HIROSHIMA MATH. J.
40 (2010), 343–355

Fractional integrals on Herz–Morrey spaces with variable exponent

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(Received December 21, 2009)
(Revised February 25, 2010)

ABSTRACT. Our aim in this paper is to prove the boundedness of fractional integrals from the Herz–Morrey space with variable exponent $M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\mathrm{R}^n)$ to $M\dot{K}^{\alpha,\lambda}_{q_2,p_2(\cdot)}(\mathrm{R}^n)$.

1. Introduction

The fractional integral operator I^β is defined by

$$I^\beta f(x) := \frac{1}{\gamma(\beta)} \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\beta}} dy, \tag{1}$$

where $0 < \beta < n$ and $\gamma(\beta) := \frac{\pi^{n/2} 2^\beta \Gamma(\beta/2)}{\Gamma((n-\beta)/2)}$. I^β is a bounded operator from the usual Lebesgue space $L^{p_1}(\mathbf{R}^n)$ to $L^{p_2}(\mathbf{R}^n)$ when $1 < p_1 < p_2 < \infty$ and $1/p_1 - 1/p_2 = \beta/n$. This is well-known as the Hardy–Littlewood–Sobolev theorem, and generalized results on some function spaces have been studied. Lu and Yang [8] have proved the boundedness in the setting of Herz spaces. On the other hand, recently the Hardy–Littlewood–Sobolev theorem is extended to the case of Lebesgue spaces with variable exponent. Capone, Cruz-Uribe and Fiorenza [1] and Diening [3] have independently proved that I^β is a bounded operator from the Lebesgue space with variable exponent $L^{p_1(\cdot)}(\mathbf{R}^n)$ to $L^{p_2(\cdot)}(\mathbf{R}^n)$ provided that $p_1(\cdot)$ satisfies the log-Hölder conditions, $\sup_{x \in \mathbf{R}^n} p_1(x) < n/\beta$ and $1/p_1(\cdot) - 1/p_2(\cdot) = \beta/n$.

Motivated by the results above, we consider the Hardy–Littlewood–Sobolev theorem on Herz–Morrey spaces with variable exponent in this paper. The class of Herz–Morrey spaces with variable exponent $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbf{R}^n)$ is initially defined by the author [6], and the boundedness of sublinear operators satisfying a proper size condition on $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbf{R}^n)$ is proved. We also note that Herz–Morrey spaces with variable exponent are generalizations of Morrey–Herz spaces [9] and Herz spaces with variable exponent [5].

2000 *Mathematics Subject Classification*. Primary 42B20; Secondary 42B35, 46B15.
Key words and phrases. Herz–Morrey space with variable exponent, Fractional integral.

Throughout this paper $|S|$ denotes the Lebesgue measure and χ_S means the characteristic function for a measurable set $S \subset \mathbf{R}^n$. A symbol C always means a positive constant independent of the main parameters and may change from one occurrence to another. Given a ball $B = \{y \in \mathbf{R}^n : |x - y| < R\}$, a cube $Q = \prod_{j=1}^n (x_j - R/2, x_j + R/2)$ $(x = (x_1, \dots x_n) \in \mathbf{R}^n,\ R > 0)$ and a constant $s > 0$, we write

$$sB := \{y \in \mathbf{R}^n : |x - y| < sR\}, \qquad sQ := \prod_{j=1}^n (x_j - sR/2, x_j + sR/2).$$

2. Definition of function spaces with variable exponent

In this section we define Lebesgue and Herz–Morrey spaces with variable exponent. Let E be a measurable set in $\mathbf{R}^n$ with $|E| > 0$. We first define Lebesgue spaces with variable exponent.

Definition 1. Let $p(\cdot) : E \to [1, \infty)$ be a measurable function.

(1) The Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by

$$L^{p(\cdot)}(E) := \{f \text{ is measurable} : \rho_p(f/\lambda) < \infty \text{ for some constant } \lambda > 0\},$$

where $\rho_p(f) := \int_E |f(x)|^{p(x)} dx$.

(2) The space $L^{p(\cdot)}_{\mathrm{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(E) := \{f : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset E\}.$$

$L^{p(\cdot)}(E)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(E)} := \inf\{\lambda > 0 : \rho_p(f/\lambda) \le 1\}.$$

We denote

$$p_- := \operatorname{ess\,inf}\{p(x) : x \in E\}, \qquad p_+ := \operatorname{ess\,sup}\{p(x) : x \in E\}.$$

The set $\mathscr{P}(E)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. $p'(\cdot)$ means the conjugate exponent of $p(\cdot)$, namely $1/p(x) + 1/p'(x) = 1$ holds. If $p(\cdot) \in \mathscr{P}(E)$, then the norm $\|f\|_{L^{p(\cdot)}(E)}$ is equivalent to

$$\sup\left\{\int_E |f(x)g(x)|dx : \|g\|_{L^{p'(\cdot)}(E)} \le 1\right\}.$$

More precisely, equivalence

$$\|f\|_{L^{p(\cdot)}(E)} \le \sup\left\{\int_E |f(x)g(x)|dx : \|g\|_{L^{p'(\cdot)}(E)} \le 1\right\} \le r_p \|f\|_{L^{p(\cdot)}(E)} \qquad (2)$$

holds for all $f \in L^{p(\cdot)}(E)$, where $r_p := 1 + 1/p_- - 1/p_+$. We also note that generalized Hölder's inequality

$$\int_E |f(x)g(x)|dx \le r_p \|f\|_{L^{p(\cdot)}(E)} \|g\|_{L^{p'(\cdot)}(E)}$$

is true for all $f \in L^{p(\cdot)}(E)$ and all $g \in L^{p'(\cdot)}(E)$ (see [7]).

Next we define Herz–Morrey spaces with variable exponent motivated by [5, 9]. We use the following notation. For each $k \in \mathbf{Z}$ we denote

$$B_k := \{x \in \mathbf{R}^n : |x| \le 2^k\}, \qquad R_k := B_k \backslash B_{k-1}, \qquad \chi_k := \chi_{R_k}.$$

Definition 2. Let $\alpha \in \mathbf{R}$, $0 < q < \infty$, $p(\cdot) \in \mathscr{P}(\mathbf{R}^n)$ and $0 \le \lambda < \infty$. The Herz–Morrey space with variable exponent $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbf{R}^n)$ is defined by

$$M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbf{R}^n) := \{f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbf{R}^n \backslash \{0\}) : \|f\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbf{R}^n)} < \infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbf{R}^n)} := \sup_{L \in \mathbf{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha q} \|f\chi_k\|^q_{L^{p(\cdot)}(\mathbf{R}^n)} \right)^{1/q}.$$

It obviously follows that $M\dot{K}^{\alpha,0}_{q,p(\cdot)}(\mathbf{R}^n)$ coincides with the Herz space with variable exponent $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbf{R}^n)$ defined in [6].

3. Properties of variable exponent

Given a function $f \in L^1_{\mathrm{loc}}(E)$, the Hardy–Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{r>0} r^{-n} \int_{B(x,r) \cap E} |f(y)|dy \qquad (x \in E),$$

where $B(x,r) := \{y \in \mathbf{R}^n : |x-y| < r\}$. $\mathscr{B}(E)$ is the set of $p(\cdot) \in \mathscr{P}(E)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(E)$. In this section we state some properties of variable exponents belonging to the class $\mathscr{B}(E)$. Cruz-Uribe, Fiorenza and Neugebauer [2] and Nekvinda [10] proved the following sufficient conditions independently. We remark that Nekvinda [10] gave a more general condition in place of (4).

Proposition 1. *Suppose that E is an open set. If $p(\cdot) \in \mathscr{P}(E)$ satisfies*

$$|p(x) - p(y)| \le \frac{-C}{\log(|x-y|)} \qquad \text{if } |x-y| \le 1/2, \tag{3}$$

$$|p(x) - p(y)| \le \frac{C}{\log(e+|x|)} \qquad \text{if } |y| \ge |x|, \tag{4}$$

then we have $p(\cdot) \in \mathscr{B}(E)$.

The next proposition is due to Diening [4, Theorem 8.1]. We remark that Diening has proved general results on Musielak–Orlicz spaces. We describe them for Lebesgue spaces with variable exponent. Given a function f and a cube Q, we write $f_Q := \frac{1}{|Q|}\int_Q f(x)dx$. Let $\mathscr{Y}$ be all families of disjoint and open cubes in $\mathbf{R}^n$.

Proposition 2. *Suppose $p(\cdot) \in \mathscr{P}(\mathbf{R}^n)$. Then the following conditions are equivalent.*

(1) *There exists a constant $C > 0$ such that for all $Y \in \mathscr{Y}$ and all $f \in L^{p(\cdot)}(\mathbf{R}^n)$,*

$$\left\| \sum_{Q \in Y} |f|_Q \chi_Q \right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)}. \tag{5}$$

(2) $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$.

(3) $p'(\cdot) \in \mathscr{B}(\mathbf{R}^n)$.

The following are the key lemmas due to the author [6].

Lemma 1. *If $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$, then there exist constants $0 < \delta < 1$ and $C > 0$ such that for all balls B in $\mathbf{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta}. \tag{6}$$

Lemma 2. *If $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$, then there exists a constant $C > 0$ such that for all balls B in $\mathbf{R}^n$,*

$$C^{-1} \le \frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbf{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbf{R}^n)} \le C. \tag{7}$$

In order to prove Lemma 1, we use the next proposition proved by Diening [4, Lemma 5.5].

Proposition 3. *If $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$, then there exist constants $0 < \delta < 1$ and $C > 0$ such that for all $Y \in \mathscr{Y}$, all non-negative numbers t_Q and all $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ with $f_Q \neq 0$ $(Q \in Y)$,*

$$\left\| \sum_{Q \in Y} t_Q \left|\frac{f}{f_Q}\right|^{\delta} \chi_Q \right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le C \left\| \sum_{Q \in Y} t_Q \chi_Q \right\|_{L^{p(\cdot)}(\mathbf{R}^n)}. \tag{8}$$

Proof (Proof of Lemma 1). Take a ball B and a measurable subset $S \subset B$ arbitrarily. $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$ implies that M satisfies the weak $(p(\cdot), p(\cdot))$ inequality, i.e., for all $f \in L^{p(\cdot)}(\mathbf{R}^n)$ and all $\lambda > 0$ we have

$$\lambda\|\chi_{\{Mf(x)>\lambda\}}\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)}.$$

If we take $\lambda \in (0, |S|/|B|)$ arbitrarily, then we get

$$\lambda\|\chi_B\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le \lambda\|\chi_{\{M(\chi_S)(x)>C\lambda\}}\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le C\|\chi_S\|_{L^{p(\cdot)}(\mathbf{R}^n)},$$

namely

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \le C\lambda^{-1}.$$

Since λ is arbitrary, we obtain

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \le C\frac{|B|}{|S|}. \tag{9}$$

Now we can take a open cube Q_B so that $B \subset Q_B \subset \sqrt{n}B$. Putting $f = \chi_S$ and $Y = \{Q_B\}$ in (8), we get

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_{Q_B}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \le C\left(\frac{|S|}{|Q_B|}\right)^{\delta}.$$

By virtue of $B \subset Q_B \subset \sqrt{n}B$ and (9), we see that

$$\begin{aligned}
\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbf{R}^n)}} &= \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_{Q_B}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \cdot \frac{\|\chi_{Q_B}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \\
&\le \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_{Q_B}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \cdot \frac{\|\chi_{\sqrt{n}B}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \\
&\le C\left(\frac{|S|}{|Q_B|}\right)^{\delta} \cdot \frac{|\sqrt{n}B|}{|B|} \\
&\le C\left(\frac{|S|}{|B|}\right)^{\delta}.
\end{aligned}$$

Hence we have proved Lemma 1.

Proof (Proof of Lemma 2). The left-side inequality in (7) is easily obtained by the generalized Hölder inequality. We will prove the right-hand side inequality. Because $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$, Proposition 2 implies that

$$\| \, |f|_Q \chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le C\|f\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}$$

for all cube Q and all $f \in L^{p(\cdot)}(\mathbf{R}^n)$. Using (2) we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}\|\chi_Q\|_{L^{p'(\cdot)}(\mathbf{R}^n)} \\
&\quad \le \frac{1}{|Q|}\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)} \sup\left\{\int_{\mathbf{R}^n} |f(x)\chi_Q(x)|dx : \|f\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le 1\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \sup\{|f|_Q \|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)} : \|f\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le 1\} \\
&\le C \sup\{\|f\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)} : \|f\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le 1\} \\
&\le C.
\end{aligned}
$$

For each ball B we can take a cube Q_B such that $n^{-1/2}Q_B \subset B \subset Q_B$. Thus we get

$$
\begin{aligned}
\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbf{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbf{R}^n)} &\le \frac{C}{|Q_B|}\|\chi_{Q_B}\|_{L^{p(\cdot)}(\mathbf{R}^n)}\|\chi_{Q_B}\|_{L^{p'(\cdot)}(\mathbf{R}^n)} \\
&\le C.
\end{aligned}
$$

Therefore we have obtained Lemma 2.

4. Boundedness of fractional integrals

In this section we prove boundedness of fractional integrals on Herz–Morrey spaces with variable exponent under proper assumptions. The next result is the Hardy–Littlewood–Sobolev theorem on Lebesgue spaces with variable exponent due to Capone, Cruz-Uribe and Fiorenza [1, Theorem 1.8]. We remark that this result is initially proved by Diening [3] provided that $p_1(\cdot)$ is constant outside of a large ball.

Theorem 1. *Suppose that* $p_1(\cdot) \in \mathscr{P}(\mathbf{R}^n)$ *satisfies conditions* (3) *and* (4) *in Proposition* 1, $0 < \beta < n/(p_1)_+$, *and define* $p_2(\cdot)$ *by*

$$
\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\beta}{n}.
$$

Then we have

$$
\|I^\beta f\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} \le C\|f\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}
$$

for all $f \in L^{p_1(\cdot)}(\mathbf{R}^n)$.

Let $p_2(\cdot) \in \mathscr{P}(\mathbf{R}^n)$ satisfy conditions (3) and (4) in Proposition 1. Then so does $p_2'(\cdot)$. In particular we see that $p_2'(\cdot) \in \mathscr{B}(\mathbf{R}^n)$. Therefore applying Lemma 1 we can take a constant $0 < r < 1/(p_2')_+$ so that

$$
\frac{\|\chi_S\|_{L^{p_2'(\cdot)}(\mathbf{R}^n)}}{\|\chi_B\|_{L^{p_2'(\cdot)}(\mathbf{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^r, \tag{10}
$$

for all balls B in $\mathbf{R}^n$ and all measurable subsets $S \subset B$.

The next theorem is the main result in the present paper.

THEOREM 2. *Suppose that* $p_2(\cdot) \in \mathscr{P}(\mathbf{R}^n)$ *satisfies conditions* (3) *and* (4) *in Proposition* 1, *and take a constant* $0 < r < 1/(p_2')_+$ *so that* (10) *holds. Let* $0 < \beta < nr$, $0 < \alpha < nr - \beta$, $0 < q_1 \le q_2 < \infty$ *and* $0 < \lambda < \alpha$. *Define the variable exponent* $p_1(\cdot)$ *by*

$$\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\beta}{n}.$$

Then we have

$$\|I^\beta f\|_{M\dot{K}^{\alpha,\lambda}_{q_2,p_2(\cdot)}(\mathbf{R}^n)} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\mathbf{R}^n)}$$

for all $f \in M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\mathbf{R}^n)$.

PROOF. Take $f \in M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\mathbf{R}^n)$ arbitrarily. Because $0 < q_1/q_2 \le 1$, we apply inequality

$$\left(\sum_{h=1}^{\infty} a_h\right)^{q_1/q_2} \le \sum_{h=1}^{\infty} a_h^{q_1/q_2} \qquad (a_1, a_2, \ldots \ge 0), \tag{11}$$

and obtain

$$\begin{aligned}\|I^\beta f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_2,p_2(\cdot)}(\mathbf{R}^n)} &= \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1}\left\{\sum_{k=-\infty}^{L} 2^{\alpha q_2 k}\|(I^\beta f)\chi_k\|^{q_2}_{L^{p_2(\cdot)}(\mathbf{R}^n)}\right\}^{q_1/q_2} \\ &\le \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1}\sum_{k=-\infty}^{L} 2^{\alpha q_1 k}\|(I^\beta f)\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\mathbf{R}^n)}.\end{aligned}$$

If we denote $f_j := f\chi_j$ for each $j \in \mathbf{Z}$, then we can write $f = \sum_{j=-\infty}^{\infty} f_j$. Thus we have

$$\begin{aligned}&\|I^\beta f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_2,p_2(\cdot)}(\mathbf{R}^n)} \\ &\quad\le C\sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1}\sum_{k=-\infty}^{L} 2^{\alpha q_1 k}\left(\sum_{j=-\infty}^{k-2}\|(I^\beta(f_j))\chi_k\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\right)^{q_1} \\ &\qquad + C\sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1}\sum_{k=-\infty}^{L} 2^{\alpha q_1 k}\left(\sum_{j=k-1}^{\infty}\|(I^\beta(f_j))\chi_k\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\right)^{q_2} \\ &\quad=: U_1 + U_2.\end{aligned}$$

First we estimate U_1. Using the generalized Hölder inequality, we have that for every $j, k \in Z$ with $k \le L$ and $j \le k-2$,

$$\begin{aligned}
|I^\beta(f_j)(x)\chi_k(x)| &\le \frac{1}{\gamma(\beta)}\int_{R_j}\frac{|f_j(y)|}{|x-y|^{n-\beta}}dy\cdot\chi_k(x)\\
&\le C2^{k(\beta-n)}\int_{R_j}|f_j(y)|dy\cdot\chi_k(x)\\
&\le C2^{k(\beta-n)}\|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\|\chi_j\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}\cdot\chi_k(x).
\end{aligned}$$

By virtue of Lemma 2 we obtain

$$\begin{aligned}
&\|I^\beta(f_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\\
&\quad\le C2^{k(\beta-n)}\|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\|\chi_j\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}\|\chi_k\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\\
&\quad\le C2^{k\beta}\|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\|\chi_j\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}\cdot 2^{-kn}\|\chi_{B_k}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\\
&\quad\le C2^{k\beta}\|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\|\chi_j\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}\cdot\|\chi_{B_k}\|^{-1}_{L^{p_2'(\cdot)}(\mathbf{R}^n)}\\
&\quad= C2^{k\beta}\|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\|\chi_j\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}\cdot\|\chi_{B_j}\|^{-1}_{L^{p_2'(\cdot)}(\mathbf{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p_2'(\cdot)}(\mathbf{R}^n)}}{\|\chi_{B_k}\|_{L^{p_2'(\cdot)}(\mathbf{R}^n)}}\\
&\quad\le C2^{k\beta}\|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\|\chi_j\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}\|\chi_{B_j}\|^{-1}_{L^{p_2'(\cdot)}(\mathbf{R}^n)}2^{nr(j-k)}.
\end{aligned}$$

Now note that

$$\begin{aligned}
I^\beta(\chi_{B_j})(x) &\ge I^\beta(\chi_{B_j})(x)\chi_{B_j}(x)\\
&= \frac{1}{\gamma(\beta)}\int_{B_j}\frac{dy}{|x-y|^{n-\beta}}\cdot\chi_{B_j}(x)\\
&\ge C2^{j\beta}\chi_{B_j}(x).
\end{aligned}$$

On the other hand, $p_1(\cdot)$ belongs to $\mathscr{P}(\mathbf{R}^n)$ and satisfies $0<\beta<n/(p_1)_+$, (3) and (4). Thus applying Theorem 1 and Lemma 2 we get

$$\begin{aligned}
\|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} &\le C2^{-j\beta}\|I^\beta(\chi_{B_j})\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\\
&\le C2^{-j\beta}\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\\
&\le C2^{-j\beta}\cdot 2^{nj}\|\chi_{B_j}\|^{-1}_{L^{p_1'(\cdot)}(\mathbf{R}^n)}\\
&\le C2^{(n-\beta)j}\|\chi_j\|^{-1}_{L^{p_1'(\cdot)}(\mathbf{R}^n)}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\|I^{\beta}(f_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\\
&\quad\le C2^{k\beta}\|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\cdot 2^{(n-\beta)j}\|\chi_{B_j}\|^{-1}_{L^{p_2(\cdot)}(\mathbf{R}^n)}\cdot\|\chi_{B_j}\|^{-1}_{L^{p_2'(\cdot)}(\mathbf{R}^n)}2^{nr(j-k)}\\
&\quad= C2^{(\beta-nr)(k-j)}\|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}(2^{-jn}\|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|\chi_{B_j}\|_{L^{p_2'(\cdot)}(\mathbf{R}^n)})^{-1}\\
&\quad\le C2^{(\beta-nr)(k-j)}\|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}.
\end{aligned}$$

Hence we have

$$U_1\le C\sup_{L\in\mathbf{Z}}2^{-L\lambda q_1}\sum_{k=-\infty}^{L}\left(\sum_{j=-\infty}^{k-2}2^{\alpha j}2^{(\beta-nr+\alpha)(k-j)}\|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\right)^{q_1}.$$

Remark that $\beta-nr+\alpha<0$. We consider the two cases "$1<q_1<\infty$" and "$0<q_1\le 1$".

If $1<q_1<\infty$, then we use the Hölder inequality and obtain

$$\begin{aligned}
U_1&\le C\sup_{L\in\mathbf{Z}}2^{-L\lambda q_1}\sum_{k=-\infty}^{L}\left(\sum_{j=-\infty}^{k-2}2^{\alpha jq_1}2^{(\beta-nr+\alpha)(k-j)q_1/2}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(\mathbf{R}^n)}\right)\\
&\quad\times\left(\sum_{j=-\infty}^{k-2}2^{(\beta-nr+\alpha)(k-j)q_1'/2}\right)^{q_1/q_1'}\\
&= C\sup_{L\in\mathbf{Z}}2^{-L\lambda q_1}\sum_{k=-\infty}^{L}\sum_{j=-\infty}^{k-2}2^{\alpha jq_1}2^{(\beta-nr+\alpha)(k-j)q_1/2}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(\mathbf{R}^n)}\\
&= C\sup_{L\in\mathbf{Z}}2^{-L\lambda q_1}\sum_{j=-\infty}^{L-2}2^{\alpha jq_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(\mathbf{R}^n)}\sum_{k=j+2}^{L}2^{(\beta-nr+\alpha)(k-j)q_1/2}\\
&\le C\sup_{L\in\mathbf{Z}}2^{-L\lambda q_1}\sum_{j=-\infty}^{L-2}2^{\alpha jq_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(\mathbf{R}^n)}\\
&\le C\|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\mathbf{R}^n)}.
\end{aligned}$$

If $0<q_1\le 1$, then we apply (11) replacing q_1/q_2 by q_1 and get

$$\begin{aligned}
U_1&\le C\sup_{L\in\mathbf{Z}}2^{-L\lambda q_1}\sum_{k=-\infty}^{L}\sum_{j=-\infty}^{k-2}2^{\alpha jq_1}2^{(\beta-nr+\alpha)(k-j)q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(\mathbf{R}^n)}\\
&= C\sup_{L\in\mathbf{Z}}2^{-L\lambda q_1}\sum_{j=-\infty}^{L-2}2^{\alpha jq_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(\mathbf{R}^n)}\sum_{k=j+2}^{L}2^{(\beta-nr+\alpha)(k-j)q_1}
\end{aligned}$$

$$\leq C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1}$$

$$\leq C\|f\|_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\mathbf{R}^n)}^{q_1}.$$

Next we estimate U_2. Using Theorem 1 we have

$$U_2 \leq C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} 2^{\alpha q_1 k} \left(\sum_{j=k-1}^{\infty} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \right)^{q_1}$$

$$= C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} \left(\sum_{j=k-1}^{\infty} 2^{\alpha j} 2^{\alpha(k-j)} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \right)^{q_1}.$$

Now we consider the two cases "$1 < q_1 < \infty$" and "$0 < q_1 \leq 1$".

Because $\lambda < \alpha$, we can take a constant $\delta > 1$ so that $\lambda - \alpha/\delta < 0$. If $1 < q_1 < \infty$, then we use the Hölder inequality and obtain

$$U_2 \leq C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} \left(\sum_{j=k-1}^{\infty} 2^{\alpha j q_1} 2^{\alpha(k-j)q_1/\delta} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1} \right)$$

$$\times \left(\sum_{j=k-1}^{\infty} 2^{\alpha(k-j)q_1'(\delta-1)/\delta} \right)^{q_1/q_1'}$$

$$= C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} \sum_{j=k-1}^{\infty} 2^{\alpha j q_1} 2^{\alpha(k-j)q_1/\delta} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1}$$

$$= C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} \sum_{j=k-1}^{L-1} 2^{\alpha j q_1} 2^{\alpha(k-j)q_1/\delta} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1}$$

$$+ C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{\alpha j q_1} 2^{\alpha(k-j)q_1/\delta} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1}$$

$$=: I_1 + I_2.$$

Because $\alpha > 0$, we get

$$I_1 = C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-1} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1} \sum_{k=-\infty}^{j+1} 2^{\alpha(k-j)q_1/\delta}$$

$$= C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-1} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1}$$

$$\leq C\|f\|_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\mathbf{R}^n)}^{q_1}.$$

On the other hand, it follows from $\lambda - \alpha/\delta < 0$ that

$$\begin{aligned}
I_2 &= C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{\alpha(k-j)q_1/\delta} \cdot 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1} \\
&\le C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{\alpha(k-j)q_1/\delta} \cdot 2^{j\lambda q_1} \\
&\quad \times 2^{-j\lambda q_1} \left(\sum_{m=-\infty}^{j} 2^{\alpha m q_1} \|f_m\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1} \right) \\
&\le C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{\alpha(k-j)q_1/\delta} \cdot 2^{j\lambda q_1} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbf{R}^n)}^{q_1} \\
&= C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \left(\sum_{k=-\infty}^{L} 2^{\alpha k q_1/\delta} \right) \left(\sum_{j=L}^{\infty} 2^{j q_1(\lambda-\alpha/\delta)} \right) \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbf{R}^n)}^{q_1} \\
&= C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \cdot 2^{\alpha q_1 L/\delta} \cdot 2^{L q_1(\lambda-\alpha/\delta)} \cdot \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbf{R}^n)}^{q_1} \\
&= C\|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbf{R}^n)}^{q_1}.
\end{aligned}$$

If $0 < q_1 \le 1$, then we use inequality (11) replacing q_1/q_2 by q_1 again and get

$$\begin{aligned}
U_2 &\le C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} \sum_{j=k-1}^{\infty} 2^{\alpha j q_1} 2^{\alpha q_1(k-j)} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1} \\
&= C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} \sum_{j=k-1}^{L-1} 2^{\alpha j q_1} 2^{\alpha q_1(k-j)} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1} \\
&\quad + C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{\alpha j q_1} 2^{\alpha q_1(k-j)} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1} \\
&=: J_1 + J_2.
\end{aligned}$$

The estimate of J_1 is obtained by

$$\begin{aligned}
J_1 &= C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-1} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1} \sum_{k=-\infty}^{j+1} 2^{\alpha q_1(k-j)} \\
&= C \sup_{L\in\mathbf{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-1} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1} \\
&\le C\|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbf{R}^n)}^{q_1}.
\end{aligned}$$

Because $\alpha > 0$ and $\lambda - \alpha < 0$, we have the estimate of J_2 by

$$
\begin{aligned}
J_2 &= C \sup_{L \in \mathbf{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{\alpha q_1 (k-j)} \cdot 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}^{q_1} \\
&\le C \sup_{L \in \mathbf{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{\alpha q_1 (k-j)} \cdot 2^{j\lambda q_1} \|f\|_{M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbf{R}^n)}^{q_1} \\
&= C \sup_{L \in \mathbf{Z}} 2^{-L\lambda q_1} \left(\sum_{k=-\infty}^{L} 2^{\alpha q_1 k} \right) \left(\sum_{j=L}^{\infty} 2^{q_1 j (\lambda - \alpha)} \right) \|f\|_{M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbf{R}^n)}^{q_1} \\
&= C \sup_{L \in \mathbf{Z}} 2^{-L\lambda q_1} \cdot 2^{\alpha q_1 L} \cdot 2^{L q_1 (\lambda - \alpha)} \cdot \|f\|_{M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbf{R}^n)}^{q_1} \\
&= C \|f\|_{M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbf{R}^n)}^{q_1}.
\end{aligned}
$$

Consequently we have proved the theorem.

Acknowledgement

The author is thankful to Professor Naohito Tomita for his comment and encouragement. He is also grateful to the referee whose valuable advices and suggestions improved the first manuscript.

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