$

A straightforward computation leads us to that

$$\begin{aligned}
\mu(B_j) &< \int_{B_j} \frac{f(x)}{\lambda} \log^{\varepsilon}\left(\mathrm{e}+\frac{f(x)}{\lambda}\right) \mathrm{d}\mu(x) \\
&= \int_{\{x\in B: f(x)\leq \lambda/2\}\cap B_j} \frac{f(x)}{\lambda} \log^{\varepsilon}\left(\mathrm{e}+\frac{f(x)}{\lambda}\right) \mathrm{d}\mu(x) \\
&\quad + \int_{\{x\in B: f(x)>\lambda/2\}\cap B_j} \frac{f(x)}{\lambda} \log^{\varepsilon}\left(\mathrm{e}+\frac{f(x)}{\lambda}\right) \mathrm{d}\mu(x) \\
&\leq \frac{1}{2}\log^{\varepsilon}(\mathrm{e}+1)\mu(B_j) + \int_{\{x\in B: f(x)>\lambda/2\}\cap B_j} \frac{f(x)}{\lambda} \log^{\varepsilon}\left(\mathrm{e}+\frac{f(x)}{\lambda}\right) \mathrm{d}\mu(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mu(\Omega_\lambda) &\leq C\sum_j \mu(B_j) \\
&\leq C\sum_j \int_{\{x\in B: f(x)>\lambda/2\}\cap B_j} \frac{f(x)}{\lambda} \log^{\varepsilon}\left(\mathrm{e}+\frac{f(x)}{\lambda}\right) \mathrm{d}\mu(x) \\
&\leq C\int_{\{x\in B: f(x)>\lambda/2\}} \frac{f(x)}{\lambda} \log^{\varepsilon}\left(\mathrm{e}+\frac{2f(x)}{\lambda}\right) \mathrm{d}\mu(x),
\end{aligned}$$

which in turn implies that

$$\begin{aligned}
&\int_B M_{L(\log L)^{\varepsilon}} f(y)\mathrm{d}\mu(y) \\
&\quad = \int_0^1 \mu(\Omega_\lambda) d\lambda + \int_1^{\infty} \mu(\Omega_\lambda) d\lambda \\
&\quad \leq \mu(B) + C\int_1^{\infty} \int_{\{x\in B: f(x)>\lambda/2\}} \frac{f(x)}{\lambda} \log^{\varepsilon}\left(\mathrm{e}+\frac{2f(x)}{\lambda}\right) \mathrm{d}\mu(x) d\lambda \\
&\quad \leq \mu(B) + C\int_{\{x\in B: f(x)>\lambda/2\}} \int_1^{2f(x)} \frac{f(x)}{\lambda} \log^{\varepsilon}\left(\mathrm{e}+\frac{2f(x)}{\lambda}\right) d\lambda \mathrm{d}\mu(x) \\
&\quad \leq \mu(B) + C\int_B f(x)\log^{1+\varepsilon}(\mathrm{e}+f(x))\mathrm{d}\mu(x) \\
&\quad \leq C\mu(B),
\end{aligned}$$

and then yields the estimate (11).

For each fixed $x\in\mathscr{X}$ and a ball B containing x, decompose f as

$$f(y) = f(y)\chi_{2\kappa B}(y) + f(y)\chi_{\mathscr{X}\setminus 2\kappa B}(y) = f_1(y) + f_2(y).$$

Write

$$\frac{1}{[\mu(B)]^{1-\alpha}}\int_B M_{L(\log L)^\varepsilon}f(y)\mathrm{d}\mu(y) \le \frac{1}{[\mu(B)]^{1-\alpha}}\int_B M_{L(\log L)^\varepsilon}f_1(y)\mathrm{d}\mu(y) + \frac{1}{[\mu(B)]^{1-\alpha}}\int_B M_{L(\log L)^\varepsilon}f_2(y)\mathrm{d}\mu(y) = \mathrm{I}_1 + \mathrm{I}_2.$$

The inequality (3) together with the inequality (11) gives us that

$$\begin{aligned}\mathrm{I}_1 &\le C[\mu(B)]^\alpha \frac{1}{\mu(2\kappa B)}\int_{2\kappa B} M_{L(\log L)^\varepsilon}f_1(y)\mathrm{d}\mu(y)\\ &\le C[\mu(B)]^\alpha \|f\|_{L(\log L)^{1+\varepsilon},2\kappa B}\\ &\le CM_{\alpha,L(\log L)^{1+\varepsilon}}f(x).\end{aligned}$$

On the other hand, it follows from an estimate of Bernardis et al. (see [2, Lemma 4.4]) that for any $y \in B$,

$$[\mu(B)]^\alpha M_{L(\log L)^\varepsilon}f_2(y) \le C \inf_{z\in B} M_{\alpha,L(\log L)^\varepsilon}f_2(z).$$

Applying the fact that $M_{\alpha,L(\log L)^\varepsilon}f(x) \le M_{\alpha,L(\log L)^{1+\varepsilon}}f(x)$, we have

$$\mathrm{I}_2 \le C \inf_{z\in B} M_{\alpha,L(\log L)^\varepsilon}f_2(z) \le CM_{\alpha,L(\log L)^\varepsilon}f_2(x) \le CM_{\alpha,L(\log L)^{1+\varepsilon}}f(x),$$

and then completes the proof of Lemma 10.

PROOF (Proof of Theorem 3). It suffices to prove that there exists a constant $C > 0$ such that for any weight w and $\lambda > 0$,

$$w(\{x \in \mathscr{X} : M_\alpha f(x) > \lambda\}) \le \frac{C}{\lambda}\int_{\mathscr{X}} |f(x)|M_\alpha w(x)\mathrm{d}\mu(x). \tag{12}$$

If we can do this, our desired result follows from Lemma 9, the estimate (12) and Lemma 10 directly.

We now prove (12). The argument is familiar and standard. For any $\lambda > 0$ and $x \in \mathscr{X}$ with $M_\alpha f(x) > \lambda$, there exists a ball B_x containing x such that

$$\frac{1}{[\mu(B_x)]^{1-\alpha}}\int_{B_x} |f(y)|\mathrm{d}\mu(y) > \lambda.$$

Our hypotheses on the function f guarantee that $\mu(\{x \in \mathscr{X} : M_\alpha f(x) > \lambda\}) < \infty$. By Lemma 5, we can obtain a sequence of disjoint balls $\{B_j\}_j$ such that

$$\{x \in \mathscr{X} : M_\alpha f(x) > \lambda\} \subset \bigcup_j c_4 B_j$$

and

$$\frac{1}{[\mu(B_j)]^{1-\alpha}}\int_{B_j}|f(y)|\mathrm{d}\mu(y) > \lambda.$$

Therefore,

$$\begin{aligned} w(\{x\in\mathscr{X} : M_\alpha f(x) > \lambda\}) &\le \sum_j w(c_4 B_j) \\ &\le C\sum_j [\mu(B_j)]^{1-\alpha}\inf_{x\in B_j} M_\alpha w(x) \\ &\le \frac{C}{\lambda}\sum_j\int_{B_j}|f(x)|M_\alpha w(x)\mathrm{d}\mu(x) \\ &\le \frac{C}{\lambda}\int_{\mathscr{X}}|f(x)|M_\alpha w(x)\mathrm{d}\mu(x). \end{aligned}$$

4. Proof of Theorem 1

For each fixed $1 < p \le q < \infty$ and $\gamma > 0$, set $\Phi(t) = t\log^{1+\varepsilon}(\mathrm{e}+t)$ with $0 < \varepsilon < \gamma/q$. Note that if we choose $\delta = \gamma - \varepsilon q$, then

$$\begin{aligned} \Phi^{-1}(t) &\approx \frac{t}{\log^{1+\varepsilon}(\mathrm{e}+t)} = \frac{t^{1/q}}{\log^{(2q-1+\gamma)/q}(\mathrm{e}+t)} \times t^{1/q'}\log^{(q-1+\delta)/q}(\mathrm{e}+t) \\ &\approx \Psi^{-1}(t)\Theta^{-1}(t), \end{aligned}$$

where $\Psi(t) = t^q\log^{2q-1+\gamma}(\mathrm{e}+t)$ and $\Theta(t) = t^{q'}\log^{-1-\delta(q'-1)}(\mathrm{e}+t)$. It is easy to verify that $\Psi(t^{1/q}) \approx t\log^{2q-1+\gamma}(\mathrm{e}+t)$, Θ is doubling and satisfies the $B_{q'}$ condition. We then obtain from Theorem 2 that $M_{\alpha,\Phi}$ is bounded from $L^{q'}(\mathscr{X}, u^{-q'/q})$ to $L^{p'}(\mathscr{X}, v^{-p'/p})$.

On the other hand, for each $\lambda > 0$, set

$$\Omega_\lambda = \{x\in\mathscr{X} : |I_\alpha f(x)| > \lambda\}.$$

The set is bounded, then $u(\Omega_\lambda) < \infty$. By duality, there exists a nonnegative function $g\in L^{q'}(\mathscr{X})$ with $\|g\|_{L^{q'}(\mathscr{X})} = 1$ such that

$$\begin{aligned} u(\Omega_\lambda)^{1/q} &= \|u^{1/q}\chi_{\Omega_\lambda}\|_{L^q(\mathscr{X})} \\ &= \int_{\Omega_\lambda} u(x)^{1/q}g(x)\mathrm{d}\mu(x) \\ &\le \frac{C}{\lambda}\int_{\mathscr{X}}|f(x)|M_{\alpha,\Phi}(u^{1/q}g)(x)\mathrm{d}\mu(x) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{C}{\lambda}\left(\int_{\mathscr{X}}|f(x)|^p v(x)\mathrm{d}\mu(x)\right)^{1/p}\\
&\quad\times\left(\int_{\mathscr{X}}(M_{\alpha,\Phi}(u^{1/q}g)(x))^{p'}v(x)^{-p'/p}\mathrm{d}\mu(x)\right)^{1/p'}\\
&\le \frac{C}{\lambda}\left(\int_{\mathscr{X}}|f(x)|^p v(x)\mathrm{d}\mu(x)\right)^{1/p}\left(\int_{\mathscr{X}}g(x)^{q'}\mathrm{d}\mu(x)\right)^{1/q'}\\
&= \frac{C}{\lambda}\left(\int_{\mathscr{X}}|f(x)|^p v(x)\mathrm{d}\mu(x)\right)^{1/p},
\end{aligned}
$$

where the first inequality follows from Theorem 3, the second inequality follows from the Hölder inequality, and the last one follows from the boundedness of $M_{\alpha,\Phi}$.

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